Limit and Continuity

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Content:

1. Learning Outcome:

After reading this chapter you will be able to know the concept of limit. Limits of a rational function, asymptote. In addition to limit the concept of continuity and intermediate value theorem is explain in detail.

2. Limit:

Observe the function given below

$$f(x) = \frac{x^3 - 1}{x - 1}$$

The function is not defined for x=1, since the result is % which makes no sense. However we try to see what happens to f(x) when x is slightly below or above 1. Take a calculator and try to find out the values f(x), when x taking values which are slightly move than 1 and slightly less then 1. Some of the values are given below in table 1.

X	.5	.6	.9	.99	.9999	1	1.0009	1.009	1.09
Y	1.75	1.96	2.71	2.970	2.9997	.	3.00270087	3.02708	3.27

As x approaches 1, f(x) takes values which are closer and closer to 3. So, we can say that f(x) tends to 3 as x tends to 1. This is written as;

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

Given the above example the idea of limit should be closed intuitively. What we are looking at is what happens to the value of the function when the independent variable x approaches a particular value.

We can make a formal statement like this

Suppose y = f(x)

Defined on the interval (a, b)

$$\lim_{x \to v} f(x) = L$$

Now x can approach v either from the right hand side (i.e. x takes values which are greater than g) or from the left hand side (i.e. x tends to a taking values which are less than v) when x approaches v from the left hand side we say L is the left hand limit of f(x)

$$\lim_{x \to v^-} f(x) = L$$

V^- means x approaches V from values which are smaller them V.

Similarly

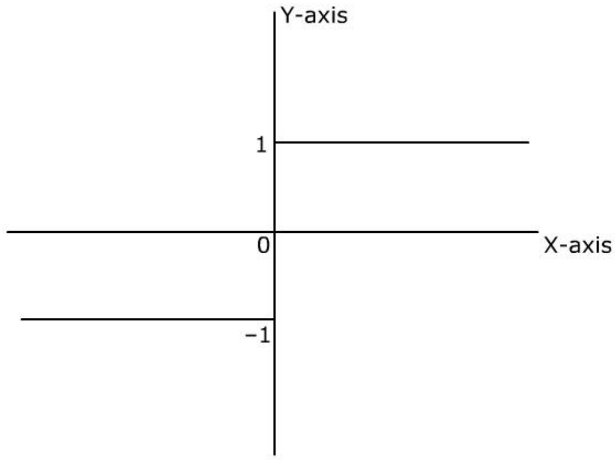

Fig. 1

$$\lim_{x \to V^+} f(x) = L$$

This is the right hand side limit of f(x)

The limit of a function exists if R.H.S limit = L.H.S limit

Look at the function $f(x) = \frac{|x|}{x}$

x	f(x)
–1	–1
–5	–1
–2.5	–1
0	
–25	1

In this case we make a table like in the previous example what do we find? As x approaches o from the left hand side the value of the function gets closer and closes to -1. On the other hand when x approaches 0 from the right hand side f(x) approaches 1. In this case the right hand limit and the left hand limit are not equal. The limit of a function exists if and only if both the L.H.S. limit and RHS limit exist and are equal. So we can say that the limit of. This can be verified from the figure and table.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist}$$

Take another example

$$f(x) = \frac{1}{x}$$

What is limit of the function when x tends to 0?

In this case when x tends to 0 from the right hand side the value of the function increases and when x is very close to zero the value of the function approaches ∞. On the other hand when x approaches 0 from the left hand side the value of the function gets closer and closes to $-\infty$.

This can be observed by glancing at the figure

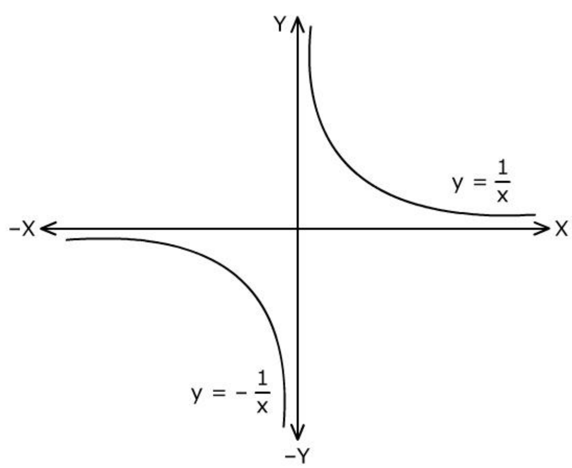

Fig. 2

From the above discussion we can make a statement about the concept of limit.

