

## **MATRICES AND MATRIX OPERATIONS**

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## 1.0 Learning Outcomes

In the present chapter you will learn about the following aspects;

- ❖ Understand the concept of matrices
- ❖ You will be able to apply matrix operations.
- ❖ Understand the properties of matrix.
- ❖ Rules of matrix multiplication
- ❖ Understand the concept of power of matrix
- ❖ Understand the concept of invertible matrix

## 1.1 Introduction

The subject of matrices had its origin in various types of problems. Of these, solutions of a given system of equations and linear transformations in geometry are extremely interesting. In 1857, the British mathematician Arthur Cayley formulated the general theory of matrices. He developed the properties of matrices as pure algebraic structure, though matrices as arrays of coefficients in homogeneous linear equation were recognized long before. A matrix is a very useful tool to analyse various problems in different subjects.

### 1.1.1 Matrices

A matrix is an ordered set of numbers listed in rectangular form; i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

OR

A matrix is simply a rectangular table of numbers written in either ( ) or [ ] brackets. These symbols are also called the notation of matrix. Matrix has many applications in science, engineering, computing and Economics. In Economics, matrices are useful to study of stock market trends, optimization of profit, minimization of loss, input - output analysis etc. Do not confuse a matrix with determinants which use vertical bars, i.e.  $\begin{vmatrix} \end{vmatrix}$ . Basically, matrix is a simple

pattern of numbers on the other hand determinant gives us a single number. The size of matrix is written  $a_{ij}$ , where,  $i$  =row and  $j$  = columns.  $a_{ij}$  is the element of a matrix.

For examples

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow 2 \times 2 \text{ Matrix} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \end{bmatrix} \Rightarrow 4 \times 1 \text{ Matrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} 3 \times 3 \text{ matrix}$$

### 1.1.2 Types of matrix

➤ **Square Matrix:** If a matrix has 'n' rows and 'n' columns then we say it is a square matrix. For example;

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2} \quad \text{or} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

➤ **Diagonal Matrix:** It is a square matrix where all non-diagonal element is zero such that

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{13} \end{bmatrix}_{3 \times 3}$$

➤ **Row Matrix:** A matrix with one row is called a row matrix.

$$A = [a_{11}, a_{12}, a_{13}]_{1 \times 3}$$

➤ **Column Matrix:** A matrix with one column is called a column matrix.

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}_{3 \times 1}$$

➤ **Zero Matrix:** If the all elements of an matrix is zero then it is called zero matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

➤ **Opposite Matrix:** If the all elements of an matrix multiply by negative sign then we get opposite matrix

$$a_{ij} = - a_{ij}$$

➤ **Transpose Matrix:** If we convert row to column and column to row of an matrix then we get transpose of an matrix. For example;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

Then;

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}_{3 \times 3}$$

It is also true for  $m \times n$  matrix

We can also write;

$$A^T = A' = (a'_{ij}), \text{ where } a'_{ij} = - a_{ij}$$

**Example:** Given;

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \text{ find } A^T$$

**Solution:**

$$A^T = A' = \begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}_{3 \times 2}$$

**Rules of Transposition**

- $(A^1)^1 = A \dots \dots \dots (i)$
- $(A + B)^1 = A^1 + B^1 \dots \dots \dots (ii)$
- $(\alpha A^1) = \alpha A^1$  and  $(\alpha B)^1 = \alpha \beta^1 \dots \dots \dots (iii)$  ( $\alpha$  is constant)
- $(AB)^1 = B^1 A^1 \dots \dots \dots (iv)$

➤ **Symmetric Matrix:** A square matrix is said to be symmetric matrix, it is equal to its transpose matrix.

$$\begin{aligned} & \text{i.e. } a_{ij} = a_{ji} \quad \forall i \ \& \ j \\ & \text{or } A = A' = A^T \quad \{A = (a_{ij})_{n \times n}\} \end{aligned}$$

➤ **Skew - symmetric Matrix:** It is defined as;

$$\begin{aligned} & a_{ij} = -a_{ji} \\ & \text{or } A = -A' = -A^T \quad \{A = (a_{ij})_{n \times n} \ \& \ A = (a_{ji})_{n \times n}\} \end{aligned}$$

➤ **Identity Matrix:** An identity matrix (I) is an diagonal matrix with all diagonal elements is equal to one. It is also known as unit matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \quad I_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots 1 \end{bmatrix}_{n \times n}$$

