DC-1

Semester-II

Paper-IV: Mathematical methods for Economics-II

Lesson: Vectors And Vector Operations

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Contents of the Present Chapter

1.0 Learning Outcomes

In the present chapter you will learn about the following aspects;

- Understand the concept of linear equation system and Leontief Model
- Understand the concept vectors and vector operations.

- Geometric interpretations of vectors.
- Rules of scalar product.
- Understand the concept lengths and distance of vectors.
- Understand the Cauchy-Schwarz inequality and its theorems.
- Understand the concept of lines and plane

1.1 Introduction

There are many systems in mathematics, which are employed to handle problems in geometry, mechanics and other branches of applied mathematics. Vectors, Matrices and Determinants are the important part of mathematical systems, which are related to linear algebra. Basically linear algebra is branch of mathematics concerned with the study of vectors. Vector spaces and linear maps between them are the main structure of linear algebra. Most of the economic problems are based on multidimensional. Economists have used mathematical model to solve these problem in terms of system of equations. If the system of equation are linear then this area of mathematics are called linear algebra.

1.2 Linear Equation Systems

In general, equations systems are linear, if it has the form such that;

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots \quad + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots \quad + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots \quad + a_{3n}x_n = b_3$$

$$a_{m1}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots \quad + a_{mn}x_n = b_n$$

The above linear system is 'm' linear equation with 'n' unknown i.e. x_1, x_2 … x_n, where a_{11}, a_{12}, a_{13} … a_{mn} are the coefficient of equation system and b_1, b_2, b_3 … b_m are called right-hand sides constraints. The above systems is said to be consistent if it has at least one solution otherwise it is said to be inconsistent.

1.3 Leontief Model

It is also known input-output model and given by W.W. Leontief. In order to illustrate why linear equations systems are important in economics, we briefly discuss this model.

Suppose an economy has three sectors, i.e. agriculture, industry and service. The total output of a particular sector is consumed by all these sectors as input and final demand of the sectors.

Input-output process can be explained by the given table;

Output↓ / Input→	Agriculture X_1	Industry X_2	Service X_3	Final Demand F
Agriculture X_1	X_{11}	X_{12}	X_{13}	F_1
Industry X_2	X_{21}	X_{22}	X_{23}	F_2
Service X_3	X_{31}	X_{32}	X_{33}	F_3
Primary Input (Labour) L→	L_1	L_2	L_3	-

By the above table, total output of agriculture, industry and service sectors can be written as;

$$X_1 = X_{11} + X_{12} + X_{13} + F_1$$

$$X_2 = X_{21} + X_{22} + X_{23} + F_2$$

$$X_3 = X_{31} + X_{32} + X_{33} + F_3$$

and $$L = L_1 + L_2 + L_3$$

In general, we can write as;

$$X_i = \sum_{j=1}^{n} X_{ij} + F_i \quad \text{And} \quad L = \sum_{L=1}^{n} L_i$$

Here $X_i \Rightarrow$ Total output i^{th} sector

$X_{ij} \Rightarrow$ Output of i^{th} sector used as input in j^{th} sector.

$F_i \Rightarrow$ Final demand for i^{th} sector.

The above identity states that all the output of particular sector could be utilized either as an input in one of the producing sectors of the economy or as a final demand.

Now, if technological coefficient between the sectors defined as;

$$a_{ij} = \frac{X_{ij}}{X_j}$$

or $X_{ij} = a_{ij} \,.\, X_j$

Then the above equation systems can be converted as;

$$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + F_1$$
$$X_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + F_2$$
$$X_3 = a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + F_3$$

Now, write the equation systems in matrix form;

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Or, $X = AX + F$

$X\,[I-A] = F$ OR $X = [I-A]^{-1}\,F$

In general; $X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \ldots\ldots. + a_{1n}X_n + b_1$

Or; $(1-a_{11})X_1 - a_{12}X_2 - a_{13}X_3 \quad \ldots\ldots \quad - a_{1n}X_n = b_1$

Similarly, $- a_{21}X_1 + (1 - a_{22})\,X_2 - a_{13}X_3 \ldots\ldots \quad - a_{3n}X_n = b_3$

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$- a_{ni}X_1 - a_{n2}X_2 - a_{n3}X_3 \ldots\ldots + (1-a_{nn})X_n = b_n$

This is called Leontief systems of input-output. The numbers a_{11}, a_{12}, a_{13} … a_{nn} are called technical (input) coefficient and b_1, b_2, b_3 … b_n are final demand.

