Chapter :Unconstrained Optimization

Content Developers: Rakhi Arora & Vaishali Kapoor

Table of Contents

Learning outcomes:

After you have read this chapter, you should be able to:-

1. Explain concept of optimum value.
2. Calculate optimum given different functions.
3. Apply calculus of optimization in economic analysis.

Introduction

The 'minimization' of any function (like cost of producing an output) and 'maximization' (for ex : of profit function, consumer's utility function and country' economic growth) is jointly known as 'optimization' i.e. 'quest for the best. So far, you have studied optimization problem wherein only one independent variable affected function to be optimized. But in real world, there may be and there are more than one or n-variables affecting our objective function.

In case there are two or more choice variables that affect the objective function then optimisation techniques need to studied from lens of Total Differential instead of Derivative technique (used in case of one choice variable). So to create a analogy first we will study optimisation of objective function of one choice variable with help of Differential. Also, geometric characteristics would be analysed for one variable objective function.

In the first section of this chapter, we will analyse geometric characteristics and calculus of optimisation in case of one variable. In the second section, we deal with the optimisation in two variable case. In third section, quadratic forms of total differential is covered and sufficient conditions for n-variable case is derived. In last section, application of optimization technique in economics is discussed.

The problem of optimization with a single choice variable would be revisited again but in terms of differentials.

First Order Condition

Consider $f(x) = z$ then at a minimum (or maximum)(point, the necessary condition for extremum at say point A is $f'(x) = 0$ or $dz = 0$ as x varies.

$$dz = f'(x).dx$$

From derivative form we know that $f'(x)$ must be zero. $f'(x) = \frac{dz}{dx} = 0$. This is equivalent to saying that $dz = 0$ as x varies.

Second Order Condition

Consider the following Fig. 1, at each point A, B and C. $dz = 0$. Point A is a maximum, B is a minimum but C is neither so it can be stated that $dz = 0$ is necessary condition but not sufficient.

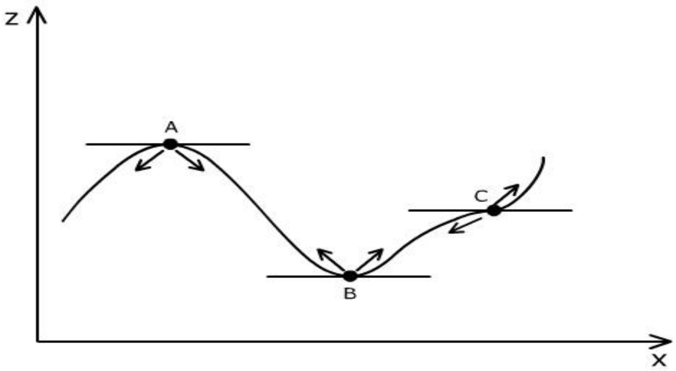


Fig. 1

For A to be maximum, $dz = 0$ and $dz < 0$ as we move in both directions i.e. dz should be decreasing as move away from A. It is equivalent to saying that $d(dz) < 0$, i.e. $d^2z < 0$.

At a minima like at B, dz is increasing as we move away from B i.e. $d^2z > 0$. This gives the sufficient condition for optimization. At C point dz is increasing when we move to the right direction and decreasing when we move to the left direction.

Let us calculate $d^2 z$.

$$d^2 z = d(dz) = d[f'(x)dx] = f''(x)dx^2$$

So the sufficient condition becomes:

For maximum of z : $d^2 z \leq 0$

For minimum of z : $d^2 z \leq 0$

I. Optimization in case of two variables

Now, the stage is set to extend the analysis of optimization to two variables:

1. Geometrical Characteristics

Techniques of optimization and geometrical characteristics could be analogously constructed from one variable to two variables. For a function of one variable, an extreme value is represented by peak of a hill or bottom of a valley in a two-dimensional graph as in Fig. 1. When z becomes function of *x* and *y* ($z = f(x, y)$), now plotting this graph in 3D space these hills and valleys will appear as domes and bowls like in Fig. 2 and Fig. 3.