Or  $A I_n = I_n A = A$

**Example:** Construct the  $3 \times 3$  matrix  $A = (a_{ij})_{3 \times 3}$  with  $a_{ij} = 2i - j$

**Solution:** Total element of matrix =  $3 \times 3 = 9$

Given;  $a_{ij} = 2i - j$

$$\begin{aligned} \therefore a_{11} &= 2 \times 1 - 1 = 1, & a_{12} &= 2 - 2 = 0, & a_{13} &= 2 - 3 = -1 \\ a_{21} &= 4 - 1 = 3, & a_{22} &= 4 - 2 = 2, & a_{23} &= 4 - 3 = 1 \\ a_{31} &= 6 - 1 = 5, & a_{32} &= 6 - 2 = 4, & a_{33} &= 6 - 3 = 3 \end{aligned}$$

Then Matrix;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

Putting the values;

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}_{3 \times 3}$$

## 2.1 Operations of Matrix

**Equality Matrix:** Suppose,  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{m \times n}$  be the two  $m \times n$  matrices.

Then A and B is said to be equal matrices if  $A = B$

$$a_{ij} = b_{ij} \forall i = 1, 2, \dots, m \text{ \& } j = 1, 2, \dots, n$$

Thus, if both matrices have same dimension then they called equal. Otherwise they called unequal matrix such that  $A \neq B$ .

**Example:** Given;

$$\begin{pmatrix} 3 & y-1 \\ 2 & x \end{pmatrix} = \begin{pmatrix} z-2 & 2 \\ 4 & 3 \end{pmatrix}$$

Solve for x, y, and z.

**Solution:** By equation of matrices

$$\begin{bmatrix} 3 & y-1 \\ 2 & x \end{bmatrix} = \begin{bmatrix} z-2 & 2 \\ 4 & 3 \end{bmatrix}$$

$$\therefore Z - 2 = 3 \text{ or } Z = 5, \quad y - 1 = 2, \quad y = 3 \text{ and } x = 3$$

### 1.2.1 Addition and Multiplication by a Scalar

Let  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})$  be the two matrices then the sum of A and B matrices is defined as;

$$A + B = (a_{ij})_{m \times n} + (b_{ij})_{m \times n}$$

$$A + B = (a_{ij} + b_{ij})_{m \times n}$$

If  $\alpha$  is a real number, then

$$\alpha A = \alpha (a_{ij})_{m \times n} = (\alpha a_{ij})_{m \times n}$$

$$\text{and } \alpha B = \alpha (a_{ij})_{m \times n} = (\alpha b_{ij})_{m \times n}$$

$$\therefore \alpha A + \alpha B = (\alpha a_{ij} + \alpha b_{ij})_{m \times n}$$

**Example :** Given,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Compute  $A + B$ , and  $2A + \frac{1}{2}B$

**Solution:**

$$\begin{aligned}A + B &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+1 & 2+0 & 3+2 \\ 4+0 & 2+0 & -3+1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 & 5 \\ 4 & 4 & -2 \end{pmatrix}\end{aligned}$$

And

$$\begin{aligned}2A + \frac{1}{2}B &= 2 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & -3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 & 6 \\ 8 & 4 & -6 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 2+\frac{1}{2} & 4+0 & 6+1 \\ 8+0 & 4+1 & -6+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2\frac{1}{2} & 4 & 7 \\ 8 & 5 & 5\frac{1}{2} \end{pmatrix}\end{aligned}$$

### 1.2.2 Rules of Matrix Addition and Multiplication by Scalars

If A, B and C are  $m \times n$  matrix and  $\alpha$  and  $\beta$  are scalar then;

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $A + 0 = A$
- $A + (-A) = 0$
- $(\alpha + \beta) A = \alpha A + \beta A$
- $\alpha (A + B) = \alpha A + \alpha B$

### 1.3 Matrix Multiplication:

Let us consider two matrices A and B, such that



## Mathematical Methods for Economics: Matrices and Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3 \times 3}$$

Then the product of A and B is denoted by AB and it is given by;

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}_{3 \times 3}$$

**Example:** Given;

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Compute AB and BA

**Solution:**

$$AB = \begin{bmatrix} 2 \times 4 & + & 3 \times 2 \\ 2 \times 4 & + & 0 \times 2 \\ 1 \times 4 & + & 2 \times 2 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 8 \end{bmatrix}$$

$$BA = [4 \times 2 + 2 \times 3 \quad 4 \times 2 + 2 \times 0 \quad 4 \times 1 + 2 \times 2] = (14 \quad 8 \quad 8)$$