Example: If the technical coefficient is given by $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$ and final demands of goods are 50 and 100. Write down the Leontief model.

Solution: Let X_1 and X_2 are goods.

Then; $X_1 = 0.5X_1 + 0.2X_2 + 50$

$X_2 = 0.1X_1 + 0.4X_2 + 100$

Or, $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

Or, $$\begin{bmatrix} 0.5 & -0.2 \\ -0.1 & 0.6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \end{bmatrix} \text{ or } \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.2 \\ -0.1 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

1.4 Vectors

'*Vectors facilitate analytic study of such physical objects as have direction in addition to magnitude'*. A vector space is a set whose elements can be added together and multiplied by scalars or numbers.

Let F be a field of vector space and a_1, a_2 ……… a_n be the numbers of F. Then the ordered set of number's is called vector of order n.

$$V = \{a_1, a_2\ a_3 \ldots a_n\}$$

Where, a_1, a_2 ……… a_n are called the components of the vector V and these numbers are also called scalars. It is also denoted by $\vec{V}$.

1.4.1 Scalars

Quantities that have only magnitude and no direction are called scalars. For example – time, population, temperature, power etc.

1.4.2 Types of Vectors

Zero Vectors: A vector whose initial and terminal points are coincided is called a zero vector. The length of zero vectors is zero.

Equal Vector: Two vectors are said to be equal if they have the same length and direction.

Unit Vector: A vector 'a' is called a unit vector if its magnitude is one. It is denoted by $\hat{a}$.

Row Vector: It is represented by a row i.e. $a = \{a_1, a_2\ a_3 \ldots a_n\}$

Column Vector: It is represented by a column, e.g. $a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

Free Vector: A vector whose direction is known but the initial point and the line of application are not known is called a free vector.

1.5 Operations on Vector

Rules for vector addition and multiplication by scalars is given below

If U, V and W are arbitrary n-vectors and α, β are arbitrary numbers, then,

$(U + V) + W = U + (V + W) \rightarrow$ Associative law. …(i)

$U + V = V + U \rightarrow$ Commutative law …(ii)

$U + 0 = U$ …(iii)

$U + (-U) = 0$ …(iv)

$(\alpha + \beta)U = \alpha U + \beta U$ …(v)

$\alpha(U + V) = \alpha U + \alpha V$ …(vi)

$\alpha(\beta U) = \alpha\beta.U$ …(vii)

I.U = U (Here I is identity Matrix) (viii)

Example: Given u = (2, -3, 5) and v = (-1, 9, -3). Compute a + v, u – v, 3u – 2 v and $-\sqrt{2}V$

Solution:

u + v = (2, -3, 5) + (-1, 9, -3)

= {2 + (-1), (- 3) + 9, 5 + (-3)} = (1, 6, 2)

u – v = {2 – (-1), -3 -9, 5 – (-3)} = (3, -12, 8)

3u – 2v = 3 (2, -3, 5) -2 (-1, 9, -3)

= (6, -9, 15) – (-2, 18, -6)

= (8, -27, 21)

$$-\sqrt{2}V = -\sqrt{2}(-1, 9, -3) = \left(\sqrt{2} - 9\sqrt{2}, 3\sqrt{2}\right)$$

Example: If 2(x, y, z) + 5 (-1, 2, 3) = (3, 1, 3) then find x, y and z.