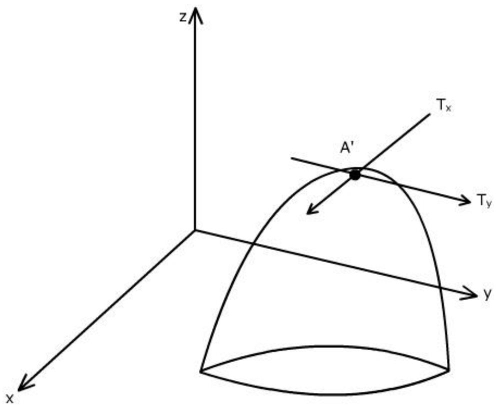


Fig. 2

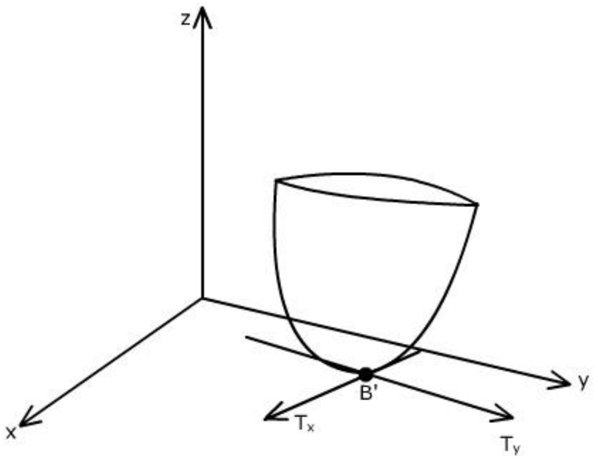


Fig. 3

In 2-D space, when a function of one variable is graphed at points, where it attains minimum or maximum, tangent is parallel to *x*-axis. i.e. axis of choice variable. At point A in Fig. 1 $f(x)$ attains maximum and tangent is parallel to x-axis. Likewise, in Figure 2; at point A′ Tangent T_x is parallel to x-axis in xz plane and also T_y is parallel to y-axis in yz plane. Alternatively, like is Figure 4, we could have a plane tangent to z at point A." The 'hill' in 2D is equivalent to dome in 3D.

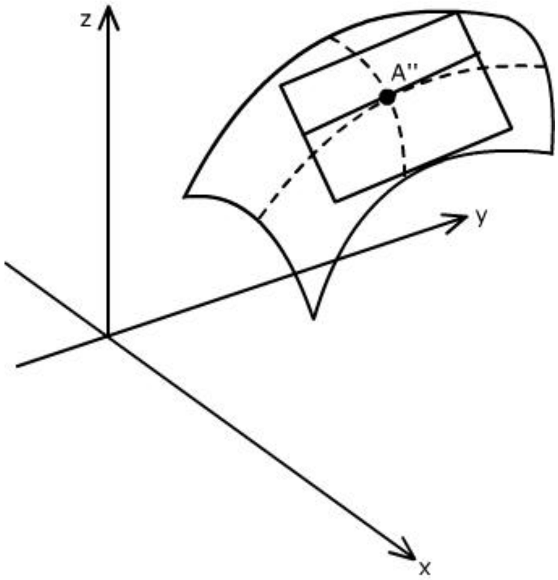
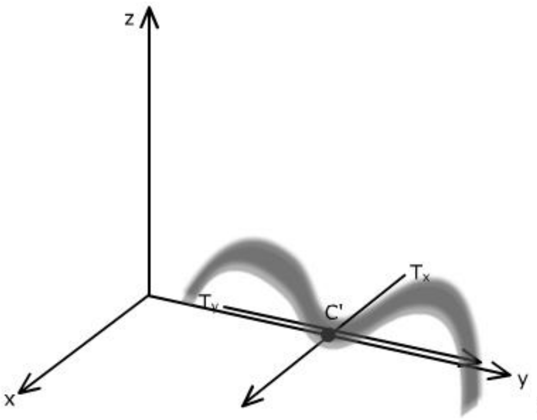


Fig. 4

Similarly, at minimum value at point B in Fig. 1, we have tangent parallel to x axis. At point B'; in Figure 3 we have T_x : tangent parallel to x-axis in xz plane and T_y is parallel to y-axis in yz plane. The valley in 2D (for minimum) is transformed as a bowl in 3D.

In Figure 1, at point C, tangent is parallel to x-axis but x is neither a maximum nor minimum but rather a point where curvature of the curve changes i.e. inflation point. Similarly, in Figure 5 and 6 points C and C" respectively, are not optimum points even when tangents T_x and T_y are parallel to x-axis in xz plane and y-axis in yz plane respectively



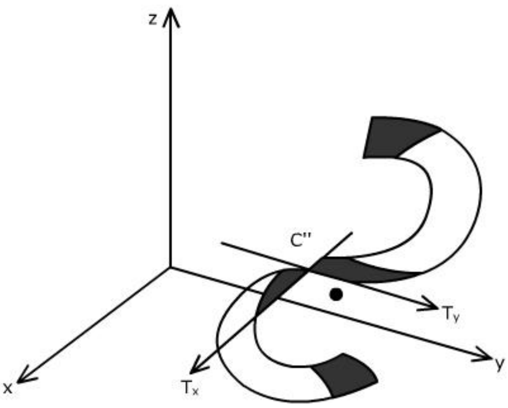


Fig. 5. **Fig. 6**

At point C' is Figure 5 it is a minimum when viewed in background of *yz* plane and maximum when viewed in background of *xz* plane. A point with such dual personality is known as 'saddle' point.

At C'' in fig. 6, then surface get twisted so it is a inflexion point like point C in Fig. 1.

2. Differential Conditions for Optimization

From here on we would consider the following functional form i.e. $z = f(x, y)$.The first order condition for an extremum is $dz = 0$. With reference to $z = f(x, y)$; dz is total differential given by

$$dz = f_x dx + f_y dy$$

and $dz = 0$ implies that an extremum point must be a stationary point and simultaneously *z* must be constant for changes of two variables *x* and *y*. This paramounts to saying that not both *dx* and *dy* are zero.

