LESSON: Consumer Optimization

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Learning Outcomes:

After studying this chapter, a student should be able to:-

i) State condition for attaining optimal choice bundle.
ii) Calculate optimal consumption bundle given information about preferences, income and prices.
iii) Analyze impact of change in price on quantity demanded of a good.
iv) Derive demand curve for a good for different kind of preferences.
v) Define & distinguish between normal good and giffen goods.
vi) Analyze impact of change in income on quantity demanded of a good.
vii) Derive engel curve for a good for different kind of preferences.
viii) Distinguish between normal good and inferior goods.

1. Introduction

Can you recall when you were given pocket money at the age of 7 or 8! You always knew how to utilize that money. Either kids at that age would spend on cola, ice-cream, or whatever toys one wanted. But I 'm sure you must have chosen whatever must have brought you joy and satisfaction. You didn't know what optimization was, what microeconomics technique to be applied and what conditions were to be met. But you were genius who did optimization subconsciously and actually every consumer does.

In this chapter, we will deal with optimization formally. This chapter is divided into three main sections. First section covers optimization's conditions for various preferences. In second section, demand curve of a good for a consumer is desired. In last section, impact of change in income on optimal quantity of good is analyzed.

2. Optimization

Optimization in context of utility would mean maximizing utility given the budget set. Given any level of income, M and prices & good x & y as p_X & p_Y a consumer maximizes his utility by choosing a consumption bundle that gives him highest satisfaction. This bundle choice is dependent on consumer's preferences. It is obvious to assume that consumer is happier consuming good x (relatively to good y), then his optimal consumption bundle would have more of good x. But, this consumer's choice is also affected by price of good x in the market and is constrained by his income. Hence, optimal choice is decided by nexus between budget set and preferences.

2.1 Optimization in case of well behaved preferences.

Preferences are well behaved if indifference curves are negatively sloped, and are convex to the origin. This section analyses 'optimal choice bundle'.

2.1.1 Diagrammatic treatment to it.

Budget line is locus of all consumption bundles which are affordable when entire income is consumed. Indifference map shows indifference curves of varying utility. Indifference curve is locus of all consumption bundles which yield some constant level of utility. Budget set and indifference map have same x-axis and y – axis labeling as x-good and y-good, respectively. Lt us super impose indifference curves on budget set, like in

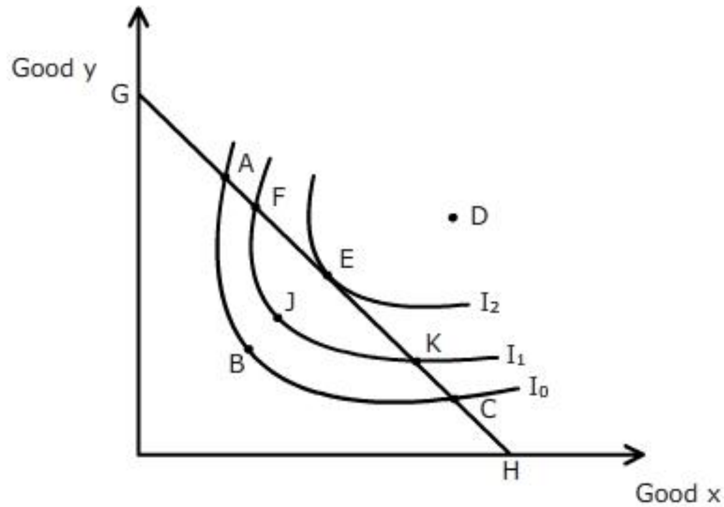


Figure1

A Few affordable bundles given some income, 'M' are marked in above figure. Point A,B and C yield utility U_0 and likewise points F,G, H yield utility U_1 & point E yield U_2[1].

