<u>Homogeneous and Homothetic Function</u>

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<u>Contents</u>

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1. Learning Outcomes

After completing of the present chapter, you should able to:-

1. Chain Rule of Differentiation
2. Implicit function Differentiation
3. Implicit function Theorem
4. Homogeneous and Homothetic Functions.
5. Euler's Theorem

2. Tools of Comparative Static Analysis

In economic analysis, the theory represents certain association between the independent variables and the dependent variables. It is harder to solve clearly by transmuting the equations to ones that reveal the dependent/endogenous variables as functions of the independent/exogenous variable of the given data. When there is change in exogenous variable then endogenous variable also change, to determine this change; the method of implicit differentiation is applied. This technique of finding rates of change of endogenous variables, as exogenous variables change, is known in economic as comparative statistics.

2.1Chain Rule:

One of the most important techniques of differentiation is chain rule. The chain rule is a rule of differentiating compositions of functions. Composition of function signifies the function of another variable. These are functions of one or several variables in which the variables themselves functions of the another basic variables.

If a function consists of two variables and both are function of common variable 't', e.g.

$y= f(x_1 (t),x_2 (t))$

then according to chain rule, differentiation y w.r.t. 't' is

$$\frac{dy}{dt} = \frac{\partial f}{\partial x1}\frac{dx1}{dt} + \frac{\partial f}{\partial x2}\frac{dx2}{dt}.$$

Example: $y=3x_1 + 5x_2$ with $x_1=t^2$ and $x_2=4t^3$

Applying chain rule gives

$\frac{dx1}{dt}=2t$; $\frac{dx2}{dt}=12t^2$

$\frac{\partial f}{\partial x1}=3$ and $\frac{\partial f}{\partial x2}=5$

$\frac{dy}{dt}=\frac{\partial f}{\partial x1}\frac{dx1}{dt}+\frac{\partial f}{\partial x2}\frac{dx2}{dt}$

$=3.(2t)+ 5.(12\ t^2)$

$=6t + 60t^2$

2.1.a Chain rule with Multivariable Function:

If x and y are both multivariable functions; i.e. x = x(u,v) and y=y(u,v); have first order partial derivatives at the point (u,v) and suppose z = f (x, y) is differential at point (x(u,v); y(u,v)) then f(x(u,v); y (u,v)) has first order partial derivatives at (u,v) given by:

$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$

and

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

Example:

Let $z=e^{x^2y;}$ where $x(u,v)=\sqrt{uv}$

and y (u,v)=1/v, find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

$\frac{\partial z}{\partial x}=e^{x^2y}\ 2xy$ and $\frac{\partial z}{\partial y}= e^{x^2y}x^2$

$\frac{\partial x}{\partial u}=v^{1/2}\frac{1}{2}\ u^{-1/2}$ and $\frac{\partial x}{\partial v}=\frac{1}{2}u^{1/2}\ v^{-1/2}$

$\frac{\partial y}{\partial u} = 0$ and $\frac{\partial y}{\partial v} = -1.v^{-1-1} = -v^{-2}$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$

$= e^{x^2 y}\,(2xy)\,\frac{1}{2}\,(\frac{v}{u}) + e^{x^2 y}.\,x^2.0$

$= e^{x^2 y}.\,2xy.\,\frac{1}{2}\,\frac{v^{1/2}}{v^{1/2}}.$

Putting values of x and y.

$\frac{\partial z}{\partial u} = e^{x^2 y}.\,xy.\,\frac{v^{1/2}}{u^{1/2}}.$

$= e^{((uv)^{1/2})2.1/v}\,(uv)^{1/2.}\,.\frac{I}{V}\,\frac{v^{1/2}}{u^{1/2}}.\,.$

$e^{(u.v).1/v} = e^u.$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\,.\,\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\,.\frac{\partial y}{\partial v}$

$=(2xy\,e^{x^2 y}).\frac{1}{2}.\,\frac{\sqrt{u}}{\sqrt{v}} + x^2 e^{x^2 y.}\,(\frac{-1}{v2})$

$= e^{x^2 y}(xy\frac{\sqrt{u}}{\sqrt{v}} - \frac{x^2}{v^2})$

$= e^{x^2 y}(\frac{\sqrt{uv}}{v}.\frac{\sqrt{u}}{\sqrt{v}} - \frac{(\sqrt{uv})^2}{v^2})$

$= e^{x^2 y}(\frac{u}{v} - \frac{u}{v})$

$= e^{x^2 y}(0)$

$= 0$

2.1.b Chain rule with ‘n’ variables:

The chain rule can also be extended to a number of variables, which is a function of other variables:

$Z=f(x_1 \ldots\ldots x_n)$ with

$X_1 = x_1(t_1 \ldots\ldots\ldots t_m)$

$X_2 = x_2(t_1 \ldots\ldots\ldots t_m)$

$X_n = x_n(t_1 \ldots\ldots\ldots t_m)$ then

$$\frac{\partial Z}{\partial tj}=\frac{\partial z}{\partial x1}.\frac{\partial x1}{\partial tj}+\frac{\partial z}{\partial x2}.\frac{\partial x2}{\partial tj}+\ldots\ldots+\frac{\partial z}{\partial xn}.\frac{\partial xn}{\partial tj}$$

2.2 Directional Derivatives

For a function z=f(x,y), the partial derivative with respect to x gives the rate of change of f in the x_0 direction and the partial derivative with respect to y gives the rate of change of f in the y_0 direction. How do we compute the rate of change of f in an arbitrary direction? The rate of change of a function of several variables in the direction u is called the directional derivative in the direction u. Here u is assumed to be a unit vector.