Then for *dz* to be zero it is necessary that partial derivatives f_x and f_y must be zero. So, the first order condition becomes:

$$f_x = f_y = 0$$

This condition was satisfies by point A' in Fig. 2. B' in Fig. 3, C' in Fig.5 and C" in Fig. 6. But at points C' and C" the function did not attain an extreme value so, this condition though necessary but is not sufficient.

As in one variable case second order total differential will determine the whether any point is extremum or not. For the function $z = f(x, y); d^2z$ is calculated as follows:

$$d^2z = d(dz)$$

$$= \frac{\partial(dz)}{\partial x} dx + \frac{\partial(dz)}{\partial y} .dy$$

$$= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy$$

$$= f_{xx}dx^2 + f_{yx}dydx + f_{xy}dxdy + f_{yy}dy^2$$

By young's theorem we know that cross partial derivatives are identical i.e. $f_{yx} = f_{xy}$. So $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$.

For maximum of *z*, $d^2z < 0$ becomes sufficient condition for maximum of *z* for *dx* and *dy* both not being zero and likewise for minimum of *z*, $d^2z > 0$. For any values of *dx* and *dy*, not both zero.

$$d^2z \begin{cases} < 0 & \text{if } f_{xx} < 0;\ f_{yy} < 0 \text{ and } f_{xx}f_{yy} > f_{xy}^2 \\ > 0 & \text{if } f_{xx} > 0;\ f_{yy} > 0 \text{ and } f_{xx}f_{yy} > f_{xy}^2 \end{cases}$$

Example 1. Consider the function $z = e^{2x} - 2x + 2y^2 + 3$. To find extreme values of z following are relevant partial derivatives:

$$f_x = ze^{2x} - 2 \qquad f_{xx} = 4e^{2x}$$

$$f_y = 4y \qquad f_{yy} = 4 \qquad f_{xy} = 0$$

First order necessary conditions become:

$$f_x = 0$$

$\Rightarrow \quad 2e^{2x} - 2 = 0$

$\Rightarrow \quad e^{2x} = 1$

$\Rightarrow \quad 2x = 0$

$\Rightarrow \quad x = 0$

and $\quad f_y = 0$

$\Rightarrow \quad 4y = 0$

$\Rightarrow \quad y = 0$

So the stationary point for z is (0, 0). To know whether z attains minimum or maximum at (0, 0), lets check second order conditions.

$$f_{xx}(0,0) = 4$$

$$f_{yy}(0,0) = 4$$

$$f_{xy}(0,0) = 0$$

$$f_{xx} f_{yy} = 16$$

$\because \quad f_{xx} > 0; \quad f_{yy} > 0 \quad$ and $\quad f_{xx} f_{yy} > f_{xy}^2$

hence z attains minimum at point (0, 0).

z at (0, 0) is $e^0 - 0 + 0 + 3 = 4$ so the maximum point is denoted by (0, 0, 4)

III. Quadratic Forms

The expression for d^2z is an example of quadratic form. Second order condition for extremum can also be studied from perspective of quadratic equations. Consider the following:

Let $\quad a = dx,\ v = dy,\ a = f_{xx},\ b = f_{yy}$ and $h = f_{xy}$

So that $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$

becomes $d^2z = au^2 + 2huv + bv^2$

Here d^2z is function of *u* and *v* in quadratic form. *u* and *v* which are *dx* and *dy* are assumed variables here. This sort of analysis is totally different when we earlier dealt with second order condition where second order partial derivatives were variables.

But here we are interested in knowing the sign that d^2z may assume at an extremum. d^2z needs to be negative for a maximum and positive for minimum. The second order total differential will assume value at a point depending upon specific value of partial derivatives for any value of *dx* and *dy*. Hence, *dx* and *dy* can vary but at a extremum value second order partial derivatives should have specific sign.

a) Detrimental Test for sign Definiteness

For two variable case,

$$d^2z = au^2 + 2huv + bv^2;$$

the sign of first and third terms are independent of the values of variables *u* and *v* since they are squared in the above equation. Thus, for positive or negative definiteness of these terms alone, depend on signs of *a* and *b*. But sign of middle term could turn the sign of d^2z.

If the entire polynomial could be made as function of *u* and *v* wherein *u* and *v* appear in squares. Let us complete the squares

$$d^2z = au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2$$

$$= a\left(u^2 + \frac{2h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2$$

$$= a\left(u + \frac{h}{a}v\right)^2 + \frac{ab - h^2}{a}v^2$$

Now, d^2z is positive definite iff $a > 0$ this would give first term as positive and second term would also be positive iff $ab - h^2 > 0$.

Also, d^2z is negative definite iff $a < 0$ and along with it $ab - h^2 > 0$.

Hence $ab - h^2 > 0$ is a prerequisite for either extremum value. It is equivalent to saying $ab > h^2$. Hence *ab* should be positive and it would be so when *a* and *b* assume identical signs.

For positive definite of d^2z (minimum of z); $a(= f_{xx})$ should be positive, $b(= f_{yy})$ should be positive and $ab - h^2 (= f_{xx} f_{yy} - f_{xy}^2)$ should be greater than zero. For negative definite of d^2z (maximum of z); $a(= f_{xx})$ and $b(= f_{yy})$ should be negative and $ab - h^2 (= f_{xx} f_{yy} - f_{xy}^2)$ should be greater than zero. This is same as derived earlier.