Suppose f is defined on an interval (a, b) except possibly at a point C $\in$ (a, b). Then $f(x) \to L$ if and only if $f(x) \to L$ as $x \to c^+$

Now we are in a position to give a formal definition of limit of a function.

Definition: Let f be a function defined over an interval containing a, except possibly at a, and L be a number.

Limit of f(x) as x approaches a, is L, written as

$$\lim_{x \to a} f(x) = L$$

If for any $\in$ > 0, however small there exists a $\delta > 0$ such that

$$|f(x) - L| < \in$$

Whenever $0 < |x - a| < \delta$

What does this formal definition of limit means?

It means that x tends to 'a'

Then limit of f(x) is L

If for every neighborhood of L that can be chosen as small, there can be found corresponding neighborhood of a (excluding the point x=a)in the domain of the function such that, for every value of x in the a-neighborhood, its image lies in the chosen L-neighborhood.

This can be explained with the following example.

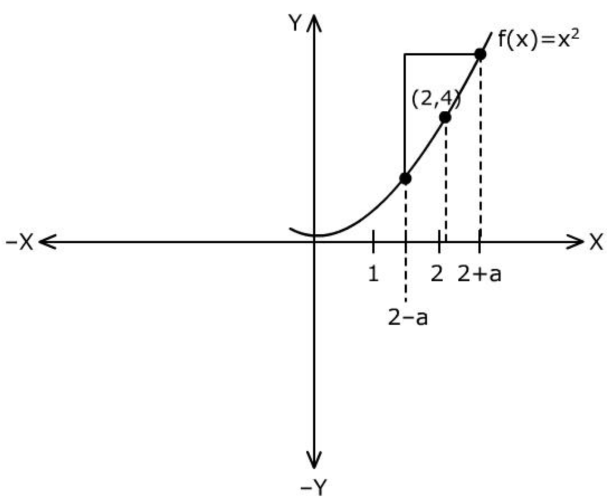

Fig. 3

Let $f(x) = x^2$

Now that

$$\lim_{x \to 2} x^2 = 4$$

Select a small neighbor of L (Here L=4) so the neighborhood of 4 (4-∈, 4+∈). Now we construct a neighborhood of 2, say $(2-\delta, 2+\delta)$ such that the two neighborhoods define a rectangle (see the diagram). With two of its converse lying on the curve. It can be seen that for every value of x lying in the neighborhood of 2, the corresponding value of the f(x) lies in the neighborhood of 4. Thus 4 fulfills the definition of limit.

Example:

$$f(x) = x^2 \qquad x \to 2$$

$$\lim_{x \to 2} f(x) = 4$$

We can make x closes and closes to 2 from both sides (left hand side and write hand side)

$$x \to 2^- \to x = 1.8,\ 1.9,\ 1.92,\ 1.98,\ 1.99..$$

$$x \to 2^+ \to x = 1.5,\ 2.25,\ 2.10,\ 2.05,\ 2.005..$$

As we put this values in the function the value of function gets closes used closes to 4.

This is happen even thought the function may not be defined for when x= 2.

In order to prove that the limit of the function is 4. We use the formal definition of limit.

We must show that given any ∈>0 we can find $\delta > 0$ such that

$$\left| x^2 - 4 \right| < \in \quad when \quad 0 < \left| x - 2 \right| < \delta$$

Choose $\delta \le 1$ so that $0 < |x-2| < 1$

$\Rightarrow \qquad -1 < (x-2) < 1$

$\Rightarrow \qquad 1 < x < 3 \qquad x \neq 2$

Thus $|x^2 - 4| = |(x-2)(x+2)| = |x-2|\,|x+2| < \delta |x+2|$

So $\qquad \delta\, |x + 2| < 5\,\delta$

Take δ as 1 or $\in$/5 whichever is smaller. Then we have $|x^2 - 4| < \in$ whenever $0 < |x - 2| < \delta$ and required result L is proved.