Solution: Given;

(2x, 2y, 2z) + (-5, 10, 15) = (3, 1, 3)

Then, 2x – 5 = 3 or x = 4

2y + 10 = 1 or y = -9/2

2z + 15 = 3 or z = -6

Example: Prove that vector equation $u\begin{pmatrix}2\\-3\end{pmatrix}+V\begin{pmatrix}4\\6\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ represents two equation in two unknown u, v, find the solution.

Solution: Given; $u\begin{pmatrix}2\\-3\end{pmatrix}+V\begin{pmatrix}4\\6\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

i.e. $2u + 4v = 1$ ……………………..…(1)

$-3u + 6v = 0$ ……………………..…(2)

Solving (1) and (2) v = 1/8, u = ¼

Example: (i) When is the vector b is said to be a linear combination of vectors x, y and z?

(ii) Consider the vector a = (1, 2, 3) and b = (2, 3, 1) then Find k such that w = (1, k, 4) is a linear combination of ‘a’ and ‘b’.

Solution: (i) Let c_1, c_2 and c_3 are real numbers, then;

$$b = c_1x + c_2y + c_3z$$

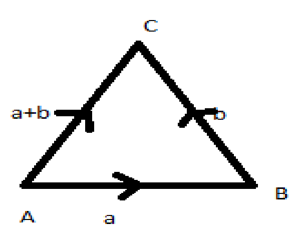
(ii) Let $a\begin{bmatrix}1\\2\\3\end{bmatrix}+b\begin{bmatrix}2\\3\\1\end{bmatrix}=\begin{bmatrix}1\\k\\4\end{bmatrix}$

Then; a + 2b = 1, 2a + 3b = k and 3a + b = 4,

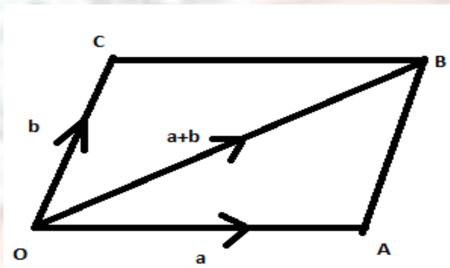
Solving above equation then a = 7/5, b = -1/5 and k = 11/5

1.6 Geometric Interpretations of Vectors

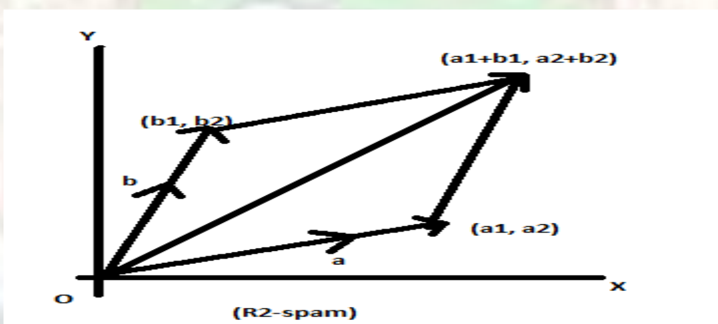
The triangle law of vectors: The figure shows the triangle law of vectors. It is represented by $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ or a + b = (a + b)



Parallelogram law of vectors: The given figure shows the parallelogram law of vectors. It is represented by $\overrightarrow{OA}+\overrightarrow{OC}=\overrightarrow{OB}$.

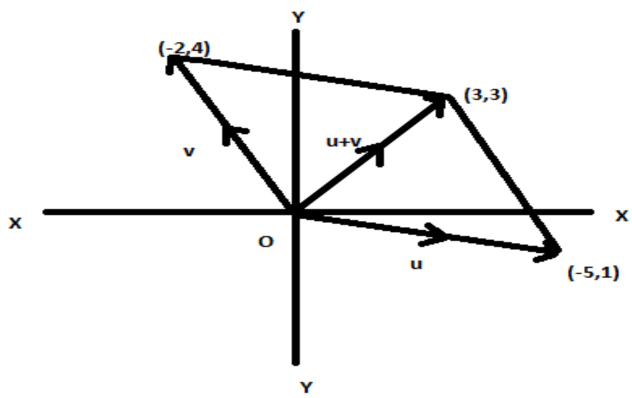


Some other geometric interpretation of vectors operations in 2-space and 3-space is given below;



Example: Given u = (5, -1) and v = (-2, 4) compute u + v with the help by geometric vectors starting of origin.

