DC-1
Semester-II
Paper-IV: Mathematical methods for Economics-II
Lesson: Higher Order Differentiation and Its Applications
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CONTENTS

1. Learning Outcomes

After completing of the present chapter, you should able to:-

1. Higher Order Differentiation
2. Examples of Higher Order Differentiation
3. Partial Differentiation
4. Examples of Partial Differentation
5. Clairant Theorm/Young's theorm

2. Higher Order Differentiation

If f (x) be differentiable function of x, then f'(x) or $\frac{dy}{dx}$ is the first derivative or first order derivative of y = f (x) w.r.t 'x'. Since the derivative of function is also a function, therefore another derivative can also be find. The second order derivative, or second derivative, is the derivative of the first derivative of the function f(x). Other notations are:

$$\frac{d\,(^{dy}/_{dx})}{dx} \text{ or } \frac{d\,(f'(x))}{dx} \text{ or } \frac{d^2y}{dx^2} \text{ or } f''(x)$$

Since f '' (x) is also a function, therefore, its derivative can also be find which is denoted as f ''' (x). For higher order derivatives superscripts can be used i.e. f^4 = fourth derivative etc.

Example: -f(x) = $4x^5 + 6x^3+2x+1$

$f''(x) = 20x^4 + 18x^2 + 2$

$f''(x) = 80x^3 + 36x$

$f'''(x) = 240x^2 + 36$

3. Partial Derivatives

Given a function y = f(x), the derivative f '(x), represents the rate of change of the function as x changes. For a function of two variables, such as z = f (x,y), one variable could be changing faster than the other variables. It will be completely possible for the function to be changing differently.

For a function of two independent variables, z = f(x,y), the partial derivative 'z' with respect to x may be found as normal rule of differentiation. The only difference is that, whenever or wherever the second independent variable 'y' appears, it will be treated as constant in every respect. Also the partial differentiation of y can be found by treating x variable as constant. Notations of partial differentiation are given below:

Notations of Partial Differentiation

Partial derivative of z w.r.t x	$\frac{\partial z}{\partial x}$	fx
Partial derivative of z w.r.t y	$\frac{\partial z}{\partial y}$	Fy

Example: $Z = x^4 y^2 - x^2 y^6$

$\frac{\partial z}{\partial x} = 4x^3y^2 - 2xy^6$

$\frac{\partial z}{\partial y} = 2x^4y - 6x^2 y^5$

Partial derivative can be defined as:-

If z = f(x,y) is a function of two variables, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, called partial derivatives of z with respect to x and y respectively, be the derivative z w.r.t.x by keeping y as constant and the derivative z w.r.t.y by keeping x as constant. All the rules of differentiation can be applied when partial differentiation can be calculated.

Symbolically if f =f (x,y) then

$fx = \frac{\partial f}{\partial x} = Lt_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$fy = \frac{\partial f}{\partial y} = Lt_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,)y)}{\Delta y}$

provided these limits exist.

Example 1.z = f (x, y) = $x^3y+x^2y^2 +xy +x+y^2$

$\frac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y + 1$

$\frac{\partial z}{\partial y} = x^3 + 2x^2y + x +2y$

3.1 Higher order partial derivative

For a function z = f (x, y); $f^{'}$ (x) & $f^{'}$ (y) are the two first order partial derivatives with respect to x and y respectively. Since ‘z’ is a function hence $f^{'}$ (x) and $f^{'}$ (y) are also a function, hence, second order partial differentiation can also be found.

The second order partial derivatives are called mixed partial derivative because derivatives of more than one variable are to be observed. e.g differentiating a function with respect to ‘x’ first and then ‘y’ is called as mixed partial derivative. The various notation of partial derivative are given in table:

Notations of second order partial derivatives

Partial derivatives of z	Notation 1	Notation 2	Notation 3
w.r.t. x twice	$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$	$\frac{\partial^2 z}{\partial x^2}$	fxx

w.r.t. y twice	$\frac{\partial}{\partial y}\ (\frac{\partial z}{\partial y}\ )$	$\frac{\partial^2 z}{\partial y^2}$	fyy
w.r.t. x first than y	$\frac{\partial}{\partial y}\ (\frac{\partial z}{\partial x}\ )$	$\frac{\partial^2 z}{\partial y\,\partial x}$	fxy
w.r.t. y y first than x	$\frac{\partial}{\partial x}\ (\frac{\partial z}{\partial y}\ )$	$\frac{\partial^2 z}{\partial x\,\partial y}$	fyx

A function has four possible second partial derivatives ones that are obtained by differentiating function w.r.t 'x' twice, w.r.t. y twice, w.r.t. x first than y and w.r.t. y first then x. All derivatives have sign (+ or -) interpretation of these signs are as follows.