Theorems of limit.

If $\lim_{x \to a} f(x) = h_1 \quad and \quad \lim_{x \to a} g(x) = h_2$

(i) $\lim_{x \to a} [f(x) + g(x)] = h_1 + h_2$

(ii) $\lim_{x \to a} [f(x) - g(x)] = h_1 - h_2$

(iii) $\lim_{x \to a} [f(x)\, g(x)] = h_1\, h_2$

(iv) $\lim_{x \to a} \left[f(x) / g(x) \right] = h_1 / h_2$, provided L_2 is not 0 ($L_2 \neq 0$)

Limit of a constant is constant itself.

Suppose $f(x) = a$ where a is constant

Them $\lim_{x \to B} f(x) = a$

Theorem:

For any polynomial

$$f(x) = a_0 + a_1 x + a_2\, x^2 + \ldots\ldots + a\, x^n$$

And for any real number a

$$\lim_{x \to 0} f(x) = a_0 + a_1\, a + a_2\, a^2 + \ldots\ldots a_n\, a^n$$

$$= f(a)$$

This means that limit of a polynomial f(x) at x=a is the same as the value of the polynomial at x= a. In the case of polynomial, to find out the limit at x = a we just are required to evaluate the polynomial at x = a.

3. Limit of a Rational function:

A rational function is the ratio of two polynomials. The above mentioned theorem can be used for computing the limits of rational functions.

Example

$$f(x) = \frac{x^4 + 4}{x - 3}$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^4 + 4}{x - 3} = \frac{\lim_{x\to 2} (x^4 + 4)}{\lim_{x\to 2} (x - 3)}$$

$$= \frac{20}{-1} = -20$$

We can say if

$$f(x) = \frac{h(x)}{g(x)}$$

Where h(x) and g(x) are polynomial

For any real number a

(i) $if \; g(a) \neq 0 \; the \; \lim_{x\to a} f(x) = f(a)$

(ii) $if \; g(a) = 0 \; but \; h(a) \neq 0, \; \lim_{x\to a} f(x) \; does \; not \; exist$

There is a useful principle for polynomial which in simple words states that.

'The end behavior of a polynomial matches the end behavior of its highest degree term'.

$$\lim_{x\to +\infty} \left(a_0 + a_1 x + a_2 x^2 + ...a_n x^n\right) = \lim_{x\to +\infty} a_n x^n$$

$$\lim_{x \to -\infty} \left(a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n \right) = \lim_{x \to -\infty} a_n x^n$$

This can be seen by looking at the following:

$$f(x) = \left(a_0 + a_1 x + a_2 x^2 + \ldots\ldots a_n x^n \right)$$

Factoring one the highest power of x from the polynomial

$$f(x) = x^n \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + \ldots\ldots + a_n \right)$$

Now as x approach ∞ or –∞ all the terms with positive powers of x in the denominator tend to '0'. So, the above mentioned principles are valid.

4. Asymptote

Definition: a line x = a is called a vertical asymptote of the graph of the function if f(x) → + ∞ or f(x) → –∞ as x approaches a from left or right.

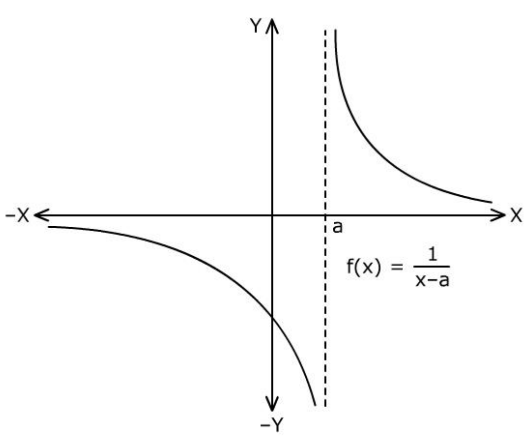

Fig. 4(a)

(a)