Solution:



1.7 The scalar product

The scalar product of any two n-vectors $u = (u_1, u_2 \ldots u_n)$ and $v = (v_1, v_2 \ldots v_n)$ is defined as;

$$u.v = u_1.v_1 + u_2.v_2 + \ldots + u_n.v_n = \sum_{i=1}^{n} u_i v_i$$

If the commodity vector $a = (a_1, a_2 \ldots a_n)$ and price of commodity vector $P = (P_1, P_2 \ldots P_n)$ then the scalar product of P and a is called total value of the entire commodity vector. It is defined as;

$$p_1a_1 + p_2a_2 + \ldots + p_na_n = p.a$$

Example: If u = (1, 2, 3) and v = (-2, 3, 5) then compute u.v

Solution: u.v = 1.(-2) + 2.3 + 3.5

= -2 + 6 + 15 = 19

Rules for the scalar product

Let u, v & w are n-vectors and α is a scalar, then

$$u.v = v.u \quad \ldots(i)$$

$$u.(v + w) = u.v + u.w \quad \ldots(ii)$$

$$\alpha\,(u.v) = (\alpha u).v = u.(\alpha v) \quad \ldots(iii)$$

$$u.u > 0 \Leftrightarrow u \neq 0 \qquad \text{------(iv)}$$

1.8 Lengths and distance of vectors

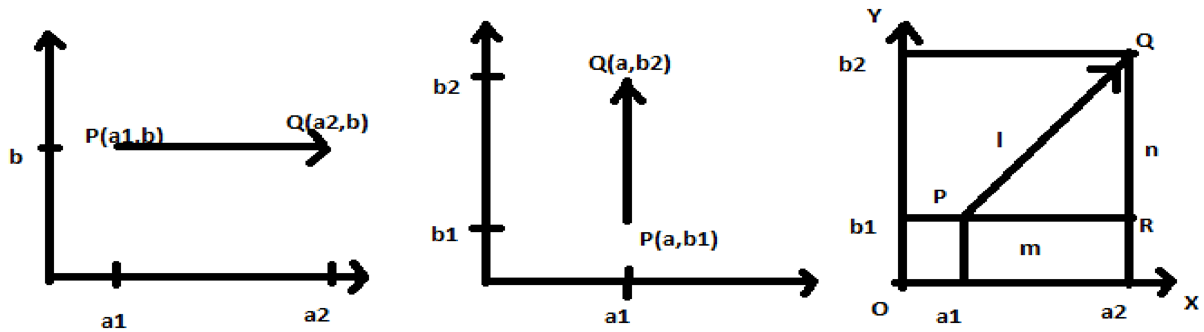
If a = (a_1, a_2, a_3 … a_n) be an n-vector then the length (norm) of a vector 'a' is given by:

$$\|a\| = \sqrt{a.a} = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

If u = (u_1, u_2 … u_n) and v = (v_1, v_2 … v_n) be on the vectors then the distance (Euclidean) between the vectors is given by;

$$d = \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

These aspects can be explained by the help of given below diagrams:



$\|\overrightarrow{PQ}\| = |a_2 - a_1|$ $\qquad$ $\|\overrightarrow{PQ}\| = |b_2 - b_1|$ $\qquad$ By Pythagorean Theorem,

$$l^2 = m^2 + n^2$$

$$= |a_1 - a_2|^2 + |b_1 - b_2|^2$$

$$\text{Or } |\overrightarrow{PQ}| = l = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

In general, $\|\overrightarrow{PQ}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} = \|x - y\|$.