Amongst all such affordable bundles, the point that maximized utility is point E that gives utility U_2. There are two remarkable things that point 'E':-

i) **It lies on the budget line**
Optimal point lies on the budget line. Points B & G yield utility U_0& U_1 and hence utility could still rise till E is reached. Any point above E (like D) is unaffordable.

ii) **Point of tangency of indifference curve with Budget line**
If optimal point has to be on budget line, then all points like A,C, F,H,& E all are such points but optimal is only E. Highest possible achievable indifference curve is I_2.

2.1.2 Algebraic expression for optimal Bundle.

As discussed in last section, optimal bundle choice requires tangency of indifference curve with budget line. This implies that slope of budget line equals slope of indifference curve, which is given as:

$$\frac{-P_x}{P_y} = MRS_{xy} = \frac{-MU_x}{MU_y}$$

Slope of budget line measures that rate at which market is willing to substitute good y for good x. The above equation implies that rate of substitution in a market should be equal to marginal rate of substitution of two good by a consumer.

---

[1] Assumed here that $U_2>U_1>U_0$ and '0','1'&'2' subscripts create correspondence between indifference curve with their respective utility level.

2.1.3 Lagrangian Technique of utility maximization[2]

The objective function is the utility function and constraint here is budget. The problem can be written as follows:-

Max : U(x,y)

Subject to: $p_x x + p_y y = M$

Lagrange, $\ell = U(x,y) - \lambda(p_x x + p_y y - M)$

where λ is Lagrange multiplier.

For optimization put

$$\frac{\partial \ell}{\partial x} = 0, \frac{\partial \ell}{\partial y} = 0 \; and \; \frac{\partial \ell}{\partial y} = 0$$

$U'_x - \lambda P_x = 0$ 1

$U'_y - \lambda P_y = 0$ 2

$-(p_x x + p_y - M) = 0$ 3

On solving equation 1 and 2, we get:

$$\frac{U'_x}{P_x} = \frac{U'_y}{P_y} = \lambda$$ [3]

Also ,it would be written as:

$$\frac{U'_x}{P_x} = \frac{U'_y}{P_y}$$

[2] optional

[3] λ is lagrange multiplier and here it becomes ratio of befits to cost. Additional benefit from each good is MU and cost is its price. So, condition implies that marginal benefit to cost ratio must be equal for all goods.

Gossen's laws, named for Hermann Heinrich Gossen (1810 – 1858), are three laws of economics:

Gossen's First Law is the "law" of diminishing marginal utility: that marginal utilities are diminishing across the ranges relevant to decision-making.

Gossen's Second Law, which presumes that utility is at least weakly quantified, is that in equilibrium an agent will allocate expenditures so that the ratio of marginal utility to price (marginal cost of acquisition) is equal across all goods and services.

$$\frac{\partial U/\partial x_i}{p_i} = \frac{\partial U/\partial x_j}{p_j} \forall (i,j)$$

where

- U is utility
- x_i is quantity of the i-th good or service
- p_i is the price of the i-th good or service

Gossen's Third Law is that scarcity is a precondition for economic value.

Source: Wikipedia

Where, U'_x is marginal utility of x and U'_y is marginal utility of y. It is same equilibrium condition required for optimal choice bundle, which was attained in last section. Equation 3 implies that this choice bundle (x,y) must end up entire income.

2.2 Interior solution and boundary optimums

Lagrangian technique and equality of slope can only be applied when indifference curves are smooth(without kinks) and are convex .In case of kinks ,though interior solution can be obtained but this calculus does not work. In other than these cases, even boundary points act as solution to optimization problem. But technique of calculus is of no use in such cases. We will analyze them in this section.

2.2.1 Kinked preferences

In case of kinked preference, optimal point would be where indifference curve's kink touches the budget line. Like in figure 2,point E is optimal point.