If z=f (x,y), the partial derivatives $f'_1(x,y)$ and $f'_2(x,y)$ Choose a particular point (xo,yo) in the domain. Any nonzero vector (h,k) is then a direction in which we can move away from (xo, yo) in a straight line to points of the form.

$(x,y) = (x(t)) = (x_o+th, y_o+tk)$

Given the point (xo, yo) and the direction (h, k) ≠ (o,o), define the directional function g by

$g(t) = f(x_o+th, y_o+tk)$ …..(1)

By using the chain rule, the derivative of this directional function can be calculated as

$$g'(t)=f'_1(x,y)\frac{dx}{dt} + f'_2(x,y)\frac{dy}{dt}$$

$$= f'_1(x_o+th, y_o+tk)h+f'_2(x_o + th, y_o + tk)k \qquad \ldots\ldots.(2)$$

If t=0 then

$$g'(0)=f'_1(x_o, y_o)h +f'_2(x_o,y_o)k \qquad \ldots\ldots.(3)$$

For the case when the vector (h,k) has length 1, the derivative of f in the direction (h,k) is called the directional derivative of f in the direction (h,k) at (x_o,y_o). It is denoted by $D_{h,k}\, f(x_0,y_0)$. Hence, the directional derivative of f(x,y) at (x_0,y_0) in the direction of unit vector (h,k) (where $h^2+k^2=1$) is

$$D_{h,k}\, f(x_0,y_0) = f_1(x_0,y_0)\, h + f_2(x_0+y_0)k$$

Any move from (x_0,y_0) to (h,k) changes the value of f by approximately $D_{h,k}\, f(x_0,y_0)$. The vector (f_1(x0,y0), f_2(x0,y0)) is called as gradient of the function f (x,y) at (x_0,y_0). Therefore, it is the scalar product of gradient with vector (h,k).

Now, differentiating (2) with respect to t, we get second derivative of the directional function g. i.e,

$$g''(t)= \frac{d}{dt} f_1'(x,y)h + \frac{d}{dt} f_2'(x,y) \qquad (4)$$

where x= x0 +th, and y= y0 +tk. Again, applying the chain rule, the above equation becomes:

$$\frac{d}{dt} f_1'(x,y)= f''_{11}(x,y) \frac{dx}{dt} + f''_{12}(x,y) \frac{dy}{dt} = f''_{11}(x,y)h+ f''_{12}(x,y)k$$

$$\frac{d}{dt} f_1'(x,y)= f''_{21}(x,y) \frac{dx}{dt} + f''_{22}(x,y) \frac{dy}{dt} = f''_{21}(x,y)h+ f''_{22}(x,y)k$$

Suppose, $f_{12}'' = f_{21}''$,then equation (4) becomes:

$$g''(t) = f''_{11}(x,y)h^2 + 2f''_{12}(x,y).hk + f''_{22}(x,y)\, k^2$$

again x= x+ t.h and y= y+t.k. Assuming t=0 and (h,k) has length 1, then above equation becomes:

$$D^2f(x,y)= f''_{11}(x,y)h^2 + 2f''_{12}(x,y).hk + f''_{22}(x,y)\, k^2$$

Example: If f(x,y) = xy.

Compute the first and second directional derivatives of f at $(x_o\ y_o)$ in the directions:

(a) $(h,k)=(1\sqrt{2},\ 1/\sqrt{2})$ and

(b) $(h,k)= (1\sqrt{2},\ -1/\sqrt{2})$.

Solution: We have

$f'_1(x,y)=y$, $f'_2(x,y)=x$, $f'_{11}(x.y)=0$.

$f'_{12}(x,y)= f'_{21}(x,y)=1$, $f'_{22}(x.y)=0$.

Thus, if $(h,k)=(1/\sqrt{2}.1/\sqrt{2})$, then

$D_{h.k} f(x_o.y_o) = y_0\frac{1}{\sqrt{2}}+ x_0\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x_o+y_o)$

and

$D^2_{h,k}f(x_o.y_o)= 0(\frac{1}{\sqrt{2}})^2+ 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} +0(\frac{1}{\sqrt{2}})^2 =1$

If $(h,k) = (\frac{1}{\sqrt{2}}, - \frac{1}{\sqrt{2}})$; then

$D_{h.k} f(x_o,y_o) = y_0 \frac{1}{\sqrt{2}} + x_0 \frac{-1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y_o- x_o)$

and

$D^2_{h,k}f(x_o.y_o)= 2\left(\frac{1}{\sqrt{2}}\right)(-\frac{1}{\sqrt{2}}) = -1$

2.3 Implicit Differentiation

An application of chain rule to determine the derivative of a function defined implicit.