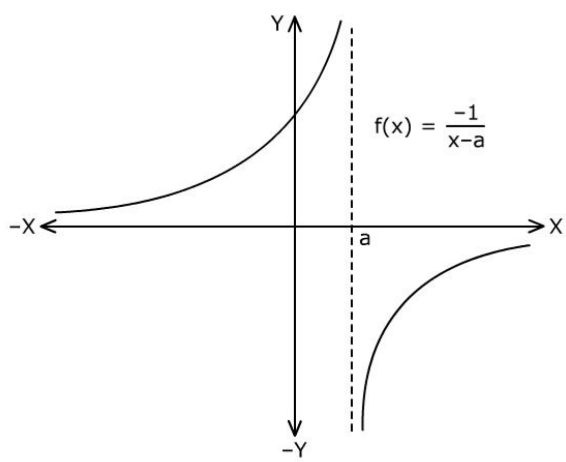

Fig. 4(b)

(b)

$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty \qquad \lim_{x \to a-} \frac{-1}{x-a} = -\infty$$

In the case of the above two functions the vertical asymptote is the line x =a

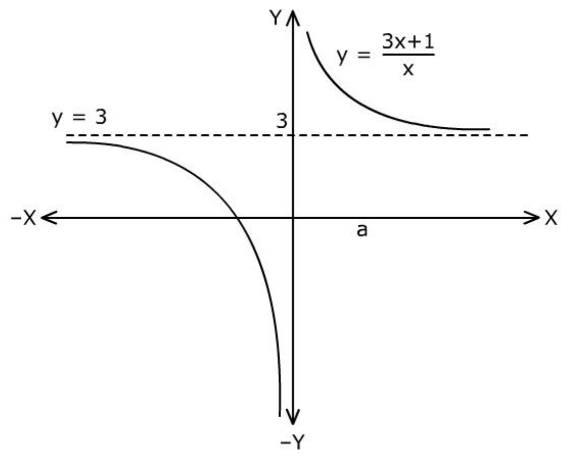

Fig. 4(c)

Look at the function $y = \dfrac{3x+1}{x}$

$$f(x) = 3 + \frac{1}{x}$$

$$\lim_{x\to+\infty} f(x) = 3 \qquad \lim_{x\to-\infty} f(x) = 3$$

As x tends to ± ∞ the graph of the function y = f(x) gets closer and closer to line y = 3. Same thing happens when the x tends to–∞. In either case we call the line y = L, a horizontal asymptote of the graph of the function f.

We can define:

A line y= L is called a horizontal asymptote of the graph of the function f if

$$\lim_{x\to+\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Very often when the limit of the function does not exist we may be interest in finding how the function f(x) behavior when x tends to (∞) and x tends to 0 (zero) or x tends to a value N.

Vertical asymptote is a vertical line x = c to which the graph of the function gets closer and closer as x approaches c from the write or from the left. We are able to get the vertical asymptote by setting the denominator of the function equal to zero.

Horizontal Asymptote: It is a line y= d to which the graph gets closer and closer as x→+∞ or x→-∞

Example: $\left(4x-5\right)\big/\left(3x+2\right)$

Vertical Asymptote

$$3x + 2 = 0 \qquad\qquad \therefore\ x = \frac{-2}{3}$$

This only vertical asymptote is $x = \frac{-2}{3}$. As x approaches $\frac{-2}{3}$ from the left or right f(x) approach the vertical line $x = \frac{-2}{3}$,

But, the Horizontal asymptote $\lim\limits_{x\to\infty} \frac{4x-5}{3x+2}$

$$\lim_{x\to\infty} \frac{4 - 5/x}{3 + 2/x} = 4/3$$

$y = 4/3$ is the horizontal Asymptote.

Example: $\dfrac{\left(2x+3\right)}{\sqrt{x^2-2x-3}}$

Vertical Asymptote

$$x^2 - 2x - 3 = 0$$

$$\left(x-3\right)\left(x+1\right) = 0$$

Hence the denominator is zero when x=3 or x=-1

So these lines are vertical asymptotes. The numerator is not zero when x = 3 or x = -1

Horizontal Asymptote

$$\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x\to\infty} \frac{2+3/x}{\sqrt{1-2/x-3/x}}$$

$$= \frac{2}{1} = 2$$

Note $\quad x = -\sqrt{x^2}$ when x<0

$$\lim_{x\to-\infty} \frac{2x+3}{\sqrt{x^2}-2x-3}$$

$$= \lim_{x\to-\infty} \frac{2+3/x}{-\sqrt{\left(x^2-2x\right)-3/x^2}}$$

$$= \lim_{x\to-\infty} \frac{2+3/x}{-\sqrt{1-2/x}-3/x^2} = \frac{2-0}{-\sqrt{1-0-0}} = -2$$

Hence y = 2 and y = –2 are horizontal asymptote.