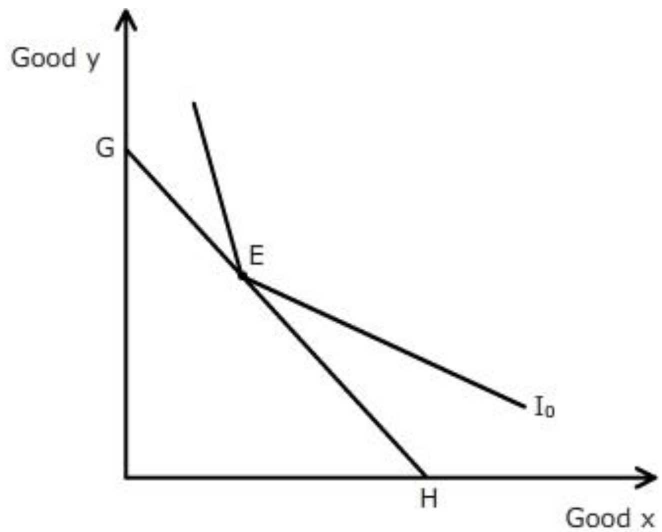


Figure 2

When two goods are perfect complements, then indifference curves are L shaped. Indifference curve for perfect complements also have kink.

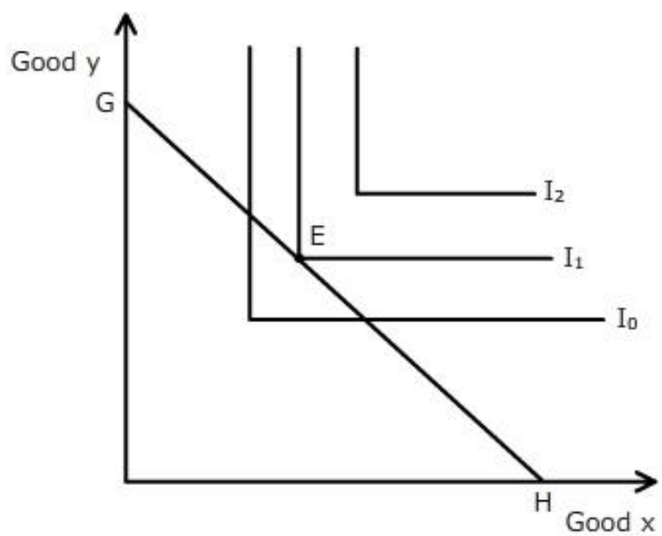


Figure 3

Optimal bundle for perfect complements indifference curve is where kink touches the budget line like at point E in figure 3 .Since at kink slope cannot be calculated so there has to be alternate method to complete optimal bundle.

Consider a consumer's preferences that 'a' units of x are consumed with 'b' units of y .The indifference curves would appear as in figure 4.The kinks would be on line OA whose slope is b/a .Origin is also one point and one indifference curve is an L at origin (that is x axis and y axis itself is an indifference curve).

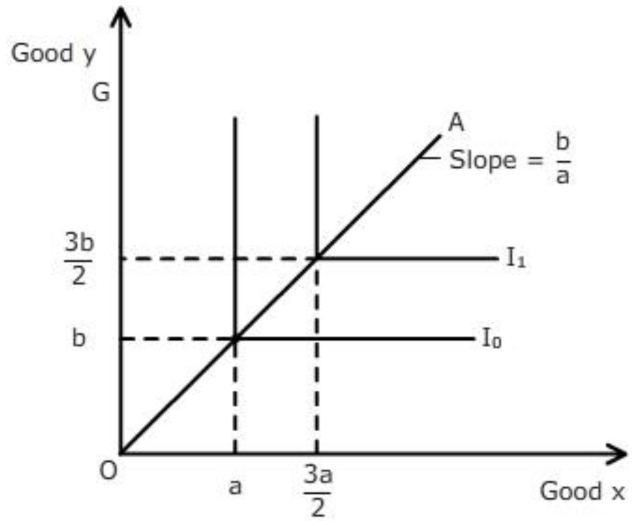


Figure 4

Optimal point is hence at intersection of line OA and budget line. Budget line is given by $p_x x+p_y y=M$ and line OA's equation is y=(b/a) x. Solving these two equations yield :