Partial derivative	Sign	Interpretation
$\frac{\partial z}{\partial x}$	+ -	Slopes in x direction is positive Slopes in x direction is negative
$\frac{\partial^2 z}{\partial x^2}$	+ -	Slopes in x direction increases as x increases(y constant) Slopes in x direction decreases as x decreases(y constant)
$\frac{\partial z}{\partial y}$	+ -	Slopes in y direction is positive Slopes in y direction is negative
$\frac{\partial^2 z}{\partial y^2}$	+ -	Slopes in y direction increases as y increases(x constant) Slopes in y direction decreases as y decreases(x constant)
$\frac{\partial^2 z}{\partial y\,\partial x}$	+ -	Slopes in x direction increases as y increases(x constant) Slopes in x direction decreases as y decreases(x constant)
$\frac{d^2 z}{\partial x\,\partial y}$	+ -	Slopes in y direction increases as x increases(y constant) Slopes in y direction decreases as x decreases(y constant)

Example $z = x^{0.5}\ y^{0.5} - 60$ find $\frac{\partial z}{\partial x^2}$ interprate the result

sol. $\frac{\partial z}{\partial x} = 0.5\ x^{0.5}\ y^{0.5}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}$ (0.5 $x^{0.5}$ $y^{0.5}$)

= (0.5) (-0.5 $x^{-0.5}$ $y^{0.5}$)

= - 0.25 $x^{-1.5}$ $y^{0.5}$

Since x and y are positive, positive number raised to any power is positive; hence $y^{0.5}$ and $x^{-1.5}$ are positive , the term -0.25 in equation show that second order differentiation of z with respect to x twice is negative meaning that the slope in the x direction decreases as x increases when y is constant.

Example : z = f(x, y) = $x^3y + x^2 y + 2x + xy + x + y^2$

fx = $x^3 + 2x^2y + 2xy^2 + y + 1$

fy= $x^3 + 2x^2y + x + 2y$

fxx= $6xy + 2y^2$

fyy = $2x^2 + 2$

fxy = $3x^2 + 4xy + 1$

fyx = $3x^2 + 4xy + 1$

.: fxy = fyx .

The two mixed second order partial derivatives (also called as cross partial derivatives) are always equal when fxy and fyx are continuous. It is explained by the following theorem given by Alexis Clairant also know as Young's theorem.

Theorem: Suppose f is defined on a disk D, which contains the point (a, b). If the partial derivatives fxy and fyx are both continuous on disk D, then
fxy (a, b) = fyx (a, b).

Example:- Verify Young's theorem f(x,y) = x $e^{-x^2y^2}$

Solution:-

$fx(x,y) = e^{-x^2y^2}$ -2 x^2 y^2 $e^{-x^2y^2}$

$fy(x,y)$ = -2 x^3 y $e^{-x^2y^2}$

Now, compute the two mixed partial derivatives.

$fxy(x,y)$ = -2 x^2 y $e^{-x^2y^2}$ -4 x^2 y $e^{-x^2y^2}$ +4 $x^4y^3e^{-x^2y^2}$

= -6 x^2 y $e^{-x^2y^2}$ +4 $x^4y^3e^{-x^2y^2}$

$fyx(x,y)$= -6 x^2 y $e^{-x^2y^2}$ +4 $x^4y^3e^{-x^2y^2}$

.: fxy = fyx

Hence proved.

3.2 Partial derivative with many variables

If z = f (x_1, x_2 ……. x_n) then

$\frac{\partial z}{\partial xi}$ is the differentiation of the function w.r.t. x_i when all the other variables x_j ($j\neq i$) are held constant.

i.e $\frac{\partial z}{\partial x_1} = \frac{\partial f\ (x1,x2\ldots\ldots xn)}{\partial x_1}$

and $\frac{\partial z}{\partial x_2} = \frac{\partial f\ (x1,x2\ldots\ldots xn)}{\partial x_2}$ and so on

Suppose, there is a function which consists of three variables v = f (x, y, z). For such a function, there are partial derivatives of w.r.t x, y and z. When partial derivative has to take with respect to one of x, y and z assuming other two independent variables are constant.