In particular, if we take y to be zero, then the distance from the point x = (x_1, x_2, … x_n) to the origin or the length of the vector x is given by $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

1.9 Cauchy-Schwarz Inequality

If u and v are two vectors then Cauchy-Schwarz Inequality is given by:

$$|u.v| \leq \|u\|.\|v\|$$

Example: If u = (1, 2, -3) and v = (-3, 2, 5) be the two vectors then find lengths of vectors, distance between vectors and check the Cauchy-Schwarz inequality (CSI).

Solution:

Lengths: $\|u\| = \sqrt{1+4+9} = \sqrt{14}, \|v\| = \sqrt{9+4+25} = \sqrt{38}$

Distance: $d = \|u - v\| = \sqrt{16+0+64} = \sqrt{80}$

CSI : u.v = (-3, 4, -15), then; $|u.v| = \sqrt{9+16+225} = \sqrt{250}$

Hence; $14 \leq \sqrt{14}.\sqrt{38}$, which is certainly true.

Orthogonality: If the angel between two vectors is 90° then the vector are said to be orthogonal. It is denoted by a ⊥ b. So, we can say that two vectors in R^2 or R^3 are orthogonal if and only if their scalar product is zero.

$$a \perp b \Leftrightarrow a.b = 0$$

In case of orthonormal vectors;

- Their dot product are zero, $\vec{a}.\vec{b} = 0$. Both vectors are unit vectors, $\|a\| = \|b\| = 1$, i.e. u.u = 1 and v.v = 1.

Theorem I: Prove that; $\|rV\| = |r|.\|V\|$ for all r in R^1 and V in R^n.

Proof: Let $\|rV\| = \|r(v_1, v_2 ... v_n)\|$

$$= \|rV_1, rV_2, ...rV_n\|$$

$$= \sqrt{(rV_1)^2 + (rV_2)^2 + ... + (rV_n)^2}$$

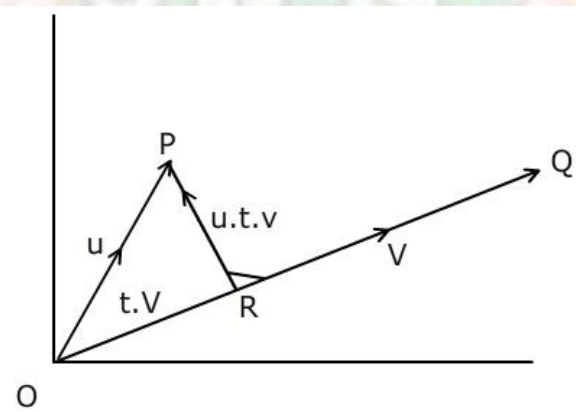
$$= \sqrt{r^2(V_1^2 + V_2^2 + ... + V_n^2)}$$

$$= |r|\sqrt{V_1^2 + V_2^2 + ... + V_n^2} \qquad \text{(since } \sqrt{r^2} = |r|\text{)}$$

$$= |r|.\|V\| \text{ proved.}$$

Theorem II: Suppose u and v are two vectors in R^n and Q be the angel between them then proved that $u.v = \|u\|\|v\|\cos Q$

Proof: Let u = OP and v = OQ be the two vectors and t.v = OR, here v is the vector and t is a scalar multiple.



By the triangular OPR;

$$\therefore \qquad \cos Q = \frac{\|tv\|}{\|u\|} = \frac{t\|v\|}{\|u\|} \qquad \ldots(1)$$

Now, applying Pythagorean Theorem in triangular OPR;

$$\|u\|^2 = \|tv\|^2 + \|u - tv\|^2$$

$$\|u\|^2 = t^2\|v\|^2 + \|u - t.v\|^2$$
$$\|u\|^2 = t^2\|v\|^2 + u.u - 2.t.u.v + t^2.v.v$$
$$\|u\|^2 = t^2\|v\|^2 + \|u\|^2 + t^2\|v\|^2 - 2.t.uv$$
$$t = \frac{u.v}{\|v\|^2} ----------------(2)$$

By equation (1) and (2)

$$\cos Q = \frac{u.v}{\|u\|.\|v\|}$$ Proved.