$$x *= \frac{aM}{aP_x + bP_y}$$

$$y *= \frac{bM}{aP_x + bP_y}$$

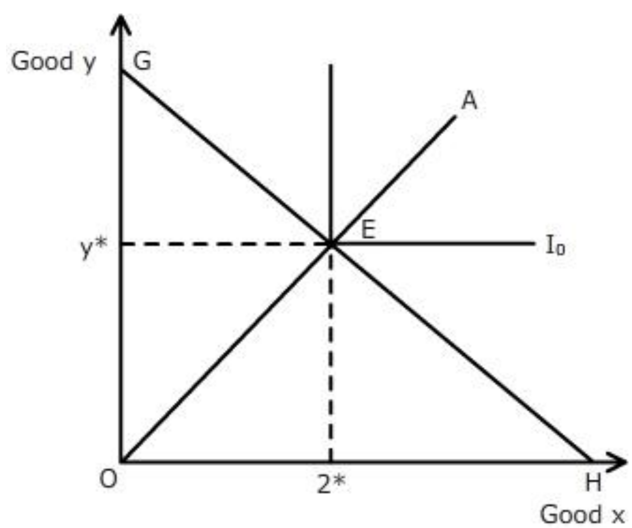


figure 5

2.2.2 Perfect substitutes

Indifference curves for perfect substitutes are straight lines. Either of the three is possible:

I. Indifference curves are steeper than budget line.
II. Indifference curves are flatter than budget line.
III. Indifference curves are parallel to budget line and one of these overlap budget line.

These cases are depicted in respective panels of figure 6.

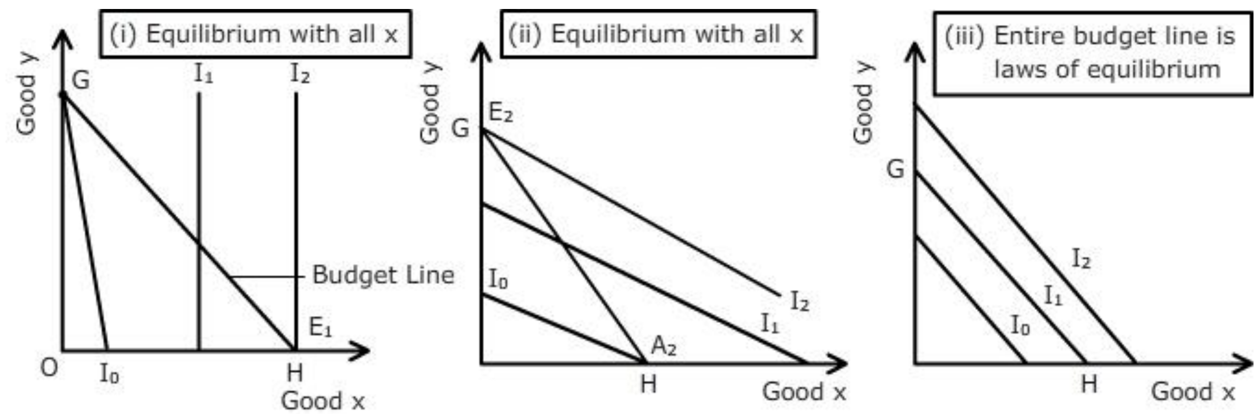


Figure 6

Case 1 Indifference curves are steeper than budget line:

This would mean that consumer is willing to substitute y for good x at greater pace .This mean consumers values good x more than good y. since two goods can be substituted easily (perfectly) and optimal choice would be at point E_1 in panel (1) of figure 6;where consumer consumes all x and zero units of good y.($\frac{M}{P_x}$,0) is boundary optimum.

