Geometric Representation of Functions: Graphs and Level Curves

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Life is Good for only two things: discovering mathematics and teaching mathematics…

- ***Simeon Denis Poisson***

1. Learning Outcomes

After going through this chapter, you should be able to:

- Understand the concept of two variable functions and three-dimensional graph.
- Get to know the various properties of differentiable functions
- Know the steps and method to graphically represent a function in two dimensions lying in a three-dimensional space.
- Understand the different types of three dimensional figures.

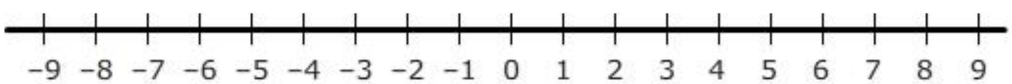
2. Introduction

Two-dimensional (2-D) geometric models are important to be studied as an undergraduate economics course. In economic theory, mathematical analysis is used for the construction of appropriate geometric and analytic generalizations of the 2-D geometric models. In this chapter, we will discuss the basic geometry of coordinates, points and displacements in n-space. Several economic variables are mathematically represented by various functions like demand function, profit function, production function, etc. which are then analyzed using economics. We shall study the complex/non-linear of several variables, in particular functions of two variables. We will be able to visualize functions of several variables.

3. Points In Euclidean Spaces

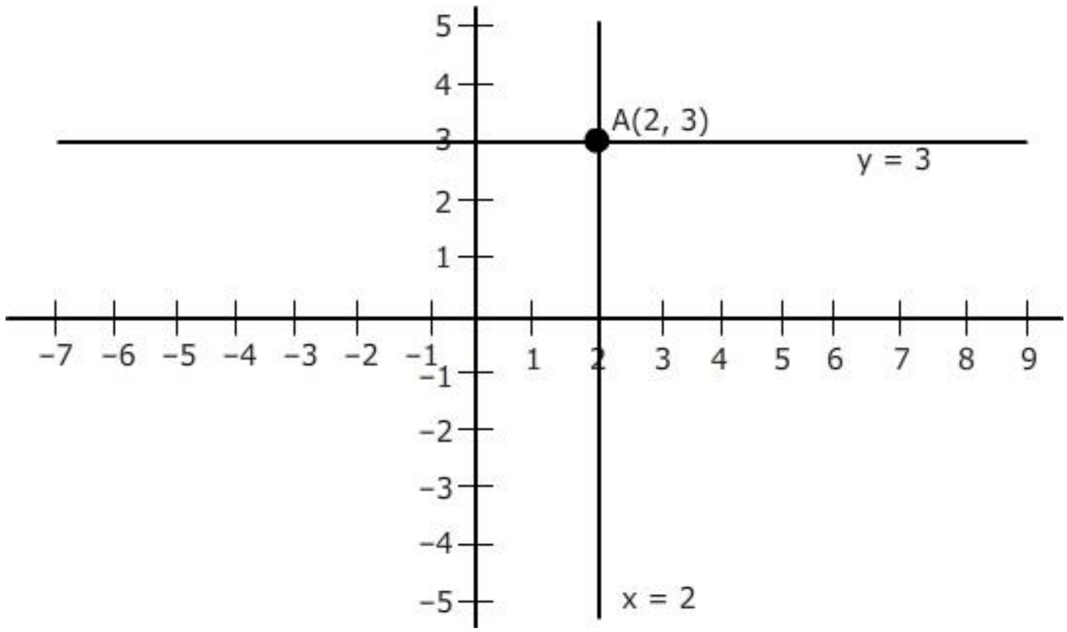
3.1 The Number Line (R^1)

The geometric representation of set of all real numbers is called the number line. Exactly one point represents every real number on the line. One and only one number is represented by those points. Following figure represents a part of a number line;



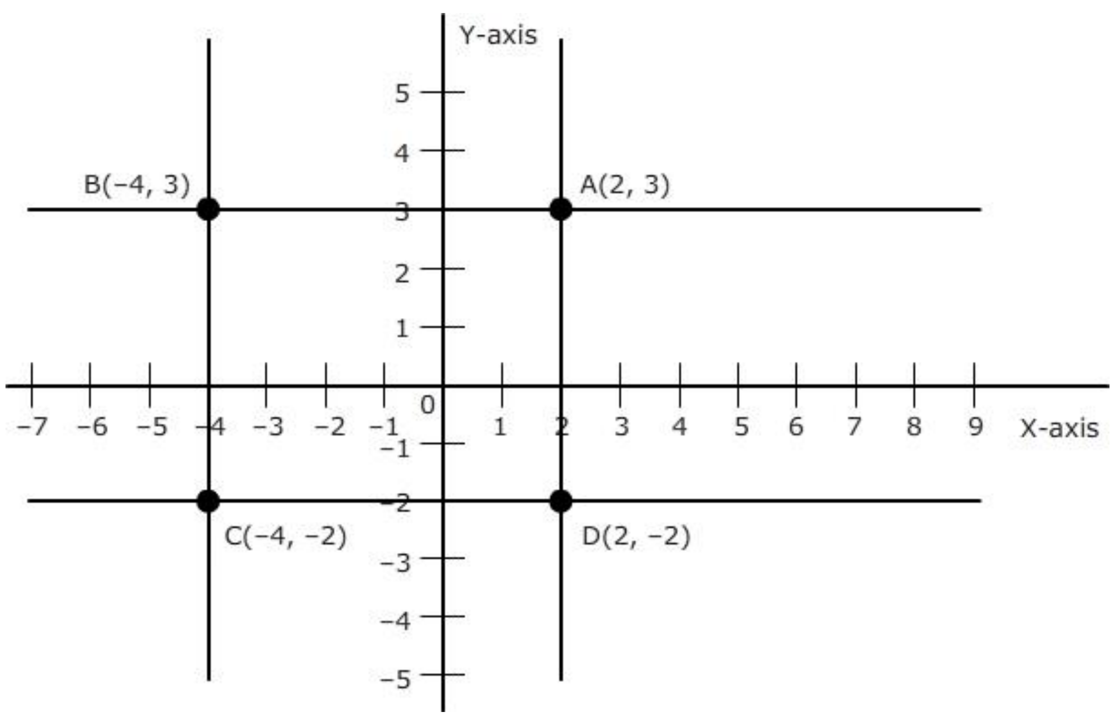
3.2 The Plane (R^2):

Economic objects like consumption bundles are represented using pairs of numbers in some of our economic examples. Cartesian plane or Euclidean 2-space is the geometric representation of such pairs of numbers. It is denoted as $\mathbf{R^2}$**.** R^2 can be shown drawing two perpendicular number lines i.e. the x-axis and the y-axis. We can find points containing x and y coordinates. For example, if we want to find a point ‘A’ with x-coordinate 2 and y-coordinate 3. It can be drawn as follows:



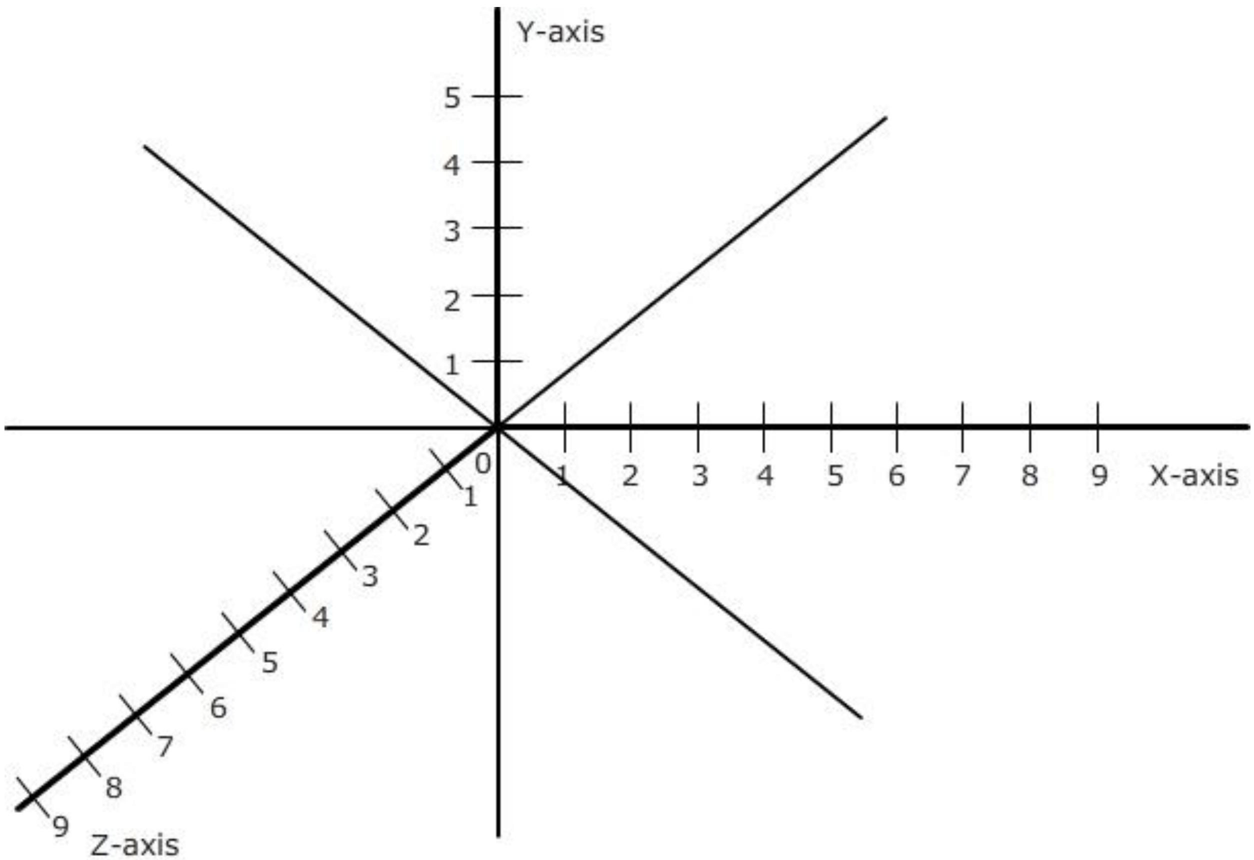
Thus, A is the point (with 2 and 3 as x and y coordinates respectively) in the plane with an ordered pair.

Similarly we can show different points A, B, C and D in R^2 as follows;



3.3 Three Dimensions and its geometric representations

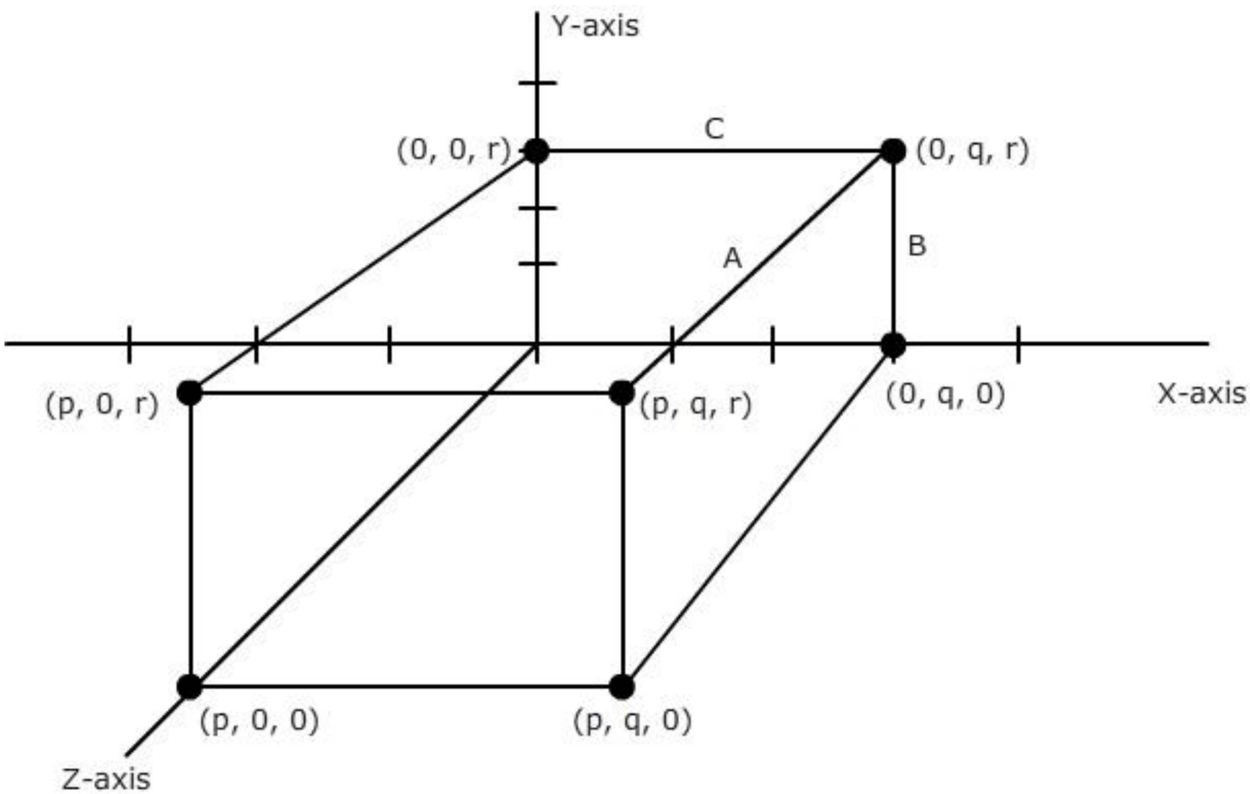
As we visualized the 2-dimensional Euclidean space R^2 by drawing two perpendicular number lines x-axis and y-axis, similarly 3-dimensional R^3 can be represented by drawing three perpendicular lines x-axis, y-axis and z-axis. Generally, x-axis is the horizontal axis and y-axis is the vertical axis on the plane of the page. Now, we draw the third axis i.e. z-axis as follows;