Another way of expressing d^2z is as follows:

$$d^2z = [u \;\; v]\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Let us denote matrix $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$ by *D*

then

$$d^2z \; is \begin{Bmatrix} \text{positive definite iff } |a| > 0 \\ \text{negagive definite iff } |a| < 0 \end{Bmatrix} \text{and } |D| > 0$$

where $|a|$ is subdeterminant of $|D|$. It is also known as first principal minor of $|D|$.

b) Three-Variable Quadratic Forms

In case z is function of three variable x, y and w,

$$d^2z = f_{xx}(dx^2) + f_{xy}dxdy + f_{xw}dxdw + f_{yx}dydx + f_{yy}d_y^2$$

$$+ f_{yw}dydw + f_{wx}dwdx + f_{wy}dwdy + f_{ww}dw^2$$

$$= [dx \quad dy \quad dw] \begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{yx} & f_{xy} & f_{yw} \\ f_{wx} & f_{wy} & f_{ww} \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dw \end{bmatrix}$$

$|D_1| = f_{xx}$; $\quad |D_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ and

$$|D_3| = \begin{bmatrix} f_{xx} & f_{xy} & f_{xw} \\ f_{yx} & f_{yy} & f_{yw} \\ f_{wx} & f_{wy} & f_{ww} \end{bmatrix}$$

where $|D_i|$ denotes ith principal minor of $|D|$.

Transforming d^2z into a form where we have completed the squares, d^2z can be written as follows:

$$d^2z = f_{xx}\left(dx + \frac{f_{xy}}{f_{xx}}dy + \frac{f_{xw}}{f_{xx}}dw\right)^2$$

$$+ \frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}}\left(dy + \frac{f_{xx}f_{yw} - f_{xy}f_{xw}}{f_{xx}f_{yy} - f_{xy}^2}dw\right)^2$$

$$+ \frac{f_{xx}f_{yy}f_{ww} - f_{xx}f_{yw}^2 - f_{yy}f_{xw}^2 - f_{ww}f_{xy}^2 + 2f_{xy}f_{xw}f_{yw}}{f_{xx}f_{yy} - f_{xy}^2}(dw)^2$$

this is equivalent to :

$$d^2z = |D_1|\left(dx + \frac{f_{xy}}{f_{xx}}dy + \frac{f_{xw}}{f_{xx}}dw\right)^2$$

$$+\frac{|D_2|}{|D_1|}\left(dy+\frac{f_{xx}f_{yw}-f_{xy}f_{xw}}{f_{xx}f_{yy}-f_{xy}^2}dw\right)^2+\frac{|D_3|}{|D_1|}(dw)^2$$

Hence, d^2z would be positive definite if

$$|D_1|>0, |D_2|>0 \text{ and } |D_3|>0$$

So, for minimum of z all principal minors must be positive. For negative definitness of d^2z;

$|D_1|<0$

$|D_2|>0$ [given that $|D_1|<0$ already]

$|D_3|<0$ [given that $|D_2|>0$ already]

Thus for maximum of z principal minors must alternate in sign in specified manner.

c) Extending it to n-variable case

Let z be function of $(x_1, x_2, ..., x_n)$ then

$$d^2z=[dx_1, dx_2, ..., dx_n]\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & \cdots & \cdots & f_{nn} \end{bmatrix}\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ \vdots \\ dx_n \end{bmatrix}$$

For minimum of z (positive definiteness of d^2z all principal minors: $|D_1|, |D_2|, ..., |D_n|$ should be positive. For maximum of z (negative definiteness of d^2z) principal minors should alternate in sign. If k is even then $|D_k|$ should be positive and if k is odd $|D_k|$ should be negative. It is equivalent to saying that $(-1)^k|D_k|>0$.

IV. Applications of Optimization Techniques in Economics

1. Profit Maximisation by a Multiproduct firm. When a firm produces more than one product; firm's profit depend on production / revenue of all products. Lets assume that a firm produces two types of goods X_1 and X_2. Let prices be p_1 and p_2 of goods X_1 and X_2 respectively. The cost function of the firm is given by

$$C = F(x_1, x_2).$$

Then the profit function π can be written as:

$$\pi = p_1 x_1 + p_2 x_2 - C$$

The above profit function has to be maximised with respect to x_1 and x_2, both. This in turn would depend on form of market. Market could have perfect competition or monopoly.

a) Perfectly competitive markets for X_1 and X_2.

The first order conditions for profit maximization are : -

$$\frac{\partial \pi}{\partial x_1} = p_1 - \frac{\partial c}{\partial x_1} = 0 \text{ or } p_1 = MC_1$$

$$\frac{\partial \pi}{\partial x_2} = p_2 - \frac{\partial c}{\partial x_2} = 0 \text{ or } p_2 = MC_2$$

The second order conditions are : -

$$\frac{\partial^2 \pi}{\partial x_1^2} < 0 \Rightarrow \frac{\partial^2 c}{\partial x_1^2} > 0, \frac{\partial^2 \pi}{\partial x_2^2} < 0 \Rightarrow \frac{\partial^2 \pi}{\partial x_2^2} \text{ and}$$

$$\frac{\partial^2 \pi}{dx_1^2} \cdot \frac{\partial^2 \pi}{dx_2^2} > \left(\frac{\partial^2 z}{\partial x_1 \partial x_2} \right)$$

b) When markets for X_1 and X_2 are not perfectly competitive. Let demand function for X_1 be $x_1 = f_1(p_1, p_2)$ and X_2 for $x_2 = f_2(p_1, p_2)$.