Suppose that x and y are related to each other with the relation; F(x,y)=0 where y = f(x) is a differentiable function of x. Find dy/dx by using chain rule method.

Consider a function:

Z= f (x,y) = F (x, f(x))

then

$dz/dx=F_X(x,y)\, dx/dx + F_Y(x,y)\, dy/dx$

Because Z=F(x,y)=0 for all x.

$Fx(x,y).dx/dx+F_Y(x,y)\, dy/dx=0$

Now, if $Fy(x,y) \neq 0$ and dx/dx=1 then

$\frac{dy}{dx} = \frac{-Fx\ (x,y)}{Fy\ (x,y)}$

$= -\frac{f'x}{f'y}$ where $F_{x\ (x,y)} = \partial z/\partial x$ and $Fy(x,y)=\partial z/\partial y$.

Example:

Find dy/dx given

$y^3 + y^2 - 5y - x^2 + 4 = 0$.

Solution:

Define a function

$F(x,y)=y^3+y^2-5y-x^2+4$.

$Fx\ (x,y)=(\partial Fx(x,y))/\ \partial x=-2x$.

$Fy(x,y)=(\partial Fy(x,y))/\partial y=3y^2+2y-5$

$$\frac{dy}{dx} = \frac{-Fx(x,y)}{Fy\ (x,y)}$$

$$= (-2x)/(3y^2 + 2y - 5)$$

2.3.a Implicit Differentiation of three or more variables:

Let's assume that there is an implicit function consists of three variables; i.e, F (x,y,v) = 0.

The above equation becomes $\partial z/\ \partial x.dx+\partial z/\ \partial y.dy+\partial z/\ \partial v.dv=0$.

In order to get derivative hold one of the variable constant. Suppose v is constant then dv=0 and $0= \partial z/\partial x.dx+\ \partial z/\ \partial y.dy.$, rearrange the equation we get $\partial y/\ \partial z = (\partial z/dx)/(\partial z/dy)$ as in two variable case, because one variable was constant, so this difference is partial. So notations have to change. Therefore,

$\partial y/\partial x=(-\partial z/\partial x)/(\partial z/\partial y)$

$\partial y/\partial x=(-fx)/fy$.

In general, the partial derivations of an implicit function $F(x_1,x_2,\ldots x_n,z)$ are given by

$\partial z/\partial xi=-(\partial F(\partial xi))/(\partial F(\partial z))$ (i=1,2…n); asumming $\partial F/\partial z \neq 0$

Example:

$x- 2y- 3z+ z^2= -2$.

Let $F(x,y,z) = x- 2y- 3z+ z^2$ and c= -2.

$Fx= \partial v/\partial x=1$, $Fz = 2z -3$

$Fy= -2$.

$\partial z/\partial x= z'x= - (-F_X)/(-F_Z)$ and $z'y=(-F'y)/ (-F'z)$

$= \frac{-1}{2z-3}$ [F'z ≠0, so assuming that z≠3/2]

$\frac{\partial z}{\partial y} = Z'y = \frac{-(-2)}{2z-3} = \frac{2}{2z-3}$.

Example: Given the demand function

$Q_2= 4850 – 5P_2 +1.5P_1 + 0.1 Y$

Where Y= 10,000, P_2= 200 and P_1= 100. Find the income elasticity of demand and cross elasticity of demand for first commodity.

Solution:

(a) Income elasticity of demand is given by:

$$e_y= \frac{\partial Q_2}{Q_2} / \frac{\partial Y}{Y}$$

$$= \frac{\partial Q_2}{\partial Y} (\frac{Y}{Q_2})$$

Given the demand function: $Q_2 = 4850 – 5P_2 +1.5P_1 + 0.1 Y$

$\frac{\partial Q_2}{\partial Y} = 0.1$ ……..(1)

$Q_2= 4850- 5(200) + 1.5(100) +0.1 (10000) = 5000$……..(2)

Putting the values of 1 and 2, hence;

e_y= 0.1(10000/5000) = 0.2

Since the value of $e_y<1$, therefore the good is income elastic.

(b) Cross elasticity of demand is given by:

$$e_c = \frac{\partial Q_2}{\partial P_1}\left(\frac{P_1}{Q_2}\right)$$

given $Q_2 = 4850 - 5P_2 + 1.5P_1 + 0.1\,Y$

$$\frac{\partial Q_2}{\partial P_1} = 1.5$$

$$e_c = 1.5\left(\frac{100}{5000}\right) = 0.03$$

2.3.b Implicit functional theorem (for 2 variables): Let F(x,y)=0 be an implicit function with continuous first derivatives, which is satisfied at some point, (x_0, y_0) and is defined in some neighborhood of this point. If Fy≠0 at this point, then there is a function y=f(x) defined in some neighborhood of x=xo corresponding to the relationship defined by F(x,y)=0 such that:

(i) $y_o=f(x_o)$ and

(ii) $f'(x_o)=-Fx/Fy$.