We can use these number lines to find a point with a particular triple of numbers. We can plot the points in the same way as in R^2. Now, ignoring the coordinate p, we can plot coordinates q and r easily on x and y axis as we did earlier. Now, from the point (q,r), move A units in the direction parallel to z-axis. Move forward of the plane; if A is positive and move behind the plane if A is negative. Remain still if A is 0. We finish at a point (p,q,r). Now, we can then take y-axis and z-axis and move B units to the right if B is positive and left if B is negative. Similarly, in x-axis and z-axis, we can move C units up if C is positive and move down if it is negative. We see that whichever method we use, we end up at same point (p,q,r).

The diagram below shows the three dimensional figure with coordinates p, q and r where the coordinates q and r are found using 2-space technique. We have already seen how moving parallel, up, down, right, left, from point (p,q,r) we form a three-dimensional figure with different coordinates as seen in the figure below.

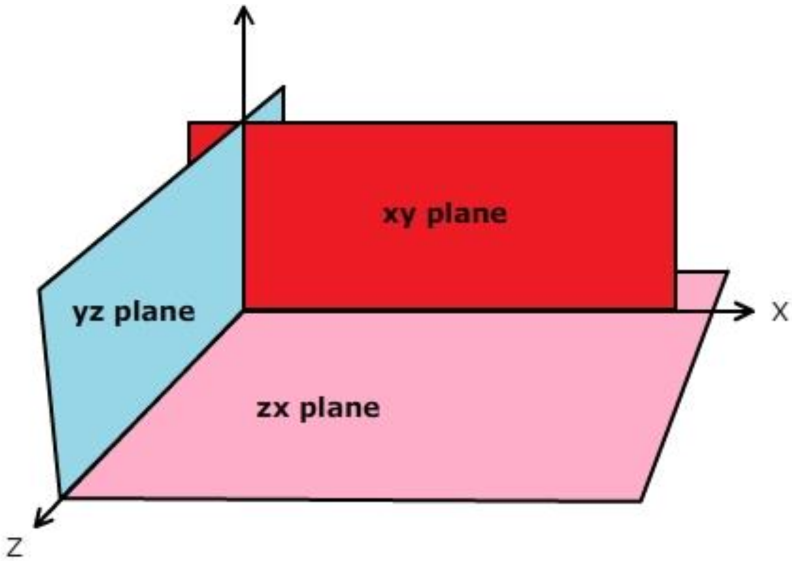
We have thus seen how to plot R^1, R^2 and R^3 using Euclidean space. R^1 consists of single real numbers, represented by a number line. R^2 consists of ordered pairs, represented by a graph i.e. a point set in a plane and R^3 consists of ordered triples, represented by a graph i.e. a point set in 3-space, forming a surface in a space. Thus, Euclidean n-spaces consist of n-tuples of numbers i.e. ordered list of n numbers. Thus, Euclidean n-space is represented as R^n. The number n, called the dimension of R^n, is used to describe how many numbers are required to describe each location, for example, R^3 has three dimensions.



Thus,
Points in three-dimensional space have three coordinates as shown in the figure above.

- The Z-coordinate of a point (p) is its distance in front of the XY-plane.
  (If the Z-coordinate is negative, the point is behind the XY-plane.)
- The X-coordinate of a point (q) is its distance to the right of the YZ-plane.
  (If the X-coordinate is negative, the point is to the left of the YZ-plane.)

- The Y-coordinate of a point (r) is its height above the XZ-plane.
  (If the Y-coordinate is negative, the point is below the XZ-plane.)