In general, function consists of n variables. If Z = f (x_1 x_2,….,x_n) then partial derivative of z w.r.t. x_i is $\frac{\partial z}{\partial xi}$ when all the other variables x_j ($j\neq i$) are held constant.

$fx_i = \frac{\partial f}{\partial xi} = Lt_{h\rightarrow 0}\frac{f\ (\text{x1 x2,...xi+h...,xn}) - f\ (\text{x1 x2,.....xi....,xn})}{h}$

provided limit exists.

Example:- f (x, y, z) = $x^2 + y^3 + z^4$

$fx = \frac{\partial\,(x,y,z)}{\partial x} = 2x$

$fy=3y^2$

$fz = 4z^3$

Example: Find Zxxx, Zxyx, Zyyy,Zyxy of the function

$Z= 3x^2(5x+7y)$

$Zx= 3x^2(5) + (5x+7y)(6x)$

$= 45x^2 + 42xy$

Zxx= 90x+42y

Zxxx= 90.

Zxy = 0 + 42x

= 42x

Zxyx = 42

$Zy= 3x^2(7) + (5x+7y)(0)$

$=21\ x^2$

Zyy = 0

Zyyy = 0.

Zyx = 42x

Zyxy = 0.

Example: Find Zxxx, Zxyy, Zyyy,Zxxy of the function Z= (9x – 4y)(12x + 2y)

$Zx = (9x - 4y)(12) + (12x + 2y)(9)$

$= 108x - 48y + 108x + 18y$

$= 216x - 30y$

$Zxx = 216$

$Zxyy = 0$

$Zxxx = 0$

$Zxy = -30$

$Zxyy = 0$

$Zy = (9x-4y)(2) + (12x + 2y)(-4)$

$= 18x - 8y - 48x - 8y$

$= -30x - 16y$

$Zyy = -16$

$Zyyy = 0$

Example: Find Zxx, Zyy of the function $Z = \frac{x+y}{3y}$

$Zx = \frac{3y(1) - (x+y)(0)}{(3y)^2}$

$= \frac{1}{3y}$

$Zxx = 0$

$Zy = \frac{3y(1) - (x+y)(3)}{(3y)^2}$

$= \frac{-3x}{(3y)^2} = \frac{-x}{3y^2}$

$Zyy = -\frac{1}{3}(-2xy^{-3}) = \frac{2}{3}xy^{-3}$

Clairaut theorem (Young's theorem) can be extended to any function of 'n' number of variables and their mixed partial derivations. The only thing has to remember that in each derivative, we differentiate with respect to each variable the same number of times.

For three variables, according to clairaut theorem,

$$f_{xz}(x, y, z) = f_{zx}(x,y,z)$$

provided with the derivatives are continuous.

The partial derivative is approximate equal to the change in function.i.e,

$f_i(x_1,\ldots x_n) \approx f_i(x_1,\ldots,x_{i-1}, x_{i+h}, x_{i+1}\ldots x_n) - f_i(x_1,\ldots,x_{i-1}, x_i, x_{i+1}\ldots x_n)$

There are n partial derivatives of first order. For each of the first order partial order derivative of the function, there are n second order derivatives. i.e.,

$$\frac{\partial}{\partial xj}\left(\frac{\partial f}{\partial xi}\right) = \frac{\partial^2 f}{\partial xj\,\partial x\,i} = fx_ix_j \quad (i=1..n;j=1..n)$$

So, total n^2 elements are there. Therefore, n*n matrix of second order partial derivative is the Hessian matrix which is symmetric and all $f_{11}=f_{22}=\ldots=f_n$(Clairant theorem)

$$\begin{bmatrix} f11 & f12\cdots & f1n \\ f21\vdots & f22\ddots & f2n\vdots \\ fn1 & fn2\cdots & fnn \end{bmatrix}$$

Example:

If the two demand functions for the two commodities are given by

$x=\frac{q}{p} \quad y=\frac{p2}{q}$

then the marginal demand functions are

$\frac{\partial x}{\partial p}=\frac{-q}{p2} \quad \frac{\partial y}{\partial p}=\frac{2p}{q}$

$\frac{\partial x}{\partial q} = \frac{1}{P}$ $\frac{\partial y}{\partial q} = \frac{-p2}{q2}$

Since $\frac{\partial x}{\partial q} \geq 0$ and $\frac{\partial y}{\partial p} \geq 0$, therefore, two commodities are competitive.