**Problem :** Given that  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{n \times p}$  then compute the product of both matrix

$$\text{i.e. } C = AB$$

**Solution:** The product of two matrix is given by;

$$C = AB$$

$$\text{or} \quad (c_{ij})_{m \times p} = (a_{ij})_{m \times n} (b_{ij})_{n \times p}$$

$$or \quad C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

{Product of  $i^{th}$  row of A and  $j^{th}$  column of B}

In general;

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1j} & \dots & C_{1p} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ C_{i1} & C_{i2} & \dots & C_{ij} & \dots & C_{ip} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ C_{m2} & C_{m2} & \dots & C_{mj} & \dots & C_{mp} \end{bmatrix}_{m \times p} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1n} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ a_{i1} & a_{i2} & \dots & a_{ik} & \dots & a_{in} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mk} & \dots & a_{mn} \end{bmatrix}_{m \times n} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1p} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kp} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ b_{n1} & b_{n2} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix}_{n \times p}$$

### 1.3.1 Properties or Rules of Matrix Multiplication

- Matrix Multiplication is not commutative:

If A and B are two matrix, then

$$AB \neq BA$$

For example, let  $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and, if AB and BA are both defined, then,

$$\begin{aligned}
 AB &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \times 0 + 0 \times 1 & 1 \times 1 + 0 \times 0 \\ 0 \times 0 + (-1) \times 1 & 0 \times 1 + (-1) \times 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
 \end{aligned}$$

And

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 \times 1 + 1 \times 0 & 0 \times 0 + 1 \times (-1) \\ 1 \times 1 + 0 \times 0 & 1 \times 0 + 0 \times (-1) \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

So, we can say;  $AB \neq BA$

Hence commutative law does not hold

*Note: Sometime it may hold but it does not always hold*

- Matrix Multiplication is Associative;

If A, B and C are three  $m \times n$ ,  $n \times p$  and  $p \times q$  matrices respectively

Then,  $(AB)C = A(BC)$

- Matrix Multiplication is distributive with respect to addition of matrices :

Suppose A, B, C are three  $m \times n$ ,  $n \times p$  and  $p \times q$  matrices respectively, then

$$A(B + C) = AB + AC$$

- Matrix Multiplication by a unit Matrix:

If a is square matrix of order  $n \times n$  and I is the unit matrix of the same order, then

$$AI = A = IA \text{ \{Also, } II = I\}$$

- If the product of two matrices is a zero matrix, then it is possible that none of them is a zero matrix, i.e.

$$AB = 0, \text{ then } A \neq 0 \text{ and } B \neq 0$$

Where RHS '0' is zero matrix

Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 \times 2 + (-1) \times 2 & 1 \times 2 + (-1) \times 2 \\ -1 \times 2 + 1 \times 2 & -1 \times 2 + 1 \times 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the product is zero matrix.

- Cancellation law does not hold in matrix multiplication:

Suppose A, B, C are three matrices such that AB and AC are defined, then  $AB = AC$  does not imply  $B = C$

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}$$

Then find AB,  $A^2$ ,  $B^2$ , and  $(A + B)^2$ . Is  $(A + B)^2 = A^2 + B^2 + 2AB$  ?

**Solution:** We have given

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix}$$

$$\therefore A + B = \begin{bmatrix} 3+2 & 4+6 \\ 2+1 & 7+5 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 3 & 12 \end{bmatrix}$$

$$\text{and } AB = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 \times 2 + 4 \times 1 & 3 \times 6 + 4 \times 5 \\ 2 \times 2 + 7 \times 1 & 2 \times 6 + 7 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 38 \\ 11 & 47 \end{bmatrix}$$