Statement: Let F $(x_1, x_2 \ldots\ldots x_n, y)=0$ be an implicit function with continuous first derivatives which is satisfied at some point, $(x_1, x_2 \ldots\ldots x_n, y)$ is defined at some neighborhood of this point. If Fy≠0 at this point, then there is a function $y=f(x_1, x_2 \ldots\ldots x_n)$ defined in some neighborhood of $x=x_o=(x_1, x_2 \ldots\ldots x_n)$ such that

(i) $y_o=f(x_o)$

(ii) $fi(x_o)=-Fxi/Fy$.

Example: The Cobb-Douglas production function: $50\ K^{0.3}\ L^{0.7} = Q$, where Q is a given level of output, K is the amount of capital and L is the amount of labor. The isoquant associated with the function reflects the levels of capital and labor that yield a constant level of output.

a. Use the Implicit Function Theorem to derive an equation for the slope of an isoquant associated with the production function.

b. When K = 6 and L = 2, what is the slope of a line tangent to this isoquant? What is the slope of the line when K = 3 and L = 14?

c. Find the MRTS for both examples in part (b).

Solution: Given the production function 50 $K^{0.3}$ $L^{0.7}$ = Q

a. Slope of Isoquant = $\frac{dK}{dL} = -\frac{F_L}{F_K} = -\frac{\partial Q}{\partial L}\Big/\frac{\partial Q}{\partial K}$

$$= -\frac{50.0.7\, K^{0.3}L^{-0.3}}{50.\ 0.3\, K^{-0.7}L^{0.7}} = -\frac{50.0.7\, K^{0.3}K^{0.7}}{50.\ 0.3\, L^{0.3}L^{0.7}} = -\frac{7}{3}\cdot\frac{K}{L}$$

b. when K=6 and L=2 then slope becomes:

Slope of Isoquant= $-\frac{7}{3}\cdot\frac{K}{L} = -\frac{7}{3}\cdot\frac{6}{2} = -7$

When k=3 and L=14 then

Slope = $-\frac{7}{3}\cdot\frac{K}{L} = -\frac{7}{3}\cdot\frac{3}{14} = -\frac{1}{2}$.

c.i) MRTS= $\left|\frac{dK}{dL}\right| = \frac{7}{3}\cdot\frac{K}{L} = 7$

ii) MRTS= $\frac{1}{2}$

Note: The marginal rate of technical substitution (MRTS) is the rate at which the two production inputs can be substituted if output is held constant. It is the absolute value of the slope of the isoquant.

Example: Compute $\sigma_{k,L} = \frac{F_L^{'}}{F_k^{'}}$ for the Cobb-Douglas function F(K,L) = AK^aL^b

Solution: The marginal rate of substitution between K and L is

$$\sigma_{k,L} = \frac{F_L^{'}}{F_k^{'}} = \frac{dK}{dL} = \frac{\partial Q}{\partial L}\Big/\frac{\partial Q}{\partial K}$$

$= \frac{b\,A\,K^{a}L^{b-1}}{a\,A\,K^{a-1}L^{b}}$

$= \frac{b}{a}\frac{K}{L}$

Example: The implicit function $U = \sqrt{AB}$ shows what combinations of apples (A) and bananas (B) provide the levels of utility U. Find the derivative of the implicit function to determine the MRS of apples for bananas (MRS_{AB}).

Solution: Given the utility function:

$$U = \sqrt{AB}$$

$$\text{Slope of IC} = \frac{dB}{dA} = -\frac{F_A}{F_B} = -\,\frac{\partial U}{\partial A}\Big/\frac{\partial U}{\partial B}$$

$$= \frac{\frac{1}{2}A^{-\frac{1}{2}}B^{\frac{1}{2}}}{\frac{1}{2}A^{\frac{1}{2}}B^{-\frac{1}{2}}} = -\,\frac{B}{A}$$

$$MRS_{A,B} = \left|\frac{dB}{dA}\right| = \frac{B}{A}$$

Note: The absolute value of the slope of the indifference curve is the marginal rate of substitution (MRS), which measures the rate at which one good can be substituted for another, while maintaining the same level of utility.

2.3 Homogeneous and Homothetic Functions

2.3.a. Homogeneous Functions

A function is called as a homogeneous function of any degree 'n' if; when each of its elements is multiplied by any number t > 0; then the value of the function is multiplied by t^n. For instance, a function is homogeneous of degree 1 if, when all its elements are multiplied by any number t > 0, the value of the function is multiplied by the same number t.

i.e, f(x1; :::; xn) is homogenous of degree k if for all t > 0

$f(tx1; :::; txn) = t^k f(x1; :::; xn)$

To explain the concept of homogenous function, take an example, Q=f(k,L), where K,L and Q are variables. When independent variables (K,L) changes, there must be change in the dependent variable. In other words, if K and L are both increases by some factor 't' then Q also changes by some factor. If t=2 (a doubling of K and L), then Q also doubles, then the function is homogenous of degree 1.