Example:

If the demand functions for two related commodities are given by

$x=ae^{-pq}$ and $y=be^{p-q}$ where $a \geq 0$ $b \geq 0$

Solution: Since two demand functions are given as

$x=ae^{-pq}$

$y=be^{p-q}$

their marginal demand functions can be calculated as:

$\frac{\partial x}{\partial p} = -aqe^{-pq}$ $\frac{\partial y}{\partial p} = be^{p-q}$

$\frac{\partial x}{\partial q} = -ape^{-pq}$ $\frac{\partial y}{\partial q} = -be^{p-q}$

Because $\frac{\partial x}{\partial q} \leq 0$ and $\frac{\partial y}{\partial p} \geq 0$, therefore the given commodities are neither competitive nor complementary.

Example

Consider two products, A and B. the demand for good A and B, & described by following two equations

$q_a = \frac{200}{pa\, pb/2}$

$q_b = \frac{1000}{p_a 1/3\, p_3}$ find ∂q/ ∂q given the result explain A and Bare complementary or substitutes.

Solution

$q_a = \frac{200}{pap^1b^2} = \frac{200\, pb^{-1/2}}{pa}$

$\frac{\partial qa}{\partial pb} = 200((\frac{-1}{2} p_b^{-1/2}) /p_a)$

$= -100 \frac{pb^{-3/2}}{pa}$

$=-100\ pa^{-1}\ pb^{-3/2}$

$q_b = \frac{1000}{pa^{1/3}pb}$

$\frac{\partial qb}{\partial pa} = \frac{\partial}{\partial pb} = (\frac{1000}{pa^{1/3}pb})$

$= \frac{1000}{pb} (\frac{\partial}{\partial pb} (p_a^{-1/3})$

$\frac{\partial q_b}{\partial pa} = \frac{\partial}{\partial pa} (\frac{1000}{pa^{1/3} \quad p_b})$

$= \frac{1000}{p_b} (\frac{-1}{3} p_a^{-1/3-1})$

$= \frac{-1000}{3} p_a^{-4/3} . p_b^{-1}$

We know that p_a and p_b are positive because prices can never be negative therefore:

$\frac{\partial qa}{\partial pb} = - (\frac{100}{pa\ pb\ 3/2}) = -(\frac{+}{+\ +}) < 0$

$\frac{\partial qb}{\partial pa} = - \frac{1000}{3pa4/3 \quad pb} = - (\frac{+}{+\ +}) < 0$

Because both cross elasticities are – ve

Example: Two goods are complement goods.

The Stone-Geary Utility function is written as u= log U = β_1 log ($q_1 - \gamma_1$) + β_2($q_2 - \gamma_2$), where *u* is the utility index, q_i is the quantity of commodity *i*, $0 < \beta_i < 1$, $\gamma_i > 0$, $q_i - \gamma_i > 0$ and *i* = 1,2.

a. Find the marginal utility of this function with respect to q_1 and determine its sign.

b. What is the significance of a positive marginal utility?

c. Find the second derivative of this function with respect to q_1. Does the utility function exhibit diminishing marginal utility?

Solution: Utility function is: u= log U = β_1 log ($q_1 - \gamma_1$) + β_2($q_2 - \gamma_2$)

a. Marginal utility is given by :$\frac{du}{dq_1} = \frac{\beta_1}{(q_1 - \gamma_1)}$; which is greater than zero; because both the numerator and the denominator are positive.

b. Since marginal utility is positive; this implies that as utility increases monotonically with increase in q_1.

c. $\frac{d^2u}{dq_1^2} = -\frac{\beta_1}{(q_1 - \gamma_1)^2}$; which is less than zero. Since the second derivative is negative, the utility function exhibits diminishing marginal utility.