Theorem III: If u and v are the two vectors in R^n then proved that $\|u+v\| \le \|u\| + \|v\|$

Proof: We know that $\frac{u.v}{\|u\|.\|v\|} = \cos Q \le 1$

Or, $$u.v \le \|u\|.\|v\| \qquad \ldots(1)$$

Now $\|u+v\|^2 = \|u\|^2 + 2(u.v) + \|v\|^2$

Or, $\|u\|^2 + 2(u.v) + \|v\|^2 \le \|u\|^2 + 2\|u\|.\|v\| + \|v\|^2$

$$\therefore \qquad u.u + u.v + v.u + v.v \le (\|u\| + \|v\|)^2$$

$$(u+v).(u+v) \le (\|u\| + \|v\|)^2$$

$$\|u+v\|^2 \le (\|u\|+\|v\|)^2$$

Or, $\|u+v\| \le \|u\|+\|v\|$ proved.

Example: Given, $a = \begin{bmatrix} 1 \\ -15 \\ 2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$ are two-column vector.

(i) Calculate lengths of vector a and b

(ii) Find k such that a vector c = a + kb is orthogonal to vector b.

Solution: (i)

$$\|a\| = \sqrt{1+225+4} = \sqrt{230}$$
$$\|b\| = \sqrt{9+25+1} = \sqrt{35}$$

(ii)

$$c = \begin{bmatrix} 1 \\ -15 \\ 2 \end{bmatrix} + k\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3k \\ -15+5k \\ 2+k \end{bmatrix}$$

Now, c is orthogonal to vector b then; c.b = 0

$$\begin{bmatrix} 1+3k \\ -15+5k \\ 2+k \end{bmatrix} . \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$3+9k-75+25k+2+k=0$$
$$k=2$$

Linear Dependent Vectors

The vectors a, b and c are called linearly dependent vectors if scalars x, y and z exist, such that:

$$x.a + y.b + z.c = 0$$

Linear dependent vectors in a plane are precisely collinear or parallel vectors.

Linear Independent Vectors

The vectors a, b and c in a plane are called linearly independent vectors if,

$$xa + yb + zc = 0$$

It implies, $x = y = z = 0$, linearly independent vectors are precisely non-collinear vectors.

Note:

- ✓ Two collinear vectors are linearly dependent.
- ✓ Two non-collinear vectors are linearly independent.

Example: If $5a + 3b = 2c$ and $a.b = c$ then show that a and c have the same directions and a and b have opposite direction. Are the vectors a, b and c linearly independent?

Solution: Given $5a + 3b = 2c$ and $a – b = c$

Solving, $a = 5/8c$, so 'a' and 'c' have same direction.

And, $a = -5/3\, b$, so 'a' and 'b' have opposite direction.

The vectors a, b and c are not linear independent since there exists a linear combination of the vectors a, b and c.

Example: Let $a = \hat{i} + 3\hat{j}$ and $b = -2\hat{i} + 5\hat{j}$ then find a unit vector parallel to vector a + b.

Solution: $a + b = -\hat{i} + 8\hat{j}$

Now $|a + b| = \sqrt{1+64} = \sqrt{65}$

$\therefore$ Unit vector along a + b

$$= -\frac{1}{\sqrt{65}}\hat{i} + \frac{8}{\sqrt{65}}\hat{j} \qquad \left[\hat{a} = \frac{1}{|a|}\right]$$