Now

$$A^2 = \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 9+8 & 12+28 \\ 6+14 & 8+49 \end{bmatrix}$$
$$= \begin{bmatrix} 17 & 40 \\ 20 & 57 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 4+6 & 12+30 \\ 2+5 & 6+25 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & 42 \\ 7 & 31 \end{bmatrix}$$

$$\therefore A^2 + B^2 + 2AB = \begin{bmatrix} 17 & 40 \\ 20 & 57 \end{bmatrix} + \begin{bmatrix} 10 & 42 \\ 7 & 31 \end{bmatrix} + 2 \begin{bmatrix} 10 & 38 \\ 11 & 47 \end{bmatrix}$$
$$= \begin{bmatrix} 17 & 40 \\ 20 & 57 \end{bmatrix} + \begin{bmatrix} 10 & 42 \\ 7 & 31 \end{bmatrix} + \begin{bmatrix} 20 & 76 \\ 22 & 94 \end{bmatrix}$$
$$= \begin{bmatrix} 17+10+20 & 40+42+76 \\ 20+7+22 & 57+31+94 \end{bmatrix}$$
$$= \begin{bmatrix} 47 & 158 \\ 49 & 182 \end{bmatrix} \quad \text{----- (1)}$$

Also,

$$(A+B)^2 = \begin{bmatrix} 5 & 10 \\ 3 & 12 \end{bmatrix} \begin{bmatrix} 5 & 10 \\ 3 & 12 \end{bmatrix} = \begin{bmatrix} 5 \times 5 + 10 \times 3 & 5 \times 10 + 10 \times 12 \\ 3 \times 5 + 12 \times 3 & 3 \times 10 + 12 \times 12 \end{bmatrix}$$
$$= \begin{bmatrix} 55 & 170 \\ 51 & 174 \end{bmatrix} \text{----- (ii)}$$

from (i) and (ii)

$$(A + B)^2 \neq A^2 + B^2 + 2AB$$

#### 1.4 Power of Matrices

Suppose A is a square matrix, then the power matrix is defined as;  $A^2 = AA$

$A^3 = AAA$  and so on-----

In general  $A^n = AAA..... A$  {Here, A is repeated n times}

**Example:** Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  then prove that;

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

**Solution:** Given

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A.A = \begin{bmatrix} 1 & 1+1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2 \\ 0 & 1 \end{bmatrix}$$

$$\text{Then } A^k = A^{k-1} . A = \begin{bmatrix} 1 & 1+(k-1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \text{ Proved}$$

**Idempotent Matrix:** Let A be an square matrix then the product A by itself is called Idempotent matrix. It is defined as;

$$AA = A, AAA = A^3 = A$$

$$\text{In General } A^n = A$$

**Orthogonal Matrix:** Let A is the  $n \times n$  square matrix then A is said to be orthogonal matrix if,

$$\text{or } (A)_{n \times n} (A')_{n \times n} = I_{n \times n}$$

**Example:** Given;  $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}_{2 \times 2}$

Prove that A is orthogonal iff  $a^2 + b^2 = 1$

**Solution:** Given,

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}_{2 \times 2}$$

$$\text{Than } A' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}_{2 \times 2}$$

Now by property of orthogonal matrix,

$$AA' = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & ab - ba \\ ba - ab & b^2 + a^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & ab - ba \\ ba - ab & a^2 + b^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$$

Given,  $a^2 + b^2 = 1$  then

$$AA' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad \text{Hence Proved}$$

### 1.5 Systems of Equations in Matrix form

It is defined as;

$$2x + 3y = 4 \quad \text{----- (1)}$$

$$6x - y = 2 \quad \text{----- (1)}$$

Now, these equation can be written as;

$$A = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{then;}$$

$$AX = \begin{bmatrix} 2 & 3 \\ 6 & -1 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 3y \\ 6x - y \end{bmatrix}$$

It is equivalent to the matrix equation

$$AX = b$$

**Example:** If  $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$  then prove by mathematical induction

$$A^n = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix}$$

**Solution:** Let  $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 1 & -3 \end{bmatrix}$$



$$= \begin{bmatrix} 1 + 2 \times 2 & -4 \times 2 \\ 1 \times 2 & 1 - 2 \times 2 \end{bmatrix}$$

$$A^3 = A.A = \begin{bmatrix} 1 + 2n & -4n \\ n & 1 - 2n \end{bmatrix} \text{ proved}$$

**Example: (i)** A matrix P is orthogonal if  $P'P = 1$ . Prove that if P is an  $n \times n$  matrix whose columns are all of length 1 and mutually orthogonal then P is orthogonal .