5. Continuity

A function can be regarded as continuous if its graph can be drawn without lifting pencil from the paper.

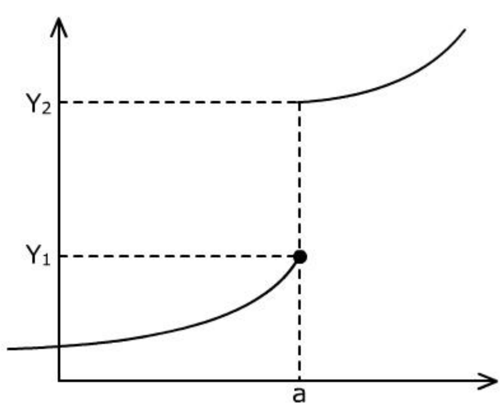

Fig. 5(a)

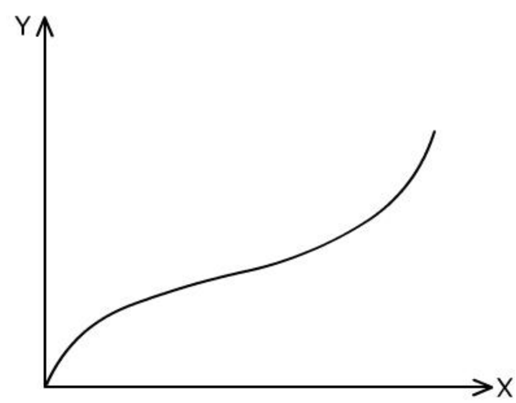

Fig. 5(b)

The graph is unbroken

In the first figure when x takes a value slightly greater them a the value of the f(x) jumps up from y_1 to y_2. The function is not continuous at x= a if the function is continuous at a point x, there will be small changes in the value of f(x) for small change in the value of x.

<u>Definition</u> :

A function f is continuous at x = a provided the following conditions are satisfied

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists.

(iii) $\lim_{x \to a} f(x) = f(a)$

If one or more of these conditions fail to hold then the functions is discontinuous at x= a

The function drawn in Fig (1a) is discontinuous at x=a. The limit does not exist.

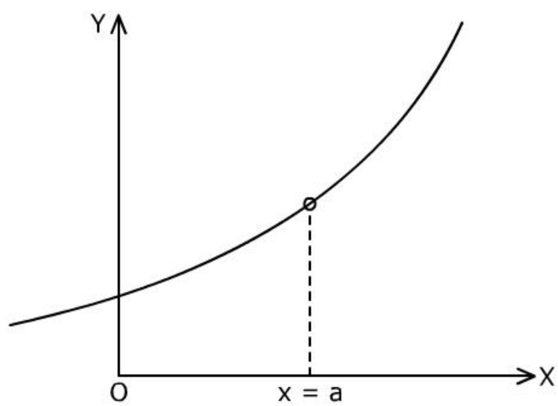

Fig. 6(a)

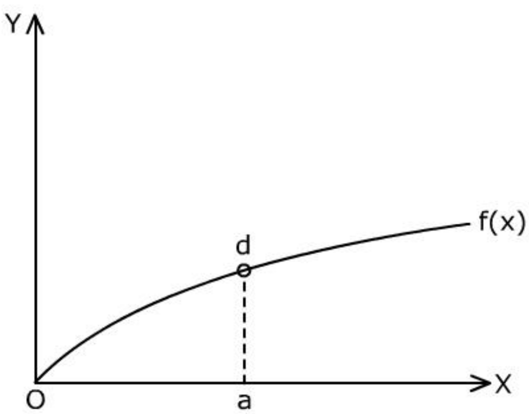

Fig.6(b)

In Fig. 2a the function is not defined at x= a. In Fig 2b the f(a) = d the $\lim_{x \to a} f(x) \neq f(a)$. In this figure this function is defined at x= a. The limit of the function at x= a exists but f(x) is not equal to the value of the function at x = a.