Example: Given the production function:

$P(L,K)=5L^{1/2} K^{1/2} +L.$

Find out the partial elasticity with respect to labor at (L,K)= (1024,27).

Solution: $P(L,K)=5L^{1/2} K^{1/2} +L$

$\epsilon_L = P'_L (L,K).L\frac{L}{P(L,K)}$

$= (L^{-4/5} K^{-1/3} +1)\frac{L}{5L^{1/5}k^{1/3}+L}$

$=\frac{L^{1/5}k^{1/3}+L}{5L^{1/5}k^{1/3}+L}$

Therefore, at (L,K)=(1024,27) we have $\epsilon_L = \frac{1024^{1/5}27^{1/3}+1024}{5.1024^{1/5}27^{1/3}+1024} = \frac{259}{271}$

This explains that if capital remains constant at K=27 and at L=24 labour increases with 1 percent, then output will increase by $\frac{259}{271}$ percent.

Example: Utility function is given:

$U = U = X^{0.5}Y^{0.5}$.

Calculate the marginal rate of substitution between X,Y.

Function is $U = X^{0.5}Y^{0.5}$.

First, take the partial derivative of U with respect to X to get MU_x.

$$MU_x = \frac{\partial U}{\partial x} = 0.5X^{-0.5}Y^{0.5}.$$

Next, take the partial derivative with respect to Y to get MU_y.

$$MU_y = \frac{\partial U}{\partial x} = 0.5X^{0.5}Y^{-0.5}.$$

Dividing MU_x by MU_y we get

$$MRS = \frac{MUx}{MUy} = \frac{0.5X^{-0.5}Y^{0.5}}{0.5X^{0.5}Y^{-0.5}} = y/x$$

Example: Given an isoquant

$$Q = K^{1/6}L^{1/2}$$

Find out slope of isoquant.

Solution: Slope of Isoquant = $\frac{dk}{dL}$

$\frac{dk}{dL} = -\frac{\partial Q}{\partial L} / \frac{\partial Q}{\partial K}$

$\frac{\partial Q}{\partial L} = \frac{1}{2}K^{1/6}L^{-1/2}$

$\frac{\partial Q}{\partial K} = \frac{1}{6}K^{-5/6}L^{1/2}$

$\frac{dk}{dL} = -(\frac{1}{2}K^{1/6}L^{-1/2}) / (\frac{1}{6}K^{-5/6}L^{1/2})$

$= -\frac{6}{2} K^{1/6} K^{5/6}L^{-1/2}L^{-1/2}$

$= -3(K/L)$

Therefore, the slope of isoquant is 3(K,L).

Example: Given demand function Q- 90+ 2P=0; and average cost function

$$AC = Q^2 - 39.5Q + 120 + 125/Q$$

Calculate the level of output where:

(a) total revenue is maximum,

(b) marginal cost is minimum,

(c) profits is maximum.

Solution: (a) The demand function is Q- 90+ 2P=0.

Written as 2P=90 - Q

$$P= 45 – 0.5Q$$

$$TR= PQ= (45 – 0.5Q)Q$$

$$=45Q – 0.5 Q^2$$

For maximizing TR, first-order condition is:

$$\frac{dTR}{dQ} =45 – Q=0$$

$$Q=45$$

and second-order condition is, $\frac{d^2TR}{dQ^2}$= -1<0.

Therefore, at Q=45, TR is maximized.

(b) $AC= Q^2- 39.5Q+ 120+ 125/Q$

$$TC= AC.Q= (Q^2- 39.5Q+ 120+ 125/Q)Q$$

$$= Q^3- 39.5Q^2+ 120Q+ 125$$

$$MC= \frac{dTC}{dQ} = 3Q^2 -79Q+ 120$$

MC is minimum when, $\frac{dMC}{dQ}$=0 and $\frac{d^2MC}{dQ^2}$

$$\frac{dMC}{dQ} = 6Q -79=0$$

$$Q= 13.167$$

And, $\frac{d^2MC}{dQ^2}$ =6>0.

Hence, at Q= 13.167, MC is minimum.