Effect on Q when K and L are aboth doubled	Economist view	Mathematician view
Q is exactly doubled	CRTS	Function is homogenous of degree=1
Q is more than doubled	IRTS	Function is homogenous of degree>1
Q is less than doubled	DRTS	Function is homogenous of degree<1

2.2.b Homothetic Functions:

If $f(x_1, x_2,\ldots,xn)$ is a function of n variables defined in domain D. then f is called homothetic if

X,y € K, f(x)=f(y) , t>0→ F)tx)=f(ty)

If utility function is

$u(x; y) = xy$,

is a homogenous function of degree 2. Then the monotonic transformations

$g_1(z) = z + 1$;

$g_2(z) = z^2 + z$;

$g_3(z) = \log z$

generate the following homothetic (but not homogenous) functions

$v_1(x; y) = xy + 1$;

$v_2(x; y) = x^2y^2 + xy$;

$v_3(x; y) = \log x + \log y$:

Example

For the function $f (x_1, x_2) = Ax_1^a x_2^b$, test the homogeneity of function.

Solution:

$f(tx_1, tx_2) = A(tx_1)^a(tx_2)^b$

$= At^{a+b}x_1^a x_2^b$

$= t^{a+b} f(x_1, x_2)$,

so that f is homogeneous of degree a + b.

Example: Given the function, check whether function is homogeneous functions or not

$$f(x, y, z) = x^5 y^2 z^3$$

Multiplying by some factor α

$f(\alpha x, \alpha y, \alpha z) = (\alpha x)^5 (\alpha y)^2 (\alpha z)^3$

$= \alpha^{10} x^5 y^2 z^3$

$= \alpha^{10} f(\alpha x, \alpha y, \alpha z)$

Hence, this function is homogenous of degree 10.

2.3.c Partial derivatives of homogeneous functions

If f be a differentiable function of n variables that is homogeneous of degree k. Then each of its partial derivatives f'_i (for i = 1, ..., n) is homogeneous of degree k − 1.

The homogeneity of f means that

$f(tx_1, ..., tx_n) = t^k f(x_1, ..., x_n)$ for all $(x_1, ..., x_n)$ and all $t > 0$.

Now differentiate both sides of this equation with respect to x_i, to get

$t f'_i(tx_1, ..., tx_n) = t^k f'_i(x_1, ..., x_n)$,

and then divide both sides by t to get

$f'_i(tx_1, ..., tx_n) = t^{k-1} f'_i(x_1, ..., x_n)$,

so that f'_i is homogeneous of degree k − 1.

1.2.4 Euler's theorem

If the function z = f(x,y) is a homogeneous of degree 'n' then according to Euler's theorem:

$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = n.f(x,y)$

If $Z=f(x_1,x_2,x_3,\ldots,x_n)$, then according to this theoem:

$x_1.\frac{\partial f}{\partial x_1} + x_2.\frac{\partial f}{\partial x_2} + x_3.\frac{\partial f}{\partial x_3} + \ldots\ldots\ldots + x_n.\frac{\partial f}{\partial x_n} = nf(x_1,x_2,x_3,\ldots,x_n)$

Example:

Use Euler's theorem to determine the degree of homogeneity of the following functions

1). $f(x,y)= 2x^2+xy-y^2$

$\frac{\partial f}{\partial x}=f_x(x,y)= 4x+y$

$\frac{\partial f}{\partial y}=f_y(x,y)= x-2y$

According to Euler's theorem:

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y)$

$= x(4x+y)+y(x-2y)$

$=4x^2+xy+xy-2y^2$

$=4x^2+2xy-2y^2$

$=2(2x^2+xy-y^2)$

The degree of homogeneity is 2.

Example:

Use Euler's theorem to determine the degree of homogeneity of the following function

$f(L,K)=AL^{\alpha}K^{\beta}$

$\frac{\partial f}{\partial L} = f_l(L,K) = \alpha AL^{\alpha-1} K^{\beta}$

$\frac{\partial f}{\partial K} = f_k(L,K) = \beta AL^{\alpha} K^{\beta-1}$

By Euler's theorem

$L \frac{\partial f}{\partial L} + K \frac{\partial f}{\partial K} = nf(L,K)$

$= L.(\alpha A L^{\alpha-1} K^{\beta}) + K.(\beta A L^{\alpha} K^{\beta-1})$

$= \alpha A L^{\alpha} K^{\beta} + \beta A L^{\alpha} K^{\beta}$

$= (\alpha+\beta)(A L^{\alpha} K^{\beta})$

$= (\alpha+\beta) f(L,K)$

The degree of homogeneity is $\alpha+\beta$

Example: Suppose that $f(x_1, ..., x_n)$ is homogeneous of degree r. Show that each of the following functions $h(x_1, ..., x_n)$ is homogeneous, and find the degree of homogeneity.

a. $h(x_1, ..., x_n) = f(x_1^m, ..., x_n^m)$ for some number m.
b. $h(x_1, ..., x_n) = [f(x_1, ..., x_n)]^p$ for some number p.

Solution:

a. We know that $f(x_1, ..., x_n)$ is homogeneous of degree r, therefore

$f((tx_1)^m, ..., (tx_n)^m) = f(t^m x_1^m, ..., t^m x_n^m)$

$= (t^m)^r f(x_1^m, ..., x_n^m)$

$= t^{mr} h(x_1, ..., x_n),$

hence h is homogeneous of degree mr.

b. We have $h(tx_1, ..., tx_n) = [f(tx_1, ..., tx_n)]^p$

$= [t^r f(x_1, ..., x_n)]^p$

$= t^{rp} [f(x_1, ..., x_n)]^p$

$= t^{rp} h(x_1, ..., x_n),$

Therefore, h is homogeneous of degree rp.