Profit function can be written as

$$\pi = R(x_1, x_2) - C(x_1, x_2)$$

where $R(x_1, x_2)$ is revenue function for the firm.

The first order conditions for maximum profit are

$$\frac{\partial \pi}{\partial x_1} = \frac{\partial R}{\partial x_1} - \frac{\partial C}{\partial x_1} = 0 \Rightarrow MR_1 = MC_1$$

$$\frac{\partial \pi}{\partial x_2} = \frac{\partial R}{\partial x_2} - \frac{\partial C}{\partial x_2} = 0 \Rightarrow MR_2 = MC_2$$

Second order conditions are :

$$\frac{\partial^2 \pi}{\partial x_1^2} = \frac{\partial^2 R}{\partial x_1^2} - \frac{\partial^2 C}{\partial x_1^2} < 0 \Rightarrow \frac{\partial MR_1}{\partial x_1} < \frac{\partial MC_1}{\partial x_1}$$

$$\frac{\partial^2 \pi}{\partial x_2^2} = \frac{\partial^2 R}{\partial x_2^2} - \frac{\partial^2 C}{\partial x_2^2} < 0 \Rightarrow \frac{\partial MR_2}{\partial x_2} < \frac{\partial MC_2}{\partial x_2}$$

and $$\frac{\partial^2 \pi}{\partial x_1^2} . \frac{\partial^2 \pi}{\partial x_2^2} > \left(\frac{\partial^2 \pi}{\partial x_1 \partial x_2} \right)^2$$

Example 2 : Let a firm under perfect competition produces X_1 and X_2 with prices equal to Rs. 10 and Rs.15 respectively. If cost function of the firm is $C = 2x_1^2 + x_1 x_2 + 2x_2^2$ where x_1 and x_2 denote the levels of output X_1 and X_2 respectively.

The profit function of the firm becomes:

$$\pi = 10x_1 + 15x_2 - 2x_1^2 - x_1 x_2 - 2x_2^2$$

For maximum profit, $\frac{\partial \pi}{\partial x_1} = 10 - 4x_1 - x_2 = 0$

and $\frac{\partial \pi}{\partial x_2} = 15 - x_1 - 4x_2 = 0$

Solving these equations, we get

$$x_1 = \frac{5}{3} \text{ and } x_2 = \frac{10}{3}$$

Amount of maximum profit is

$$\pi_{\max} = 10 \times \frac{5}{3} + 15 \times \frac{10}{3} - 2 \times \frac{25}{9} - \frac{5}{3} \times \frac{10}{3} - 2 \times \frac{100}{9}$$

= R.s 33.3

Checking for second order conditions:

$$\frac{\partial^2 \pi}{\partial x_1^2} = -4 < 0, \frac{\partial^2 \pi}{\partial x_2^2} = -4 < 0$$

$$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = -1$$

$$\therefore \quad \frac{\partial^2 \pi}{\partial x_1^2} . \frac{\partial^2 x}{\partial x_2^2} > \left(\frac{\partial^2 \pi}{\partial x_1 \partial x_2} \right)^2$$

Hence profit is maximised at $\left(\frac{5}{3}, \frac{10}{3} \right)$.

Example 3. A monopolist produces two commodities x_1 and x_2, jointly. The relevant cost function is $C = x_1^2 + 2x_1 x_2 + 3x_2^2$. The demand function are $P_1 = 36 - 3x_1$ and $P_2 = 50 - 5x_2$.

Profit function for the monopolist, $\pi = x_1 p_1 + x_2 p_2 - C$

$$\Rightarrow \quad \pi = x_1(36 - 3x_1) + x_2(50 - 5x_2) - x_1^2 - 2x_1 x_2 - 3x_2^2$$

$$= 36x_1 - 3x_1^2 + 50x_2 - 5x_2^2 - x_1^2 - 2x_1x_2 - 3x_2^2$$

$$= 36x_1 - 4x_1^2 + 50x_2 - 8x_2^2 - 2x_1x_2$$

For maximum of π;

$$\frac{\partial \pi}{\partial x_1} = 36 - 8x_1 - 2x_2 = 0$$

$$\frac{\partial \pi}{\partial x_2} = 50 - 16x_2 - 2x_1 = 0$$

Solving above two equations, we get $x_1 = \frac{119}{31}$ and $x_2 = \frac{82}{31}$

$$P_1 = 36 - 3 \times \frac{119}{31} = \frac{759}{31}$$

and $P_2 = 50 - 5 \times \frac{82}{31} = \frac{1140}{31}$

$$\pi_{\max} = 36 \times \frac{119}{31} - 4\frac{(119)^2}{(31)^2} + 50 \times \frac{82}{31} - 8 \times \left(\frac{82}{31}\right)^2$$