Actually the third condition implies the first two conditions, since it means, $\lim_{x \to a} f(x) = f(a)$. This actually means limit exists and the function is defined at x=a which is f(a) = d

Example:

$$\frac{x^2 - 4}{x - 2} \qquad at\ x = 2$$

F(2) is not defined

$\lim_{x \to 2} f(x)$ exists. $\lim_{x \to 2} f(x) \neq 2$

the f(x) is not continuous at x = a

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

The function is discontinuous at x = α

Since $\lim_{x \to 2} f(x) \neq f(2) = 3$

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

F(2) = 4 which is the same as $\lim_{x \to 2} f(x) = 4$

Notes :

(1) If a function is continuous at each number in an open interval (a, b) then we say f is continuous on (a, b).

If the function is continuous on ($-\infty$, ∞) we can say that f is continuous everywhere.

The general method of showing that the function is continuous everywhere is to show that it is continuous at an arbitrary real number.

While discussing limit of a polynomial we saw that

$$\lim_{x \to a} f(x) = f(a)$$

Thus we can make the following statement.

Polynomials are continuous everywhere.

All rational functions are continuous on any interval not containing zero of the denominator. In other words rational function is continuous on the interval on which it is defined.

Now we shall state briefly certain theorems of continuous function which will be useful while finding out whether the function is continuous or not.

Theorem : If the functions f and g are continuous at c, then,

(i) f + g is continuous at c

(i) f – g is continuous at c

(ii) fg is continuous at c

(iv) f/g is continuous at g if g(c) $\neq$ 0

It will have a discontinuity at c if g(c) = 0

Example : $\dfrac{x^2 - 9}{x^2 - 5x + 6}$

The function is a rational function. The denominator becomes o when x = 2 and at x = 3

If implies that g (c) = g(2) and g (3) = 0. So the function is discontinuous at x= 2 and x=3.

Theorem: If $\lim_{x \to c} g(x) = L$ and if t he function f is continuous at L then

$$\lim_{x \to h} f\left[g(x)\right] = f(h).$$

That is

$$\lim_{x \to c} f\left[g(x)\right] = f\left[\lim_{x \to c} g(x)\right]$$

Example: f(x) = 5- x^2 is continuous everywhere. It is continuous at 3.

$$\lim_{x \to 3}\left(5 - x^2\right) = 5 - 9 = -4$$

$$f(3) = -4$$

$$\lim_{x \to 3} f(x) = f(3)$$

Now $f(x) = \left|5 - x^2\right|$ is also continuous at 3

$$\lim_{x \to 3}\left|5 - x^2\right| = \left|\lim_{x \to 3}\left(5 - x^2\right)\right| = \left|-4\right| = 4$$

The absolute values of a continuous function is continuous.

Properties of a function defined over closed interval [a, b]. A function f(x) is said to be continuous on a closed interval [a, b] if the following conditions are satisfied.

(a) F(x) is continuous an (a, b)

(b) F (x) is continuous from right to a that is $\lim_{x \to a^+} f(x) = f(a)$

(c) F(x) is continuous from left at b that is $\lim_{x \to b^-} f(x) = f(b)$

Example: $f(x) = \sqrt{4 - x^2}$

The natural domain is the closed interval [-2, 2] we will have to find out the f continuity on open interval (-2, 2) and at two end points -2 and 2. Take an arbitrary point C

$$\lim_{x\to c} f(x) = \sqrt{\lim_{x\to c}\left(4-x^2\right)} = \sqrt{4-c^2} = f(c)$$

Which proves f(x) is continuous on (–2, 2)

The function f(x) is also continuous an end points.

$$\lim_{x\to 2^-} f(x) = \sqrt{\lim_{x\to 2^-}\left(4-x^2\right)} = \sqrt{4-4} = 0 = f(2)$$

$$\lim_{x\to -2^+} = \sqrt{\lim_{x\to -2^+}\left(4-x^2\right)} = 0 = f(-2)$$

Then f(x) is continuous on closed interval [-2, 2]