Example: Solve the following:

a. Is the function $(x^3 - y^3)/(x^{1/2} + y^{1/2})$ homogeneous of any degree? (If so, which degree?)

b. Is the function $x^3y^3 + x^{1/2}$ homogeneous of any degree? (If so, which degree?)

c. A consumer's (differentiable) demand function for some good is $f(p_1, ..., p_n, w)$, where p_i is the price of the ith good, and w is the consumer's wealth. This function f is homogeneous of degree 0. Is there any necessary relationship between $\sum_{i=1}^{n}(p_i f_i'(p_1, ..., p_n, w))$ and $w f_{n+1}'(p_1, ..., p_n, w)$?

Solution: a. Given the function $(x^3 - y^3)/(x^{1/2} + y^{1/2}) = ((tx)^3 - (ty)^3)/((tx)^{1/2} + (ty)^{1/2})$

$$= t^{5/2}(x^3 - y^3)/(x^{1/2} + y^{1/2})$$

Therefore, the function is homogenous of degree 5/2.

b .Given the function $x^3y^3 + x^{1/2} = (tx)^3(ty)^3 + (tx)^{1/2} = t^kx^3y^3 + x^{1/2}$.

Hence, function is not homogeneous of any degree.

Suppose, to the contrary, it is homogeneous of degree k. Then for some value of k we have $(tx)^3(ty)^3 + (tx)^{1/2} = t^kx^3y^3 + x^{1/2}$ for all t and all (x, y). In particular, taking t = 4 we have $4096x^3y^3 + 2x^{1/2} = 4^k(x^3y^3 + x^{1/2})$ for all (x,y), and hence $2 = 4^k$ (taking (x, y) = (1, 0)) and $4098 = 2(4^k)$ (taking (x, y) = (1, 1)), which are inconsistent.

c .Given the function, $f(p_1, ..., p_n, w)$; which is homogeneous of degree 0

then according to Euler's theorem we have

$\sum_{i=1}^{n}p_i f_i'(p_1, ..., p_n, w) + w f'_{n+1}(p_1, ..., p_n, w) = 0$. (Note that f has n + 1 arguments.)

Example: Consider the production function $Q=AK^{\alpha}L^{\beta.}$

a. Using Euler's Theorem, prove that this production function exhibits constant returns to scale when $\alpha+\beta = 1$.

b. What condition on $\alpha+\beta$ is necessary for increasing returns to scale? For decreasing returns to scale?

Solution: Given Cobb-Douglas Production Funtion:

$Q = AK^{\alpha}L^{\beta}$

$\frac{\partial Q}{\partial K} = \alpha\ A\ K^{\alpha-1}L^{\beta}$

$\frac{\partial Q}{\partial L} = \beta\ A\ K^{\alpha}L^{\beta-1}$

According to Euler's theorm:

$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K.\alpha\ A\ K^{\alpha-1}L^{\beta} + L.\ \beta\ A\ K^{\alpha}L^{\beta-1}$

$= (\alpha+\beta)\ (A\ K^{\alpha}L^{\beta})$

$= (\alpha+\beta).\ Q$

a. If $(\alpha+\beta) = 1$, then $K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = Q$. If the value of K and L doubled, i.e., 2K and 2L, then output also doubles; then there is a constant return to scale in the production.
b. If $(\alpha+\beta)$ is not equal to 1, then $K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = (\alpha+\beta).\ Q$. Doubling the value of K and L in the right hand side adds up to $2(\alpha+\beta)Q$. If $(\alpha+\beta) > 1$, the output more than doubles, i.e, there are increasing returns to scale. If $(\alpha+\beta) < 1$, the output is less than double, or the decreasing returns to scale in production.

Note: A proportional increase in all the values of inputs in a production function increases the scale of production. If there are constant returns to scale, then output will increase equi-proportionally to the increase in all inputs. If there are increasing returns to scale, an increase in all inputs will lead to a more than proportionate increase in output. If there are decreasing returns to scale, then output will increase less than proportionately with an increase in all inputs.

Example: Consider the following Cobb-Douglas production function, which is homogeneous of degree 1 in capital and labor $Q = 50K^{0.4}L^{0.6}$. The value of the output (Q) includes the payment made to the labor, i.e., the wages paid to the labor (wL), which is equal to $\frac{\partial Q}{\partial L}.L$ in a competitive labor market. Also, the value of the output includes the payment made to the capital suppliers (rK), which is equal to $\frac{\partial Q}{\partial K}.K$. Show that the sum of the total factor payments (wL + rK) equals the value of the output, i.e., wL + rK = Q, such that $wL + rK = \alpha\ Q + (1-\alpha)\ Q$, where $\alpha = 0.6$.