$$-2 \times \frac{119}{31} \times \frac{82}{31} = \frac{129952}{(31)^2}$$

= Rs. 135.23

Checking for second order conditions:

$$\frac{\partial^2 \pi}{\partial x_1^2} = -8 < 0; \qquad \frac{\partial^2 \pi}{\partial x_2^2} = -16 < 0$$

$$\frac{\partial^2 \pi}{\partial x \partial y} = -2$$

$$\therefore \qquad \frac{\partial^2 \pi}{\partial x_1^2} \cdot \frac{\partial^2 \pi}{\partial x_2^2} > \left(\frac{\partial^2 \pi}{\partial x_1 \partial x_2}\right)^2$$

Hence profit is maximum at $\left(\frac{119}{31}, \frac{82}{31}\right)$.

Example 4: A manufacturer can produce a commodity at two location. The selling price per unit is given by $P = 200 - 0.8x$, where $x = x_1 + x_2$. The cost functions at the two locations are $C_1 = 0.3x_1^2 + 60x_1 + 5000$ and $C_2 = 0.5x_2^2 + 30x_2 + 8000$ respectively.

Total revenue, $TR = [200 - 0.8(x_1 + x_2)](x_1 + x_2)$

$$= 200(x_1 + x_2) - 0.8(x_1 + x_2)^2$$

Total cost, $TC = C_1 + C_2$

$$= 0.3x_1^2 + 60x_1 + 0.5x_2^2 + 30x_2 + 13000$$

$$\pi = 200(x_1 + x_2) - 0.8(x_1 + x_2)^2 - 0.3x_1^2 - 60x_1 - 0.5x_2^2 - 30x_2 - 13000$$

$$\frac{\partial \pi}{\partial x_1} = 200 - 1.6(x_1 + x_2) - 0.6x_1 - 60 = 0$$

$$\Rightarrow \quad 2.2x_1 + 1.6x_2 = 140$$

$$\frac{\partial \pi}{\partial x_2} = 200 - 1.6(x_1 + x_2) - 1.0x_2 - 30 = 0$$

$$\Rightarrow \quad 1.6x_1 + 2.6x_2 = 170$$

$$x_1 = 29.1 \text{ and } x_2 = 47.5$$

$$\frac{\partial^2 \pi}{\partial x_1^2} = -2.2, \quad \frac{\partial^2 \pi}{\partial x_2^2} = -2.6$$

$$\frac{\partial^2 \pi}{\partial x_1^2} \cdot \frac{\partial^2 \pi}{\partial x_2^2} > \left(\frac{\partial^2 \pi}{\partial x_1 \partial x_2}\right)^2$$

Thus the profit maximization conditions are satisfied.

2. Price Discrimination

Let a monopolist who could sell its produce in two markets and charge different prices in two markets. Let inverse demand functions $P_1 = f(x_1)$ and $P_2 = f(x_2)$. Let total cost functions be $C(x)$ where $x = x_1 + x_2$.

Profit function be $\pi = p_1 x_1 + p_2 x_2 - C(x) = R_1 + R_2 - C$, where R_i is total revenue from *i*th market. The first order conditions for maximum profit are

$$\frac{\partial \pi}{\partial x_1} = \frac{\partial R_1}{\partial x_1} - \frac{\partial C}{\partial x_1}.\frac{\partial x}{\partial x_1} = R_1' - C' = 0$$

$$\frac{\partial \pi}{\partial x_2} = \frac{\partial R_2}{\partial x_2} - \frac{\partial C}{\partial x_2}.\frac{\partial x}{\partial x_2} = R_2' - C' = 0$$

Solving the above necessary conditions for maximum profit are :

$$R_1 = R_2 = C'$$

$$MR_1 = MR_2 = MC$$

Second order conditions become:

$$\frac{\partial^2 \pi}{\partial x_1^2} < 0 \Rightarrow \frac{\partial^2 R_1}{\partial x_1^2} < \frac{\partial^2 C}{\partial x_1^2}$$

and $$\frac{\partial^2 \pi}{\partial x_2^2} < 0 \Rightarrow \frac{\partial^2 R_2}{\partial x_2^2} < \frac{\partial^2 C}{\partial x_2^2}$$

and $$\frac{\partial^2 \pi}{\partial x_1^2}.\frac{\partial^2 \pi}{\partial x_2^2} > \left(\frac{\partial^2 \pi}{\partial x_1 \partial x_2} \right)^2$$

Example 5: A discriminating monopolist is able to separate its customers into two markets with respective demand functions as $x_1 = 16 - 0.2 p_1$ and $x_2 = 9 - 0.05 p_2$. The total cost function is $C = 20 + 20x$ where $x = x_1 + x_2$.