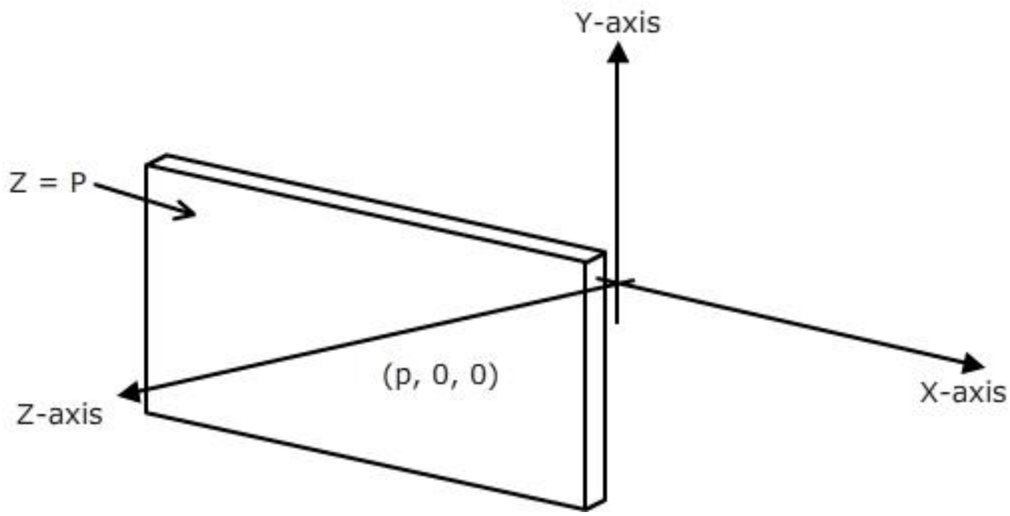


3.4 Surfaces in a Space and its geometric representation

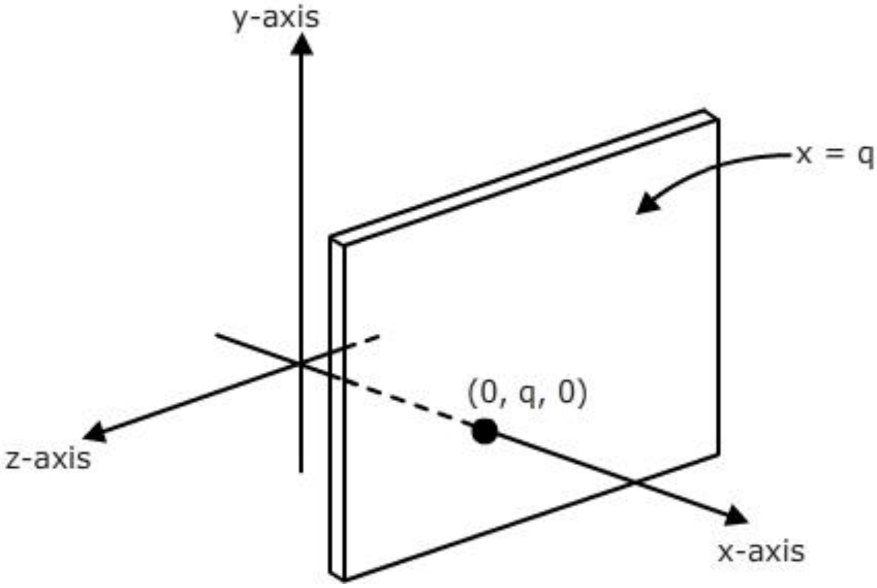
Thus, we can see that the figure above consists of ordered triples (p,q,r), represented by a graph i.e. a point set in 3-space, forming a 3-dimensional surface in a space. The following equations define three simple cases:

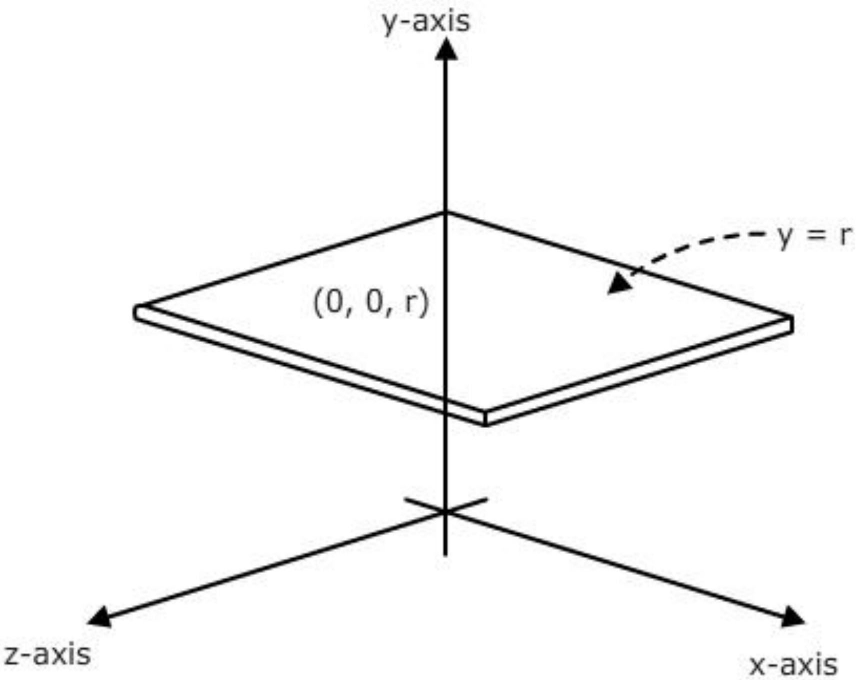
(i) z=p, (ii) x=q, (iii) y=r; which are the only requirements on the variables mentioned.

Thus, we can see that the point (p,q,r) in space which satisfies z=p (with no requirements on x and y) lie in the plane indicated in the figure below:



Similarly, the figures below show the pieces of other two equations:

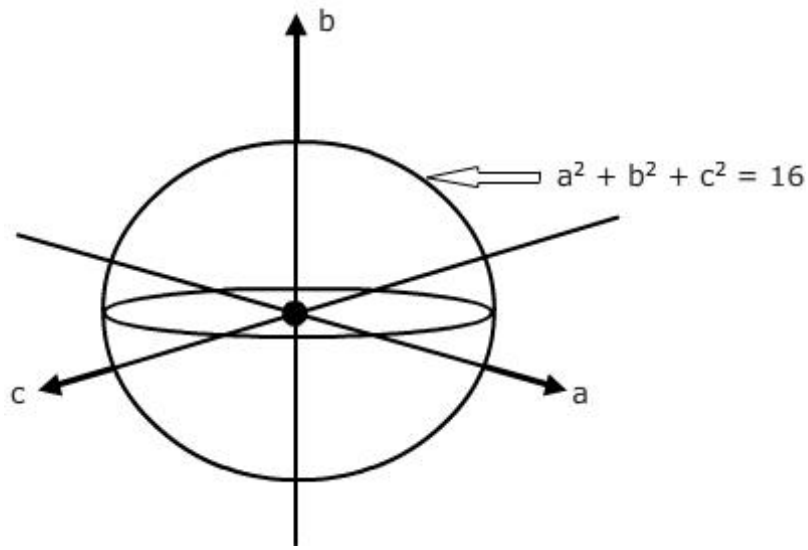




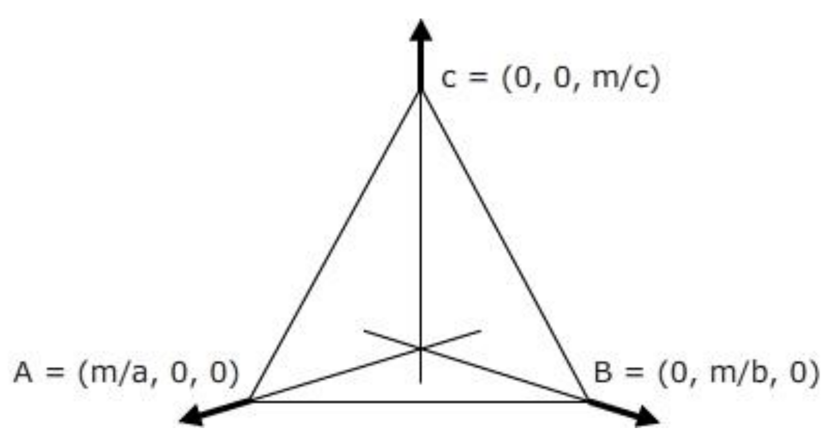
EXAMPLES

1) **SPHERE**: let us consider the equation $\mathbf{a^2 + b^2 + c^2 = 16.}$ Now,

$a^2 + b^2 + c^2 = (a^2-0) + (b^2-0) + (b^2-0)$ is a square of the distance to the point (a,b,c) from the origin (0,0,0). Thus, the equation $a^2 + b^2 + c^2 = 16$ consists of those points (a,b,c) whose distance is 4 from the origin. Thus, it represents a sphere with radius=4 and centre as (0,0,0).