Solution: Given the production function

$Q= 50\ K^{0.4}\ L^{0.6}$

$\frac{\partial Q}{\partial K}= 0.4(50\ K^{0.4-1}\ L^{0.6})$

$\frac{\partial Q}{\partial L}= 0.6(50\ K^{0.4}\ L^{0.6-1})$

$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K.0.4(50\ K^{0.4-1}\ L^{0.6})+ L.0.6(50\ K^{0.4}\ L^{0.6-1})$

Given, $(Kr)= K\frac{\partial Q}{\partial K}$ and $Lw= L\frac{\partial Q}{\partial L}$

Therefore,

$rK+wl= K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K.0.4(50\ K^{0.4-1}\ L^{0.6})+ L.0.6(50\ K^{0.4}\ L^{0.6-1})$

$= 0.4\ Q + 0.6\ Q= (1-\alpha).Q + \alpha Q$

$= \alpha Q+ (1-\alpha)\ Q$ (given $\alpha=0.6$)

Hence proved.

Example: Given the following production function; find out the elasticity of substitution:

$z = A(aK^{-\rho} + bL^{-\rho})^{-m/\rho}$ (where A,b,and ρ constants) and $\rho \neq 0$ with $\rho> -1$.

Solution: partial differentiation the function $z = A(aK^{-\rho} + bL^{-\rho})^{-m/\rho}$ with respect to L and K respectively,

$z'_L= A(-m/\rho)\ (aK^{-\rho} + bL^{-\rho})^{(-m/\rho)-1}b(-\rho)L^{-\rho-1}$

$z'_K= A(-m/\rho)\ (a\ K^{-\rho} + bL^{-\rho})^{(-m/\rho)-1}a(-\rho)K^{-\rho-1}$

therefore,

$MRTS_{K,L} = R_{K,L}= \frac{z'_L}{z'_K}$

$= \frac{b}{a}\ \frac{L^{-\rho-1}}{K^{-\rho-1}}\ \frac{b}{a}\ (\frac{K}{L})^{\rho+1}$

$\frac{K}{L}= \left(\frac{a}{b}\right)^{1/(\rho+1)}\ (R_{K,L})^{1/(\rho+1)}$

Hence,

$\sigma_{K,L}$= $ElR_{k,L}(\frac{K}{L}) = \frac{1}{1+\rho}$.

Example: Without solving the equation, show that $2x^2+5xy+y^2=19$ defines an implicit function y(x) for which y(2)=1, and find dy/dx when x=2. Express the answer in geometrical terms.

Solution: Putting x=2 and y=1 in given function $2x^2+5xy+y^2=19$, we see that equation satisfied,

$$2(2)^2 + 5(2)(1)+ (1)^2 =19.$$

Using implicit differentation, we get

$$\frac{dy}{dx} = - \frac{4x+5y}{5x+2y} = - 13/12$$

When (x,y)=(2,1). In geometrical terms, this means that the slope of the contour $2x^2+5xy+y^2=c$ which passes through point (2,1) is -13/12 at that point.

Example: The function g is defined by

$$g(x, y) = f(x, y) - a\ln(x + y),$$

where a is a constant and f satisfies the condition

$$x f'_x(x,y) + y f'_y(x,y) = a \text{ for all } (x, y).$$

Show that g is homogeneous of degree 0.

Solution: Given $g(x, y) = f(x, y) - a\ln(x + y)$,

Therefore, according to Euler's theorm,

$$xg'_x(x,y) + yg'_y(x,y) = x(f'_x(x,y) - a/[x + y]) + y(f'_y(x,y) - a/[x + y])$$

$$= xf'_x(x,y) - ax/[x + y] + y f'_y(x,y) - ay/[x + y])$$

$$= xf'_x(x,y) + y f'_y(x,y) - ax/[x + y] - ay/[x + y]$$

Given, $x f'_x(x,y) + y f'_y(x,y) = a$, therefore the equation becomes…

$$xg'_x(x,y) + yg'_y(x,y) = a - a(x/[x + y] + y/[x + y])$$

$$= a – a(1)$$

$$= 0$$

for all (x,y). Thus by Euler's theorem g is homogeneous of degree 0.

Example: The twice-differentiable function $f(x, y)$ is homogeneous of degree k, and its second derivatives are continuous. Show that

$$x^2 f''_{11}(x, y) + 2xy f''_{12}(x, y) + y^2 f''_{22}(x, y) = k(k − 1) f(x, y) \text{ for all } (x, y).$$

Solution: We know that f is homogeneous of degree k which means that f'_1 and f'_2 are homogeneous of degree k − 1. Thus by Euler's theorem applied to f'_1 and to f'_2 we have,

$$x f''_{11}(x, y) + y f''_{12}(x, y) = (k − 1) f'_1(x, y) \text{ for all } (x, y)$$
$$x f''_{21}(x, y) + y f''_{22}(x, y) = (k − 1) f'_2(x, y) \text{ for all } (x, y).$$

Multiply first equation by x times and second equation by and y times.

$$x^2 f''_{11}(x, y) + x y f''_{12}(x, y) = (k − 1) x f'_1(x, y) \text{ for all } (x, y)$$
$$x y f''_{21}(x, y) + y^2 f''_{22}(x, y) = (k − 1) y f'_2(x, y) \text{ for all } (x, y).$$

Now the sum up the above two equations

$$x^2 f''_{11}(x, y) + 2xy f''_{12}(x, y) + y^2 f''_{22}(x, y) = (k − 1)[x f'_1(x, y) + y f'_2(x, y)] \text{ for all } (x, y)$$

(given that $f''_{12}(x, y) = f''_{21}(x, y)$, by Young's theorem).