Profit of the monopolist

$$\pi = (80 - 5x_1)x_1 + (180 - 2x_2)x_2 - 20 - 20x_1 - 20x_2$$

$$= 60x_1 - 5x_1^2 + 16x_2 - 20x_2^2 - 20$$

$$\frac{\partial \pi}{\partial x_1} = 60 - 10x_1 = 0$$

$\Rightarrow \quad x_1 = 6$

$$\frac{\partial \pi}{\partial x_2} = 160 - 40x_2$$

$\Rightarrow \quad x_2 = 4$

Second order conditions are :

$$\frac{\partial^2 \pi}{\partial x_1^2} = -10 < 0, \qquad \frac{\partial^2 \pi}{\partial x_2^2} = -40 < 0$$

and $\quad \dfrac{\partial^2 \pi}{\partial x_1 \partial x_2} = 0$

$$\frac{\partial^2 \pi}{\partial x_1^2} . \frac{\partial^2 \pi}{\partial x_2^2} > \left(\frac{\partial^2 \pi}{\partial x_1 \partial x_2} \right)^2$$

Hence second order condition is satisfied for maximum of profit.

Price in market I, $P_1 = 80 - 5 \times 6 =$ Rs. 50

Price in market II, $P_2 = 180 - 20 \times 4 =$ Rs. 800

Maximum profit, $\pi = 50 \times 6 + 100 \times 4 - 20 - 20(6 + 4) = Rs. 480$

Elasticity of demand in market I, $\eta_1 = -\dfrac{\partial x_1}{\partial P_1} . \dfrac{P_1}{x_1}$

$$= 0.2 \times \frac{50}{6} = 1.67$$

Elasticity of demand in market II, $\eta_2 = -\frac{\partial x_2}{\partial P_2} . \frac{P_2}{x_2}$

$$= 0.05 \times \frac{100}{4} = 1.25$$

Monopolist can charge higher price in case elasticity of demand is lower and vice-a-versa. Price charged in a market and elasticity of demand is that market are negatively related (You will be asked to derive this in exercises).

3. Duopoly

Under Duopoly market conditions, there are two sellers of a homogeneous product with inverse market demand $p = f(x)$ where p is price and *x* denotes quantity demanded and this *x* is sum of x_1 and x_2; where x_1 is output produced by first firm and x_2 by second firm.

For each duopolistic profit function would be maximised with respect to its own output. Profit functions of each duopolist are as follows:

Duopolist I : $\pi_1 = R_1 - C_1$

$$= px_1 - C_1(x_1)$$

$$= f(x)x_1 - C_1(x_1)$$

Deupolist II : $\pi_2 = R_2 - C_2$

$$= px_2 - C_2(x_2)$$

$$= f(x)x_2 - C_2(x_2)$$

Profit maximising conditions becomes:

$$\frac{\partial \pi_1}{\partial x_1} = 0$$

$$\Rightarrow \quad f(x) + x_1 \frac{\partial f(x)}{\partial x} . \frac{\partial x}{\partial x_1} - F_1'(x_1) = 0$$

$$\Rightarrow \qquad f(x) + x_1 f_1'(x) = F_1'(x_1) \qquad (1)$$

and $\qquad \dfrac{\partial \pi_2}{\partial x_2} = 0$

$$\Rightarrow \qquad f(x) + x_2 f_2'(x) = F_2'(x_2) \qquad (2)$$

Equation 1 gives level of output duopolist I would produce for a given level of output of second. This is known as reaction function of deuoplist I. Likewise Equation 2 represents reaction function of duopolist II. Solving 1 and 2 simultaneously, values for x_1 and x_2 can be calculated.

Example 6

The market demand of a product is given by *p* = 100 – 4*x*. The cost function for two duopolist are

$$C_1 = x_1^2 + 17x_1 + 40 \ \textit{and} \ C_2 = 0.5x_2^2 + 15x_2 + 90$$

profit of first duopolist is given by

$$\pi_1 = (100 - 4x)x_1 - x_1^2 - 17x_1 - 40$$

$$= 83x_1 - 5x_1^2 - 4x_1x_2 - 40$$

For maximum profit,

$$\frac{\partial \pi_1}{\partial x_1} = 83 - 10x_1 - 4x_2 = 0$$

So reaction function of firms 1 is $x_1 = \dfrac{83 - 4x_2}{10}$

Similarly, profit function of second duopolist is given by

$$\pi_2 = (100 - 4x)x_2 - 0.5x_2^2 - 15x_2 - 90$$

$$= 85x_2 - 4x_1x_2 - 4.5x_2^2 - 90$$

For maximum profit, $\frac{\partial \pi_2}{\partial x_2} = 85 - 4x_1 - 9x_2 = 0$

Reaction function of firm 2 is $x_2 = \frac{85 - 4x_1}{9}$. Solving reaction functions of two duopolist would give equilibrium output i.e. $x_1 = 5.5$ and $x_2 = 7$

4. Profit Maximising levels of inputs

A firm suppose if use not only labour but also capital in the production then production function would be $x = f(K, L)$ where x is production dependent on capital (K) and labour (L). If monopolist purchases labour and capital at constant prices of Rs. w and Rs. r per unit respectively. If demand of monopolist's output is given by $P = F(x)$; then profit function becomes:

$$\pi = p.x - (wL + rk)$$

where $wL + rk$ is cost function of the monopolist.