(c) Profit (π) = TR- TC

$$=45Q – 0.5 Q^2 – (Q^3- 39.5Q^2+ 120Q+ 125)$$

$$= -Q^3 + 39 Q^2 - 75Q- 125$$

For maximization of profit, first order condition

$$\frac{d\pi}{dQ} = - 3Q^2 + 78Q- 75=0$$

$$(-3Q + 3)(Q - 25) = 0$$

$$Q=1 \text{ and } Q= 25$$

and for second order condition,

$$\frac{d^2\pi}{dQ^2} = - 6Q + 78$$

When Q = 1 then,

$$\frac{d^2\pi}{dQ^2} = 72>0.$$

When Q=25 then,

$$\frac{d^2\pi}{dQ^2} = -72<0.$$

Therefore, profit is maximum when Q=25

Maximum $\pi = -(25)^3 + 39\,(25)^2 - 75(25) - 125 = 6750$.

Example: Two different demand functions are given:

$Q_1 = 21 - \text{o}.1P_1$ and $Q_2 = 50 - 0.4\,P_2$

TC = 2000 + 10 Q where $Q = Q_1 + Q_2$, what price will the firm charge (a) with discrimination and (b) without discrimination between markets?

Solution: since demand function in first market is, $Q_1 = 21 - \text{o}.1P_1$

Therefore, $P_1 = 210 - 10\,Q_1$

And, $TR_1 = P_1Q_{!} = (210-10Q_1)Q_1 = 210Q_1 - 10\,Q_1^2$

$MR_1 = \frac{dTR_1}{dQ_1} = 210 - 20Q_1$

Profit is maximum when MR= MC,

$MC = \frac{dTC}{dQ} = 10$

$MR_1 = MC$

$210 - 20Q_1 = 10$

$Q_1 = 10$

When $Q_1 = 10$, $P_1 = 210 - 10(10) = 110$

demand function in second market is, $Q_2 = 50 - \text{o}.4P_2$

hence, $P_2 = 125 - 2.5\,Q_2$

$TR_2 = (125 - 2.5\,Q_2)Q = 125Q - 2.5\,Q_2^2$

$MR_2 = \frac{dTR_2}{dQ_2} = 125 - 5Q_2$

When MR_2=MC

$125 - 5Q_2 = 10$

$Q_2 = 23$

When $Q_2 = 23$, then $P_2 = 125 - 2.5(23) = 67.5$

The discriminating monopoly charges a lower price in the second market where the demand is relatively more elastic, and a higher price in the first market where the demand is relatively less elastic.

Example: A producer is a price-taker on both the market for input factors labor and capital, and the market for end products. The cost of one unit of labor equals w = 2, the cost of one unit of capital equals r = 32 , while the selling price of the end products equals p =32. The production function of this producer is given by $Y(L,K) = L^{1/8}\ K^{1/2}$. Determine the maximum profit.

Solution:

The revenue function is $R(L,K) = p.Y(L,K) = 32.\ L^{1/8}\ K^{1/2}$

Cost function

$C(L,K) = wL + rK = 2L+32K$, and

Hence, profit function becomes

$\Pi(L,K) = 32\ L^{1/8}\ K^{1/2} - 2L - 32K$

Partial derivative of $\pi(L,K)$ is given by:

$\pi'_L = 4L^{-7/8}K^{1/2} - 2$ and

$\pi'_k = 16\ L^{1/8}K^{-1/2} - 32$

the stationary points of profit function are solutions of the following system

$4L^{-7/8}K^{1/2} - 2 = 0$

$16\ L^{1/8}K^{-1/2} - 32 = 0$

Hence, $K^{1/2} = 1/2L^{7/8}$ and

therefore, $K = ¼\ L^{14/8}$

Consequently, $L^{1/8}(1/4\ L^{14/8})^{-1/2} = 1$

which gives L=1 and therefore, K=1/4

Hence, (L,K) = (1,1/4) is the only stationary point. By the use of the criterion function we investigate whether or not this point is a maximum location.