Finally, the term in brackets on the right-hand side of this equation is equal to $k f(x,y)$ by Euler's theorem, yielding the required result.

Example: A firm uses two inputs to produce a single output. Its production function f is homogeneous of degree 1. An implication of the homogeneity of f, which you are not asked to prove, is that the partial derivatives f'_x and f'_y with respect to the two inputs are homogeneous of degree zero. Use Euler's theorem to find an expression for the cross partial derivative $f''_{xy}(x, y)$ in terms of x, y, and $f''_{xx}(x, y)$.

Solution: Given f'_x is homogeneous of degree 0, therefore according to Euler's theorem an expression for the cross partial derivative is :

$$x f''_{xx}(x, y) + y f''_{xy}(x, y) = 0,$$

so that $f''_{xy}(x, y) = -(x/y) f''_{xx}(x, y)$.

3. Exercise:

1. Given Q=440-8P +0.05 Y, where P=15 and Y=12,000. Find the income and price elasticity of demand.
2. Given Q_1= 110-P_1+0.75 P_2-0.25 P_3+0.0075 Y. At P_1=10, P_2=20, P_3=40, and Y=10,000, Q1=170. Find the different cross elasticities of demand.

3. Determine whether each function is homogeneous and, if so, of what degree.

$f(x,y) = \sqrt{x^2 + y}$

$f(x,y,w)= 3x^2y - \frac{3y}{w^2}$

4.Test the degree of homogeneity of a function given below:

a) $z = 10x + 5y$

b) $z = x^2 + 5xy + 12 y^2$

c) $z= x^{0.3} + y^{0.4}$

d) $z = 10 x^5 + 10x^2y^3 +y^5$

5. Assume the demand for sugar is a function of income (Y), the price of sugar (P_s) and the

price of saccharine (P_c), a sugar substitute, as follows:

$$Q_d = f(Y, P_c, P_s) = 0.05Y + 10P_c - 5P_s^2.$$

a. Find the partial derivatives of this demand function.

b. Find the elasticity of demand with respect to income $\left(\frac{\partial Q_d}{\partial Y} \cdot \frac{Y}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

c. Find the own-price elasticity of demand $\left(\frac{\partial Q_d}{\partial P_s} \cdot \frac{P_s}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

d. Find the cross-price elasticity of demand $\left(\frac{\partial Q_d}{\partial P_c} \cdot \frac{P_c}{Q_d}\right)$ when Y = 10,000, P_s = 5 and P_c = 7.

6. Consider the production function $y = f(x_1,x_2) = x_1 x_2$ defined over the domain $x_1 > 0$ and $x_2 > 0$. Also, consider the functions $g(y) = \ln(y)$ and $j(y) = y^2$.

a. Is $f(x_1,x_2)$ a homogeneous function? If so, of what degree?

b. Is g(y) a homothetic function? Is g(y) a homogeneous function in the arguments x_1 and x_2? If so, what is its degree?

c. Is j(y) a homothetic function? Is j(y) a homogeneous function in the arguments x_1 and x_2? If so, what is its degree?

7. Consider the production function $y= f(x_1,x_2)= x_1^{1/4}x_2^{1/3}$.

a. Determine whether the production function is homogeneous. If so, of what degree?

b. Find out the partial derivatives of the function and show that they are also homogeneous.

c. Show that $x_1f_1(sx_1, sx_2)+ x_2f_2(sx_1, sx_2) = ks^{k-1}f(x_1,x_2)$.

Solution:

1. Income elasticity of demand=0.652 and price elasticity of demand= -0.13.
2. e_{p1} = -0.0889; e_{p2} =0.133 and e_{p3} = -0.0889.

3.a. homogeneous of degree 1

b.not homogeneous

4. a. homogenous of degree 1

b. homogenous of degree 2

c. not homogenous

5. Answers:

a. $\frac{\partial Q_d}{\partial Y} = 0.05; \quad \frac{\partial Q_d}{\partial P_c} = 10; \quad \frac{\partial Q_d}{\partial P_s} = -10P_s$

b. 1.12

c. - 0.56

d. 0.16

5. Answers:

a. homogeneous of degree 2

b. homothetic; not homogeneous in x_1 and x_2

c. homothetic; homogeneous of degree 4 in x_1 and x_2

6. Answers:

d. homogeneous of degree 2

e. homothetic; not homogeneous in x_1 and x_2

f. homothetic; homogeneous of degree 4 in x_1 and x_2

7. Answers:

a. homogeneous of degree 7/12 (i.e., k=1/12)

b. $f_1(x_1,x_2) = \frac{\partial y}{\partial x1} = \frac{1}{4} x_1^{-3/4} x_2^{1/3}$;$f_1(x_1,x_2)$ is homogeneous of degree k-1, i.e., -5/12; and take the partial derivative of the production function with respect to x_2 and show that $f_2(x_1,x_2)$ is homogeneous of degree -5/12.

c. Apply Euler's Theorem.

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