2) **BUDGET LINE**: let us consider a budget equation ax+by+cz=m (representing a budget plane), where m is the income of a consumer which he spends on buying three goods. Let x, y and z be the units he consumes of the three goods respectively and a, b and c be the respective per unit prices of goods that he consumes. Thus, the total cost of buying the three goods is ax+by+cz. The equation ax+by+cz=m is satisfied only by the points (x,y,z) which can be bought if total expenditure is m. Now, it represents a triangle with three vertices i.e. A = (m/a,0,0), B = (0,m/b,0) and C = (0,0,m/c).



Functions between Euclidean Spaces

We generally denote a function from set X to set Y as f:X→Y, which is a rule that assigns one and only one object in Y, to each object in X.

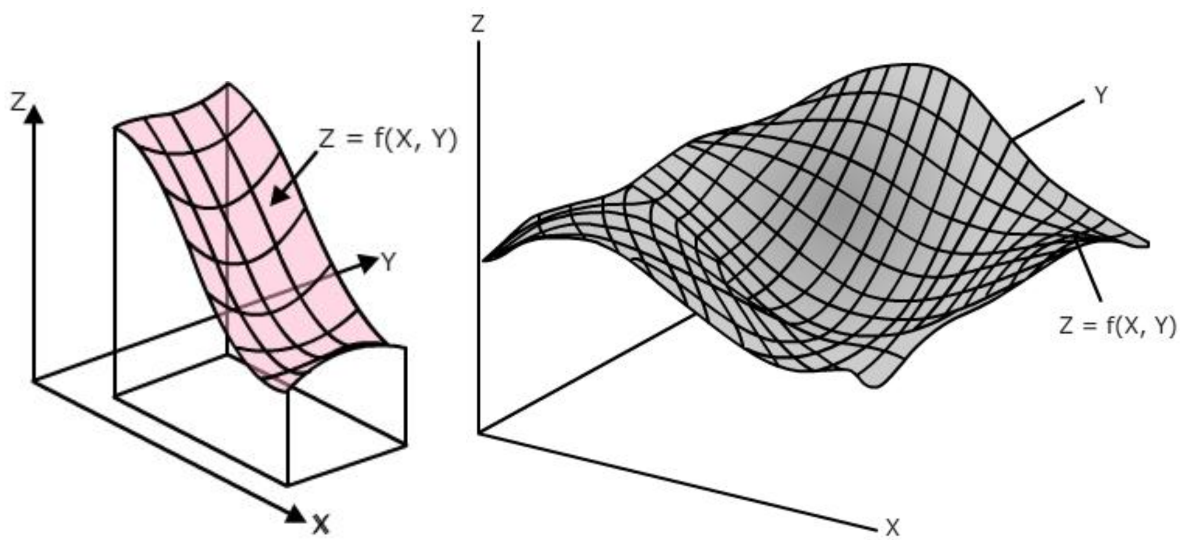
EXAMPLE: Suppose $f(x,y)=x^2 + y^2$ which defines $f:R^2 \rightarrow R^1$. The image of f is set of all non-negative real numbers. The target space of f is R^1 and the domain of f is all of R^2.

EXAMPLE: Suppose f(x) = 1/x. The domain of f(x) is all real numbers except 0. It has the same image as domain i.e. $R^1 - \{0\}$.

4. Geometric Representation Of Functions

4.1 Graph of A Function of Two Variables

Suppose in a set A in the xy-plane, a two variable function is represented by Z = f(X,Y). To construct the graph, we find f at (X,Y), for each value (X,Y) in the domain. Now, we find the point (X,Y,f(X,Y)) in R^3. We sweep out the two-dimensional graph completely if we continue this for all (X,Y). If we plot the graph of the function Z = f(X,Y), in two dimensions, that lies in a three-dimensional space, it turns out to be a smooth surface in space as follows:



However, graphical representation of a function in two dimensions lying in a three-dimensional space is difficult. Thus, we may adopt another method that makes quantitative measurements easy.

4.2 Level Curves For Arbitrary Functions (Z=f(X,Y))

Map makers draw contours to get an idea about altitude variations on earth's surface; like closer the contours, steeper the slope. For example, they draw contours or level curves connecting points on the map representing places on earth's surface at same distance (eg. 100 meters) above the sea level. We can apply the same concept for geographical representation of arbitrary functions Z=f(X,Y). We have already seen that the graph of functions in a three-dimensional space seems as being cut by horizontal planes parallel to XY-plane. This intersection onto the XY-plane is known as the **level curve for height 'a' for ' f '**, if the intersecting plane is z=a. This level curve consists of points that satisfy the equation:

$$\mathbf{f(X,Y) = a}$$

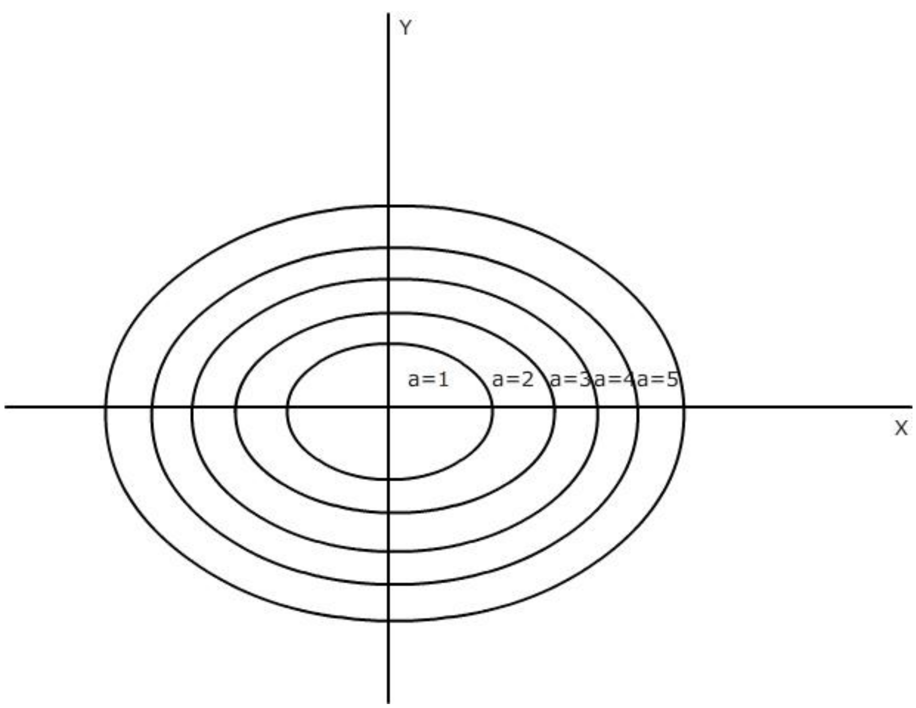
EXAMPLES

1) What are the level curves for the equation $z = f(x,y) = x^2+y^2$. Represent it graphically.