$\pi''_{LL} = -3.1/2L^{-15/8}K^{1/2}$;

π''_{KK}= -8$L^{1/8}K^{-3/2}$ and

π'_{L}= 2$L^{-7/8}K^{-1/2}$, which implies that the criterion function is given by

C(L,K) = π''_{LL}. π''_{KK} - $(\pi''_{LK})^2$

= (-3.1/2$L^{-15/8}K^{1/2}$)(-8$L^{1/8}K^{-3/2}$)- $(2L^{-7/8}K^{-1/2})^2$

= 28 $L^{-14/3}K^{-1}$ – 4$L^{-14/8}K^{-1}$

= 24 $L^{-14/8}K^{-1}$>0

Hence, as C(1,1/4) >0 and π''_{LL} (1,1/4) <0 it follows that $\pi(L,K)$ has a maximum profit at (L,K)= (1,1/4), with value π=6.

4. **Quadratic Forms**

A quadratic form of two variables is

f(x,y) = ax^2 + 2bxy +cy^2;

a,b,and c are constants. Now, using matrix notation:

f(x,y) = (x,y) $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

f''_{11}= 2a, f''_{12}= f''_{21}= 2b and f''_{22}= 2c are the second order partial derivatives of the function f(x,y)

Therefore, the Hessian of f is given by

$$2\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

The given quadratic form is said to positive definite if f(x,y) >0; for all values of x and y i.e, (x,y) ≠ (0,0), and positive semidefinite if f(x,y)≥ 0 for all values of (x,y). The given function is negative definite if f(x,y)<0; for all values of x and y; and it is negative semidefinite if f(x,y)≤0. And it is indefinite we have two different pairs of x and y; (x^-,y^-) and (x^+,y^+); and also $f(x^+,y^+)$ >0.

Example: Express the quadratic form below as a matrix form. Determine the definiteness of the equations:

a) f(x_1,x_2) = 4x^2 +8xy +5y^2
b) f(x_1,x_2) = -x^2 +xy - 3y^2

Solution: a) f(x,y) = (x,y) $\begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Therefore, symmetric matrix is $\begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}$, whose determinant is positive. Hence,f(x,y) > 0 for all values of x and y. Therefore, the quadratic form is positive definite.

$$f(x,y) = (x,y)\begin{pmatrix} -1 & 1/2 \\ 1/2 & -3 \end{pmatrix}\binom{x}{y}$$

Therefore, symmetric matrix is $\begin{pmatrix} -1 & 1/2 \\ 1/2 & -3 \end{pmatrix}$, whose determinant is negative. Hence, $f(x,y) < 0$ for all values of x and y. Therefore, the quadratic form is negative definite.

5. Exercise:

1. Find the second – order partial derivatives fxx, fyy and fxy for each of the following functions:
   (a) $Z = \frac{x+4}{2x+5y}$
   (b) $Z = (7x + 3y)^3$
   (c) $Z = (x^3 + 5y)^5$
   (d) $Z = (2x+5y)e^y$
   (e) $Z = \log(1 + x^2) + y^2$
   (f) $Z = 3x^2e^{2y}$
2. Consider the function: $f(x_1, x_2) = (3x_1^2 + 5x_1 + 1)\cdot(x_2 + 4)$.

a. Find f_1 and f_2.

b. Find f_{11}, f_{22}, f_{12} and f_{21}.

3. Assume the demand for sugar is a function of income (Y), the price of sugar (P_s) and the price of saccharine (P_c), a sugar substitute, as follows:

$$Q_d = f(Y, P_c, P_s) = 0.05Y + 10P_c - 5P_s^2.$$

a. Find the partial derivatives of this demand function.

b. Find the elasticity of demand with respect to income $\left(\frac{\partial Q_d}{\partial Y}\cdot\frac{Y}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

c. Find the own-price elasticity of demand $\left(\frac{\partial Q_d}{\partial P_s}\cdot\frac{P_s}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

d. Find the cross-price elasticity of demand $\left(\frac{\partial Q_d}{\partial P_c}\cdot\frac{P_c}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

4. Show that fxz = fzx and fxzz = fzxz = fzzx from the following function:

$F(x,y,z) = ye^{x} + x\log z$

5. The demand function of two related commodities are given by

$X_1 = p_1^{-1.7} p_2^{0.8}$

$X_2 = p_1^{0.5} p_2^{-0.2}$

What can you say about the two commodities X_1 and X_2 and also find all partial elasticties.

6. A firm produces two commodities: commodity *X* and commodity Y. the demand functions are:

$p_1 = 8 - 2x$

$p_2 = 14 - y^2$

The combined cost of production of these unit is given by $C = 10+4x+2y$. What will be the prices of two products so that joint profit will be the maximum.