6. Intermediate value Theorem:

If f(x) is continuous on the closed interval [a, b] and k is any number between f(a) and f(b) (inclusive of end points) then there is at least are number x in the interval [a, b] such that

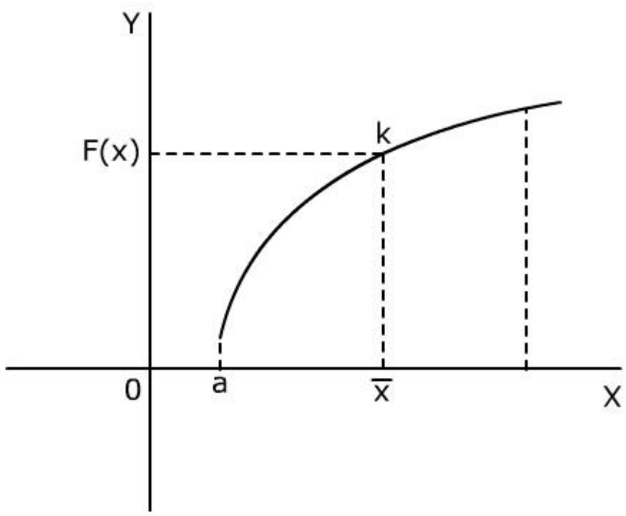

Figure 7

F(x) = k.

The theorem becomes obvious if we draw the graph of the function.

From the above we can draw another theorem.

If f is continuous on [a,b] and if f(a) and f(b) are non-zero and have opposite signs, then there is at least one solution of the equation in the interval (a, b)

$$f(x) = 0$$

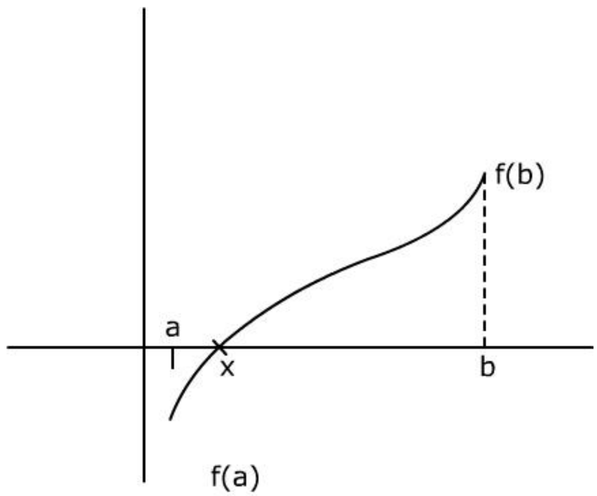

Fig. 8

Example:

$$f(x) = x^3 - x - 1$$

When $x = 1$ $f(1) = -1$

When $x = 2$ $f(2) = 5$

So the graph of the function intersect the x axis at least once.

We can make the approximation better by reducing the size of interval [1,2]

Example :

(a) $X^3 - 4x+1 = 0;$ [1, 2]

(b) $X^3 + x^2 -2x = 1;$ [–1,1]

A function f(x) is said to have a removable discontinuity at x = 0 if

$\lim_{x \to c} f(x)$ exist.

But f(x) is not continuous at x=c either because (i) or (ii)

(i) f(x) is not defined at c

(ii) f(c) differ from $\lim_{x \to c} f(x)$

Show that the following functions have removable discontinuities at x = 1

(i) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(ii) $g(x) = \begin{cases} 1 & x > 1 \\ 0 & x = 1 \\ 1, & x < 1 \end{cases}$

Solution:

(1) The $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{(x-1)(x+1)}{x - 1} = 2$

But f(1) is not defined.

So the function is discontinuous at f(x) = 1

Now if we redefine the function

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & for\ x \neq 1 \\ 2 & if\ x = 1 \end{cases}$$

Then the function becomes continuous.

(i) Similarly in the case of g(x) the limit when x $\to$ 1 is 1.

So if g(x) = 1 when x = 1

The function becomes continuous.

Note: In case the limit does not exist at C then the discontinuity is irremovable.

7. References:

K. Sydsaeter and P. Hammond, Mathematics for Economic Analysis, Pearson Educational Asia, Delhi, 2002.