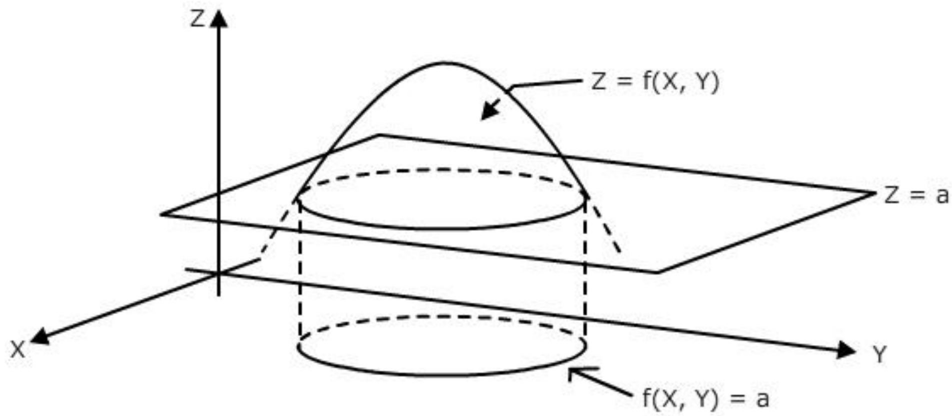
Solution: We know that Z will always be positive i.e. $z \geq 0$. The equation of the level curve is:

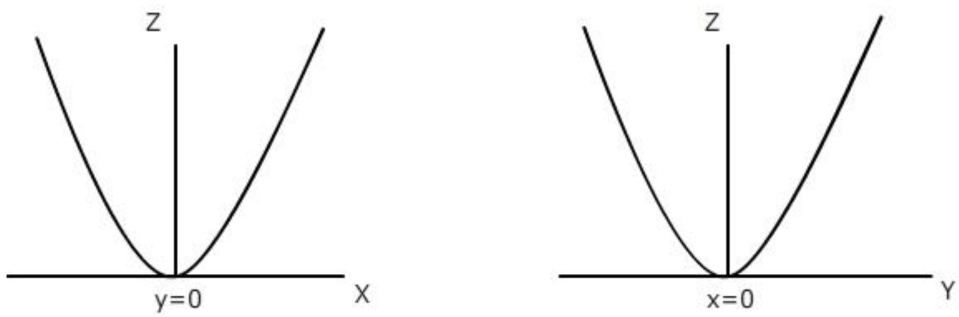
$$\mathbf{x^2+y^2 = a \geq 0}$$

From this equation, we may say that these are circles in the XY-plane with radius $\sqrt{a}$ and centre at the origin.

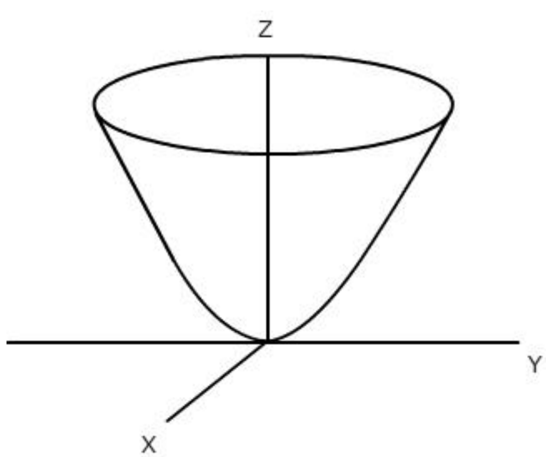


Graphical representation of Z=f(X,Y) and one of its level curves:





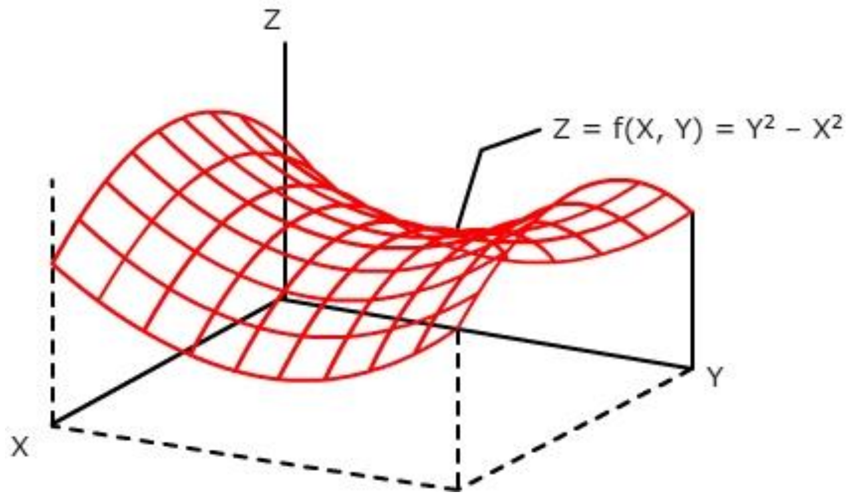
Now, we know that all the level curves are circles. $Z=Y^2$, if X=0, thus it is a parabola in the YZ-plane. Similarly, $Z=X^2$, if Y=0, thus it is a parabola in the XZ-plane. Thus, we got the above figure by rotating the parabola $Z=X^2$ around the Z-axis. The surface is a **paraboloid,** as shown in the figure below:



2) Give the geometric representation of the two variable function $Z=f(X,Y)=Y^2-X^2$.

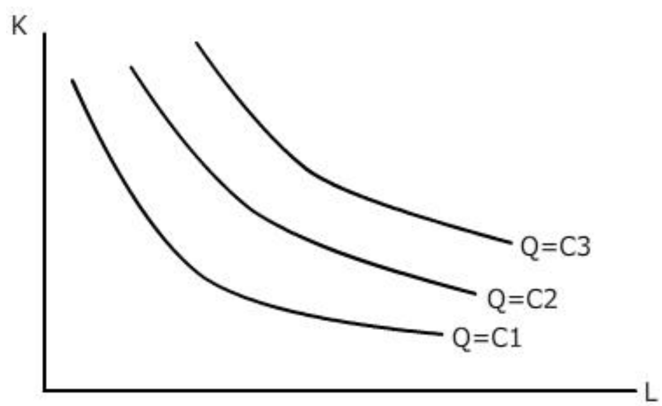
Solution: The equation of the level curve is: $\mathbf{y^2-x^2 = a}$

Now we know, if X=0, $Z=Y^2$ i.e. a parabola in the YZ-plane. Similarly, if Y=0, $Z= - X^2$ is an inverse parabola in XZ-plane. Now, after plotting the graph putting different values of Y in the XZ-plane and continuing the process, we get a saddled-shape graph as follows:

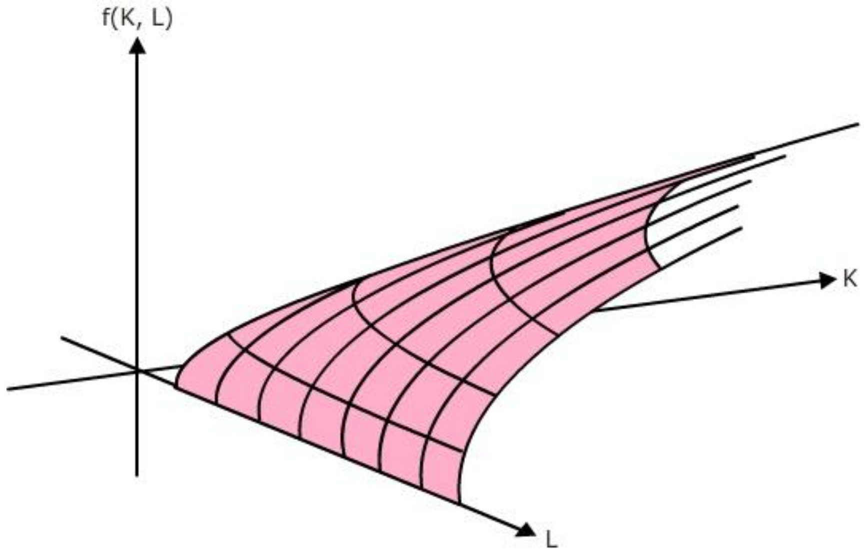


3) Suppose a firm's production function is given by Q=f(K,L)=C; where C is a constant and K and L are capital and labor inputs respectively. Represent it geometrically.

Solution: The level curve for the above production function in the KL-plane is called an iso-quant. Graphically, we can see the iso-quants of the production function Q=K.L as follows:



A Cobb-Douglas production function $f(K,L) = AK^{\alpha}L^{\beta}$; where A>0 and $\alpha+\beta<1$ representing diminishing returns to scale, have level curves similar to those in the figure above. Graphically, we can see:



4) **For the function f(a,b) =** $\frac{3(ab+1)^2}{a^4b^4 - 1}$,

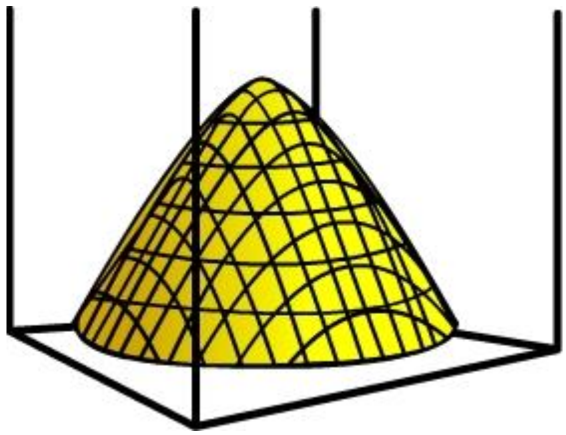
Show that all the points (a,b) satisfying ab=3, lie on a level curve.

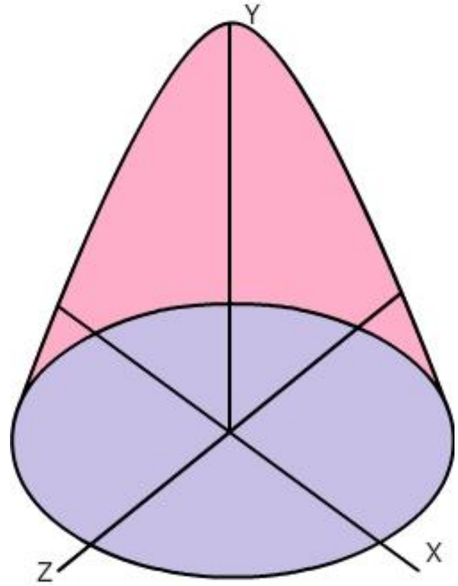
Solution: Substituting the restriction ab=3 in the function f we get,

$$\frac{3(ab+1)^2}{a^4b^4 - 1} = \frac{3(3+1)^2}{3^4 - 1} = \frac{48}{80} = \frac{3}{5}$$

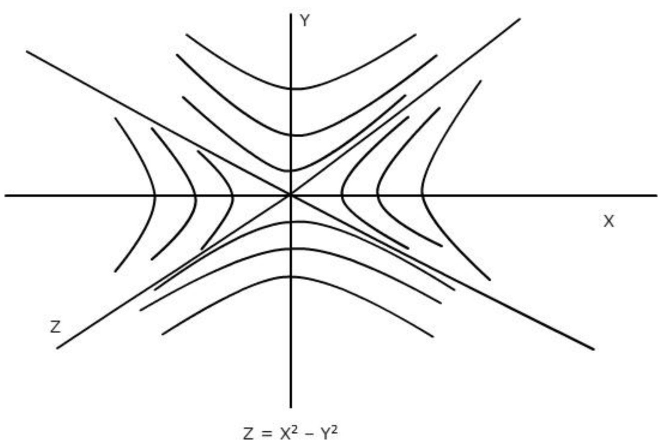
Thus, value of f(a,b) is a constant 3/5, for all values (a,b) where ab=3. Thus, ab=3 lies on a level curve for f, at height 3/5.

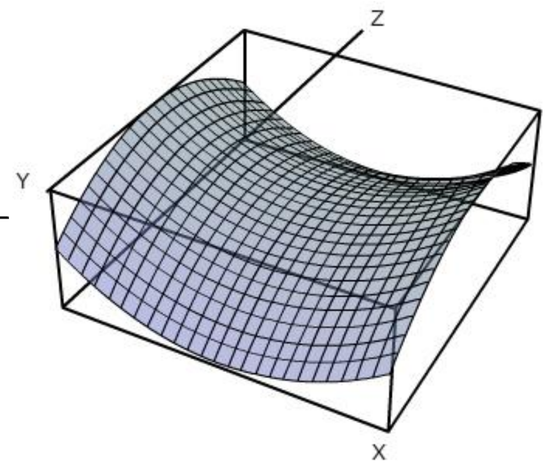
5) The function Z= $\sqrt{1 - X^2 - Y^2}$ is a hemi-sphere above the XY-plane centered at origin and unit radius.