7. Consider a production function that takes the form $y = 10L^{\frac{1}{2}}K^{\frac{1}{2}}$, and assume that capital (K) is constant at $K_0 = 64$.

a. Find the marginal product of labor, $\frac{\partial y}{\partial L}$.

b. If the labor were paid real wage equivalent to the marginal product of labor, how many labors would be employed when the going wage rate is 10?

c. What happens to the number of labor demanded when the wage declines to 8?

d. How many labor would be demanded if wage remains 8, but the capital is increased to 100?

e. Find the cross partial derivative, $\frac{\partial^2 y}{\partial K \partial L}$.

8. Example: Two different demand functions are given:

$Q_1 = 11 - 2p_1 - 2p_2$ and $Q_2 = 16 - 2p_1 - 3p_2$

TC = 10 +4x+2y. Determine the quantities that maximize the profit of monopolist and also find maximum profit.

9. Two different demand functions of discriminating monopoly are given:

$p_1 = 140 - 7q_1$ and $p_2 = 90 - 0.4\, q_2/2$

$TC = 20 + 2q + 3q^2$ where $q = q_1 + q_2$, what price will the firm charge in two markets to maximize profit?

Solution:

1. a. $f_{xx} = \frac{-4\,(5y-8)}{(2x+5y)^3}$; $f_{yy} = \frac{-50(x+4)}{(2x+5y)^3}$ and $f_{xy} = f_{yx} = \frac{10x-25y+80}{(2x+5y)^3}$

   b. $f_{xx} = 294(7x + 3y)$; $f_{yy} = 54(7x + 3y)$ and $f_{xy}=f_{yx}= 126\,(7x +3y)$

   c. $f_{xx}= 30x((x^3+5y)^4+6x^2(x^3 +5y)^3)$;$f_{yy}=50(x^3 +5y)^3$ and $f_{xy}= 300\,(x^3 +5y)^3x^2$

   d. $f_{xy}= 2e^y$ $f_{xx}=0$.

   e. $f_{xx}= -\frac{2-2x^2}{(1+x^2)^2}$; $f_{yy}= 2$ and $f_{xy}= \frac{1+2x}{1+x^2} = f_{yx}$

   f. $f_{xx}= 6x\,e^{2y}$; $f_{yy}= 4x^3e^{2y}$ and $f_{xy}= 6x^2e^{2y}$

2. Derivatives
   a. $f_1 = 6x_1x_2 + 24x_1 + 5x_2 + 20$; $f_2 = 3x_1^2 + 5x_1 + 1$
   b. $f_{11} = 6x_2 + 24$; $f_{22} = 0$; $f_{12} = 6x_1 + 5$; and $f_{21} = 6x_1 + 5$. Note both cross partial derivatives are equal, as they should be, according to the Young's Theorem.
3. Answers:
   a. $\frac{\partial Q_d}{\partial Y} = 0.05$; $\frac{\partial Q_d}{\partial P_c} = 10$; $\frac{\partial Q_d}{\partial P_s} = -10P_s$
   b. 1.12
   c. - 0.56
   d. 0.16
4. Apply Young’s theorm
5. Since $\frac{\partial x1}{\partial p2}$ and $\frac{\partial x2}{\partial p1}$ are both greater than zero. Hence, the commodities X_1 and X_2 are competitive.
6. $p_1= 3.2$ and $p_2= 3.9$; $e_{11}= -1.7$; $e_{21}= 0.8$; $e_{22}= -.2$ and $e_{12} = 0.5$.
7. Answers:

a. $\frac{\partial y}{\partial L} = 5\left(\frac{K}{L}\right)^{\frac{1}{2}} = \frac{40}{L^{\frac{1}{2}}}$

b. L = 16

c. L = 25; labor demand increases with wage decline.

d. $\frac{\partial y}{\partial L} = 5\left(\frac{K}{L}\right)^{\frac{1}{2}} = \frac{50}{L^{\frac{1}{2}}}$ = 8; i.e., L = 39.

e. $2.5L^{-\frac{1}{2}}K^{-\frac{1}{2}}$

8. x=1 and y=2; $\pi = 8$

9. p_1= 110.52 and p_2 = 85.52 and q= 13.17

6.References:

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