2.0 Lines and Plane

A line in R^n

The line L through the vectors $a = (a_1, a_2 \ldots a_n)$ and $b = (b_1, b_2 \ldots b_n)$ is the set of all $x = (x_1, x_2 \ldots x_n)$ satisfying;

$$x = (1 - t)\, a + t.b, \text{ for some real number 't'}.$$

By using the coordinates of vectors is equivalent to;

$$x_1 = (1 - t)\, a_1 + tb_1$$

$$x_2 = (1 - t)\, a_2 + tb_2$$

……………………

$$x_n = (1 - t)\, a_n + tb_n$$

Now, let $p = (p_1, p_2 \ldots p_n)$ is a point in R^n then straight line L passing through $(p_1, p_2 \ldots p_n)$ in the same direction of the vector $a = (a_1, a_2 \ldots a_n)$ is given by;

$x = p + t.a$

A Hyper plane in R^n:

A hyper plane through vector ‘a’ = $(a_1, a_2 \ldots a_n)$ that is orthogonal to ‘a’ vector $p = (p_1, p_2 \ldots p_n) \neq 0$ is the set of all points $x = (x_1, x_2 \ldots x_n)$ satisfying,

$$p.(x - a) = 0.$$

If we used coordinate representation of vectors then the equations is given by;

$$p_1 (x_1 - a_1) + p_2 (x_2 - a_2) + \ldots + p_n (x_n - a_n) = 0$$

or, $p_1 x_1 + p_2 x_2 + \ldots + p_n x_n = A$, where $A = p_1 a_1 + p_2 a_2 + \ldots + p_n a_n$.

Example: Find the equation for the plane in R^3 though vector v = (2, 1, -1) with P = (-1, 1, 3) as a normal.

Solutions: By the definition $-1(x_1 - 2) + 1 (x_2 - 1) + 3(x_3 - (-1)) = 0$

Or $-x_1 + x_2 + 3x_3 = - 4.$

Example: Given α = (1, 2, 1) and β = (-3, 0, -2), find real number x_1 and x_2 such that $(\alpha x_1 + \beta x_2) = (5, 4, 4)$

Solution: $\alpha x_1 = (x_1, 2x_1, x_1)$ and $\beta x_2 = (-3x_2, 0, -2x_2)$

Then, $2x_1 + \beta x_2 = (x_1 - 3x_2, 2x_1, x_1 - 2x_2) = (5, 4, 4)$

Now, $x_1 - 3x_2 = 5$

$2x_1 = 4$

and $x_1 - 2x_2 = 4$

Solving, $x_1 = 2$ and $x_2 = -1$

Example: Find the equation of the line in R^3 passing through the points (2, 4, -1) and (5, 0, 7). Where does the line intersect the xy plane? Using this equation to exactly describe the line segment joining the two given points.

Solution: The equation of line, x = (1 – t) . a + t.b, for some real number't'

Then, $x_1 = (1 – t) . 2 + t.5 = 2 + 3t$

$x_2 = (1 – t).4 + t.0 = 4 – 4t$

$x_3 = (1 – t).(-1) + t.7 = -1 + 8t$

These line intersects the x_1x_2-plane, when $x_3 = 0$.

So, -1 + 8t = 0 or t = 1/8, then we get;

$x_1 = 19/8$, $x_2 = 7/2$, $x_3 = 0$

∴ line of intersects x_1x_2 plane = (19/8, 7/2, 0)

Example: Show that the vectors 2a – b + c, a – 3b – 5c and 3a – 4b – 4c are coplanar.

Solution: We know that three vectors, u, v and w are coplanar if u.x + v.y + wz = 0, where x + y + z = 0 and x, y, z are not all zeros.

Let 3a – 4b – 4c = x(2a – b + c) + y(a – 3b – 5c)

= (2x + y) a + (-x-3y) b + (x – 5y).c

So, 2x + y = 3 …(i)

x + 3y = 4 …(ii)

x – 5y = -4 …(iii)

Solving the equation (i) and (ii), we get x = 1 and y = 1.

These value of x and y satisfy the third equation, i.e. x – 5y = -4

Hence the given three vectors are coplanar.

Problem: Prove that the two vectors a and b are equal if and only if their components along the x and y-axes are equal.