The necessary condition for profit maximisation would then become :

$$\frac{\partial \pi}{\partial L} = 0 \Rightarrow \frac{\partial \pi}{\partial L} = P\frac{\partial x}{\partial L} + x\frac{\partial P}{\partial x}.\frac{\partial x}{\partial L} - w = 0$$

$$\Rightarrow \quad w = \left(P + x\frac{\partial P}{\partial x}\right)\frac{\partial x}{\partial L}$$

$$\Rightarrow \quad w = P\left(1 + \frac{x}{P}\frac{\partial P}{\partial x}\right)\frac{\partial x}{\partial L}$$

$$\Rightarrow \quad w = P\left(1 - \frac{1}{\eta}\right)MP_L$$

where η is elasticity of demand and MP_L is marginal product of labour

$$\Rightarrow \quad w = MR.MP_L$$

where $P\left(1-\frac{1}{\eta}\right)=MR$

Similarly, $\frac{\partial\pi}{\partial K}=0$

$\Rightarrow \qquad r = MR.MP_K$

Example

A firm's production function is $X = 12 - \frac{L+K}{LK}$.

Let prices of labour (*L*), capital (*K*) and output (*X*) be Re.1, Rs.4 and Rs. 9 respectively.

Profit function of the firm is then given by

$$X = PX - (L + 4K)$$

$$= 9\left[12 - \frac{L+K}{LK}\right] - L - 4K$$

For Maximum profit

$$\frac{\partial\pi}{\partial L}=0$$

$$\Rightarrow \qquad -9\left[\frac{LK-(L+K)K}{L^2K^2}\right]-1=0$$

$$\Rightarrow \qquad L^2K^2 = 9K^2$$

$$\Rightarrow \qquad L^2 = 9$$

$\Rightarrow \qquad L = 3$ (neglecting negative value of *L*)

$$\frac{\partial\pi}{\partial K}=0$$

$$\Rightarrow \quad -9\left[\frac{LK-(L+K)L}{L^2K^2}\right]-4=0$$

$$\Rightarrow \quad 9L^2 = 4L^2K^2$$

$$\Rightarrow \quad K^2 = \frac{9}{4}$$

$$\Rightarrow \quad K = \frac{3}{2}$$

Output X, when L = 3 and $K = \frac{3}{2}$

$$X = 12 - \frac{3+3/2}{9/2}$$

$$= 12 - \frac{9}{2} \times \frac{2}{9}$$

= 11 Units

Exercises

1. A monopolist produces two commodities that are substitutes and having demand functions: X_1=8-P_1+P_2 and X_2=9+P_1-5P_2, Where 1,000 X_1 units of first commodity are demanded if its price is RsP_1 per unit and 1,000X_2 Units Of The Second commodity are demanded if its price is Rs-P_2 per units. It costs Rs-4 to produce each unit of the first commodity and Rs-2 to produces each unit of the second. Find the output levels and prices of the two commodities in order to have maximum profit

2. A monopolist sells two products x and y for which the demands are: x=50-0.5p_x and y-76-py

The combined cost function is C=3X^2+2XY+2Y^2+55.Find:

(i) The profit maximizing levels of output and price for each product.

(ii) The maximum profit.

3. A scooter manufacturer produces the same model of a scooter at two different production plants. The cost of production of x_1 scooters at plant I is given by

$C_1=X_1^2+1,000X_1+2,500$, and the cost of production of X_2 scooters at plant II is given by, $C_2=1.5X_2^2+2000X_2+1800$, Where x_1 and x_2 are the annual outputs of plant I and II respectively

(i) If each scooter is sold at a uniform price of Rs.20,000,find the levels of production of each plant so that profits are maximized.

(ii) If the annual demand of scooters follows the demand law x=30,000-p, where p is the price of a scooter, find the levels of production of each plant for maximum profits and the price of a scooter.

4. (a) A discriminating monopolist can separate his consumers into two distinct markets with the following demand functions:

Market I: $Q_1=16-0.2P_1$

MARKET II: $Q_2=180-2P_2$

Assume that the monopolist's total cost function takes the form TC+20Q-20=0, Where Q ($=Q_1+Q_2$) is the total output. Obtain the total profit function and determine the prices he would charge in the two markets to maximize profits. What is the total profit? Do you agree that the price charged in the market with a higher elasticity of demand would be higher? Show by calculations.

(b) Calculate the ration of prices charged by a discriminating monopolist in the two markets with price elasticities of demand equal to 3.0 and 1.5.

5. Let there be two sellers of a homogeneous product with market demand given by

$P=B-ax,(x=x_1+x_2)$a and B are positive constants. The cost functions of each of the firm is given by $C=ax_i$, i=1,2 and a is positive constant. Assuming that the conjectural variations are zero

(i) Find reaction curve of each seller.
(ii) Find equilibrium output of each seller and the industry.
(iii) Find equilibrium output if the industry becomes a monopoly.

6. the production function of a firm is given by q=12$l^{1/2}k^{1/4}$ and the prices (in Rs) of q,l and K are 9,2 and 4 respectively

(i) Find the profit maximizing values of q, L and K.
(ii) Find the amount of maximum profit
(iii) Verify second order condition.

References

K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, Pearson Educational Asia, Delhi, 2002