**(ii)** Find out if A is an orthogonal matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$$

**Solution: (i)** Let  $P = (P_1 \ P_2 \ \dots \ P_n)$  and  $P_i' P_i = 1$

$$P'P = \begin{bmatrix} P_1' & P_1'P_2 & \dots & P_1'P_n \\ P_2'P_1 & P_2'P_2 & \dots & P_2'P_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ P_n'P_1 & P_n'P_2 & \dots & P_n'P_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

$\therefore$  P is orthogonal

**(ii)** No, A is not orthogonal matrix because columns of A are not of length 1 then;

$$A'A \neq I_3$$

**Example: (i)** Let D be the  $3 \times 3$  diagonal matrix with entries  $d_1, d_2$  and along  $d_3$  along the diagonal and zero's elsewhere. Let  $A = (a_{ij})$  be an arbitrary  $3 \times 3$  matrix. Compute AD and DA.

Show that  $AD$  multiplies the  $i^{\text{th}}$  column of  $A$  by entry  $d_i$  while  $DA$  multiplies the  $i^{\text{th}}$  row of  $A$  by entry  $d_i$ .

(ii) If  $D$  is the  $3 \times 3$  diagonal matrix with entries  $d_1 = 2, d_2$  and  $d_3 = 4$ , find the  $A$  such that  $AD = DA$

**Solution:** (i) Given  $D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ ,  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$AD = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \end{bmatrix}_{3 \times 3} \quad i^{\text{th}} \text{ column multiply by } d_i \text{ ----- (1)}$$

$$DA = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \end{bmatrix}_{3 \times 3} \quad i^{\text{th}} \text{ row multiply by } d_i \text{ ----- (2)}$$

By (1) & (2)

$$AD = DA$$

(ii) Given;  $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}_{3 \times 3}$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

**Example:** For what value of  $\beta$ ,  $D = \begin{bmatrix} \beta & \beta^2 - 1 \\ 2 & \beta + 1 \end{bmatrix}$  is symmetric?

**Solution:** Given,

$$A = \begin{bmatrix} \beta & \beta^2 - 1 \\ 2 & \beta + 1 \end{bmatrix}$$

A is symmetric matrix then,

$$A = A^T = A^1$$

$$\therefore \begin{bmatrix} \beta & \beta^2 - 1 \\ 2 & \beta + 1 \end{bmatrix} = \begin{bmatrix} \beta & 2 \\ \beta^2 - 1 & \beta + 1 \end{bmatrix}$$

By equating the matrices,

$$\beta^2 - 1 = 2$$

$$\text{or } \beta^2 = 3$$

$$\text{or } \beta = \pm\sqrt{3}, \text{ the matrix is symmetric}$$

## 1.6 **Invertible matrix**

A square matrix A is called invertible matrix, if

$$AB = BA = I_n$$

$$\text{or } AB = I$$

$$\text{and } BA = I \text{ \{Hence B is inverse of matrix A\}}$$

**Problem Set**

(1) Given,

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}, \text{ find } x \text{ and } y$$

(2) Let  $B = \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}$ , then prove  $B^n = \begin{bmatrix} 1 & nq \\ 0 & 1 \end{bmatrix}$

(3) If  $A = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}$ , find  $A^2 - 5A + 7I$

(4) If  $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 5 & 1 \end{bmatrix}$  then prove that  $AB \neq BA$

(5) Let  $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ ,  $B' = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ , prove that  $(AB)' = B'A'$

(6) Given,

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \text{ then prove } A \text{ is idempotent matrix}$$

(7) For  $\mu = \frac{1}{\sqrt{2}}$  the following given matrix is orthogonal

$$A = \begin{bmatrix} \mu & 0 & \mu \\ \mu & 0 & -\mu \\ 0 & 1 & 0 \end{bmatrix}$$

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- (8) Show that  $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ , is symmetric matrix
- (9) Give an example of three matrices X, Y and Z to show that if  $XY = ZX$ , then it does not necessary that  $y = z$
- (10) If  $\alpha$  is a scalar, A and B are matrices of order  $3 \times 4$ , then show that  $\alpha(A + B) = \alpha A + \alpha B$

**Answers of the Problem Set**

(1)  $x = 1$  &  $y = 2$

(3)  $\begin{bmatrix} 11 & -6 \\ -15 & 23 \end{bmatrix}$

(9)  $X = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ ,  $y = \begin{bmatrix} -2 & 6 \\ 1 & -10 \\ 4 & 2 \end{bmatrix}$ ,  $z = \begin{bmatrix} 4 & 2 \\ -4 & -6 \\ 3 & 2 \end{bmatrix}$

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