Solution: Let $a = a_{1\hat{j}} + a_{2\hat{j}}$ and $b = b_{\hat{ij}} + b_{2\hat{j}}$ be two vectors where a_1, a_2 and b_1, b_2 are the components of a and b along the x and y-axes respectively.

Necessary Condition:

$$a = b \Rightarrow a_{1\hat{i}} + a_{2\hat{j}} = b_1\hat{i} + b_2\hat{j}$$

$$\therefore \quad (a_1 - b_1)\hat{i} = (b_2 - a_2)\hat{j}$$

It shows either $(a_1 - b_1)\ \hat{i}$ and $(b_2 - a_2)\hat{j}$ are parallel or each is a zero vector. But they are not parallel.

$$(a_1 - b_1)\hat{i} = (b_2 - a_2)\hat{j} = 0$$

$\therefore \quad a_1 - b_1 = 0$ and $b_2 - a_2 = 0$

Or $\quad a_1 = b_1$ and $b_2 = a_2$

Sufficient Condition:

In this case $a_1 = b_1$ and $a_2 = b_2$, we have show that, a = b

$\therefore \quad a_1 = b_1$ and $a_2 = b_2$

$\therefore \quad a_1 - b_1 = 0$ and $b_2 - a_2 = 0$

$$\therefore \quad (a_1 - b_1)\hat{i} = 0 = (b_2 - a_2)\hat{j}$$

$$\text{or} \quad a_1 i + a_2 j = b_1\hat{i} + b_2\hat{j}$$

a = b

<u>Problem Set</u>

1. Having bought n commodities the price being p_1, p_2 … p_n and quantities being Q_1 Q_2 …. Q_n express the total cost of purchase in vector notation.

2. The input coefficient matrix and final demand of three sector economy is given below:

$$A = \begin{bmatrix} \text{Ag.} & \text{Ind.} & \text{Ser.} \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0 & 0.5 \\ 0.1 & 0.3 & 0.1 \end{bmatrix} \text{ and } F = \begin{bmatrix} 100 \\ 40 \\ 50 \end{bmatrix} \text{ million rupees.}$$

Write down the Leontief model

3. If 3 (u, v, w) + 5 (-1, 2, 3) = (4, 1, 3), find u, v and w.

4. Solve the vector equation 4x – 7u = 2x + 8v – u for x in terms of u and v.

5. Are the following vectors independent?

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, w = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$$

If not, find the pattern of dependence between them.

6. Show that the vectors: $x = \begin{bmatrix} 5 \\ -4 \end{bmatrix}, y = \begin{bmatrix} 12 \\ 15 \end{bmatrix}$t are orthogonal and find length of vector.

7. Find the vector of unit length that is normal to the plane 3x + y – z = 10.

8. Prove that: $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

9. Can the vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1) spam the R^3 space.

10. Find the pattern of dependence of $\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}$ can they span R^3?

11. To find the point-normal equation of plane P which contains the points, p = (2, 1, 1), q = (1, 0, -3), r = (0, 1, 7).

12. Given $u = \begin{bmatrix}5\\1\end{bmatrix}$ and $v = \begin{bmatrix}0\\3\end{bmatrix}$, find 24 + 3v graphically.

13. Prove that u = (-2, 1, -1) is a point in the plane –x + 2y + 3z = 1.

14. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.

Answers of Problem set:

1. $\left[P_1Q_1 + P_2Q_2 + \ldots + P_nQ_n\right]$

2. $X = \left[I - A\right]^{-1}.F \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.4 & -0.2 \\ -0.2 & 1 & -0.5 \\ -0.1 & -0.3 & 0.9 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 40 \\ 30 \end{bmatrix}$

3. u = 3, v = -3, w = -4.

4. x = 3u + 4v

5. No

6. $\|x\| = \sqrt{41}, \|y\| = \sqrt{369}$.

9. Yes

10. linear dependence

11. 3x – 7y + z = 0.

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