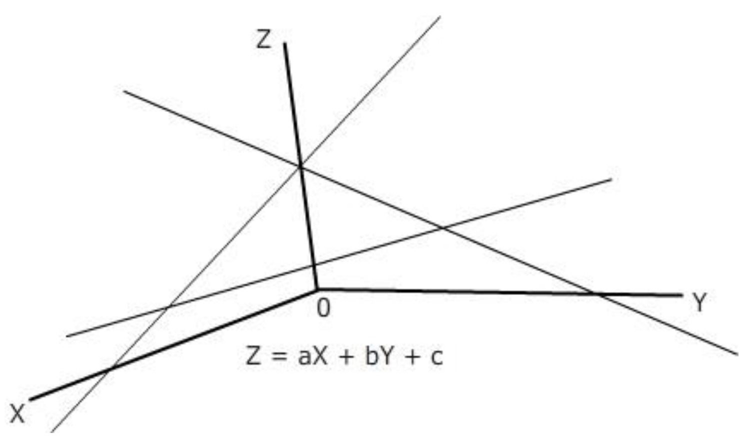


6) The functions Z=XY and Z=X^2-Y^2 are so-called hyperbolic paraboloids.





7) The linear function Z=aX+bY+c has a plane in space for its graph.



5 Differentiability of Functions Of Several Variables

In a function of several variables, if we assign different numerical values to all the variables except one i.e. suppose in a two variable function Z=f(X,Y), if we only X to vary (keeping value of Y constant), the function becomes a one variable function. Suppose we fix the value of $Y=y_0$, and differentiate the function Z at $X=x_0$, we obtain the **partial derivative of Z** w.r.t. X at the point (x_0,y_0). It is represented as $\partial z/\partial x$. Similarly, we can find the partial derivative of Z w.r.t. Y, keeping X constant.

EXAMPLES: Find the partial derivatives of the following functions;

1) Suppose $Z=f(X,Y)=Y^2-X^2$

$\partial Z/\partial X = -2X$ (keeping Y^2 as constant)

$\partial Z/\partial Y = 2Y$ (keeping X^2 as constant)

2) Suppose $Z=Y^2+XY$

$\partial Z/\partial X = Y$ (keeping Y as constant)

$\partial Z/\partial Y = 2Y+X$ (keeping X as constant)

3) $Z=X^2Y$

$\partial Z/\partial X = 2XY$ (keeping Y as constant)

$\partial Z/\partial Y = X^2$ (keeping X^2 as constant)

4) $Z=X^3+Y^2+X^2Y$

$\partial Z/\partial X = 3X^2+2XY$

$\partial Z/\partial Y = 2Y+X^2$

5) Suppose, Z=f(X,Y) and

$$Z = \begin{cases} y & \text{if } y \neq x^2 \\ 0 & \text{if } y = x^2 \end{cases}$$

We can find the partial derivatives of the function Z but the function is not differentiable at the point (0,0).

****** *Thus, we can say that if there exists partial derivatives of a function, which is also continuous at a neighboring point, then the function is differentiable at that point.*

6) Suppose, Z=f(X,Y) and

$$Z = \begin{cases} y/x & \text{if } X \neq 0 \\ 0 & \text{if } X = 0 \end{cases}$$

We can find the partial derivatives of the function Z but the function is not differentiable at the point (0,0). Thus, the function is not differentiable.

7) Find if the function a is differentiable: $a=b^2c+5b^2c^5d-bc+5d$

$\partial a/\partial b = 2bc+10b\ c^5d-c$

$\partial a/\partial c = b^2+25b^2c^4d$

$\partial a/\partial d = 5b^2c^5+5$

Thus, all the partial derivatives exist. Also we can see that the function a is defined and continuous for all the values of b,c and d. Thus, the function is differentiable.

8) Find if the function a is continuous:

$$a = \frac{xy-3}{x^2+y^2-4}$$

The function is not defined if the denominator is zero. We can see that the function a is defined for all values (X,Y) except the points lying on the circle $X^2+Y^2=4$. Thus, the function is not continuous.

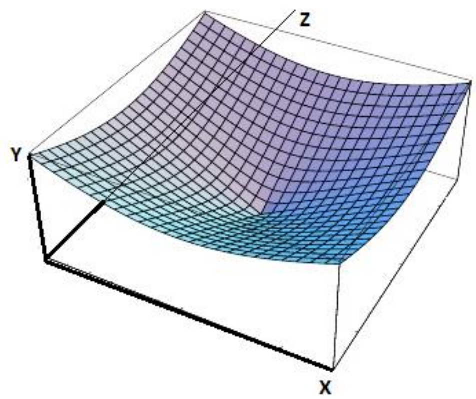
6 Exercise

1) Prove that the level curve of the function z, where

Z= $\sqrt{x^2+y^2}-x^2-y^2+2$ has level curves centered at origin. Also show that X2+Y2=6 is a level curve of the function Z.

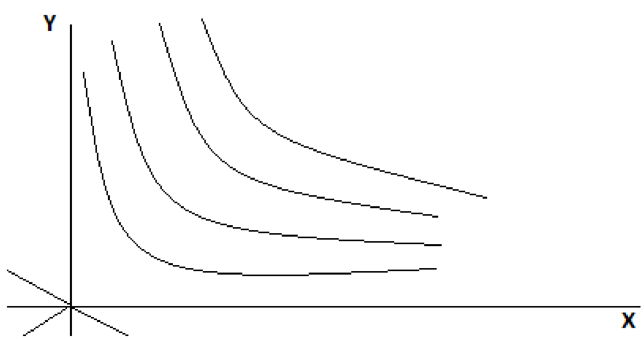
2) Plot the graph for Z= $-\sqrt{X^2+Y^2}$

[*Hint: we can trace it as absolute value functions and by staring at it we see that Z=|r|, thus it is a cone.]*



3) Plot the graph for Z=X^2-Y^2. [Hint: see section 4.2, example 6.]
4) Plot Z=$\sqrt{XY}$

Hint: The contours can be plotted as follows:



We know that this is also an indifference curve. Thus we can plot the three-dimensional figure from it.

5) Find the first partial derivatives of the following functions Z=f(X,Y):

(a) $\sqrt{x^2 + y^2}$.

(b) $\log \sqrt{(1 + x^2 + y^2)}$

(c) e^{x-y}.

(d) $\dfrac{1}{\sqrt{(1 + x + y^2 + z^2)}}$

6) Find the first partial derivatives of the following functions Z=f(X,Y):

(a) XY

(b) Log XY

(c) X^Y

(d) e^{xy}

7) How would you use the graph of Z=f(X,Y), to draw level curves of f?

8) Geometrically plot the graphs of the following functions:

(a) Z=5-X-Y

(b) $Z= - X^2 - Y^2$

7 References

K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, Pearson Educational Asia, Delhi, 2002

Carl P. Simon, Lawrence Blume, Mathematics for Economists