Case 2 Indifference curves are flatter than budget line:

This case is just reverse of above discussed case. It is depicted in panel (2) of figure 6 and in such case consumer consumes all y and nothing of good x. (0,$\frac{M}{P_y}$) is boundary optimum.

Case 3 Indifference curves slope equivalent to that of budget line:

In such a case, one indifference curve overlap budget line and hence all points starting from (0,$\frac{M}{P_y}$) and in between and including ($\frac{M}{P_x}$,0) are optimal.

Let us write down demand function of x when goods x and y perfect substitutes as follows:

$$X = \begin{cases} \frac{M}{P_x} & \text{when } \frac{MU_x}{MU_y} > \frac{P_x}{P_y} \\ 0 & \text{when } \frac{MU_x}{MU_y} < \frac{P_x}{P_y} \\ \text{Any number between } 0 \text{ \& } \frac{M}{P_x} & \text{when } \frac{M}{P_x} = \frac{M}{P_y} \end{cases}$$

2.2.3 Neutrals and Bads

Let good x be good and good y be neutral, then highest possible achievable indifference curve is where M/P_x units of x are consumed as depicted in figure 7 and zero units of good y which is neutral good.

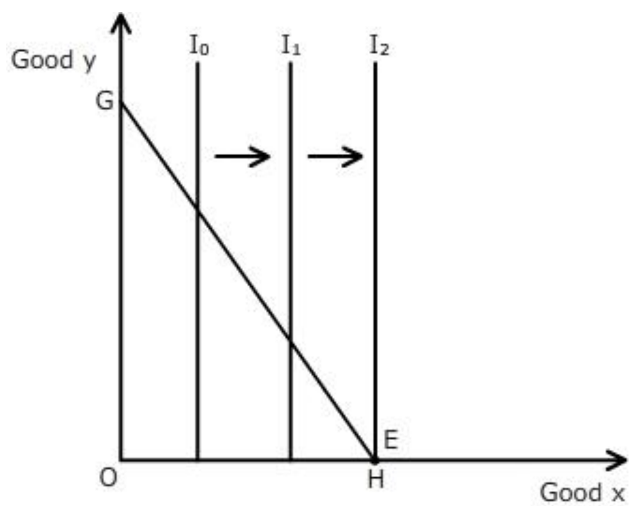


Figure 7

Now let good y be bad and good x as good. Then consumer has highest utility when bad y is not consumed. This is depicted in figure 8.

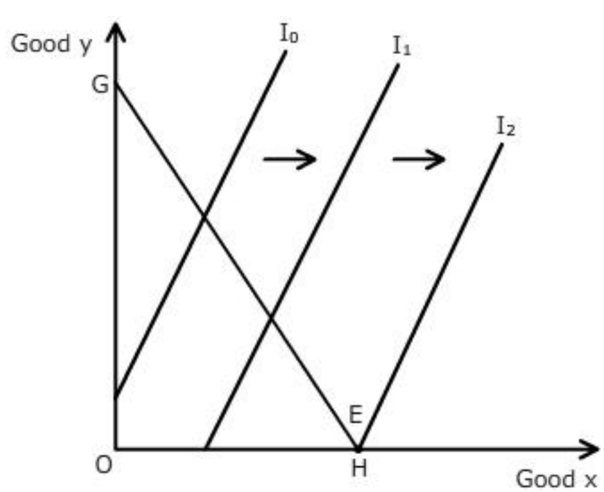


Figure 8

Demand for good x $=M/P_x$

For y =0(in either case of neutral or bad commodity)

In both the cases, boundary optimum is achieved, where all income is spent on good and nothing on bad or neutral commodity.

2.2.4 Concave preferences

When preferences are concave, then tangency condition can be met like at point F on budget line. But question is it optimum? The answer is no. The reason is yet higher indifference curve is achievable and points like E_1 in panel (1) and E_2 in panel (2) are boundary optimum in case of concave preferences.

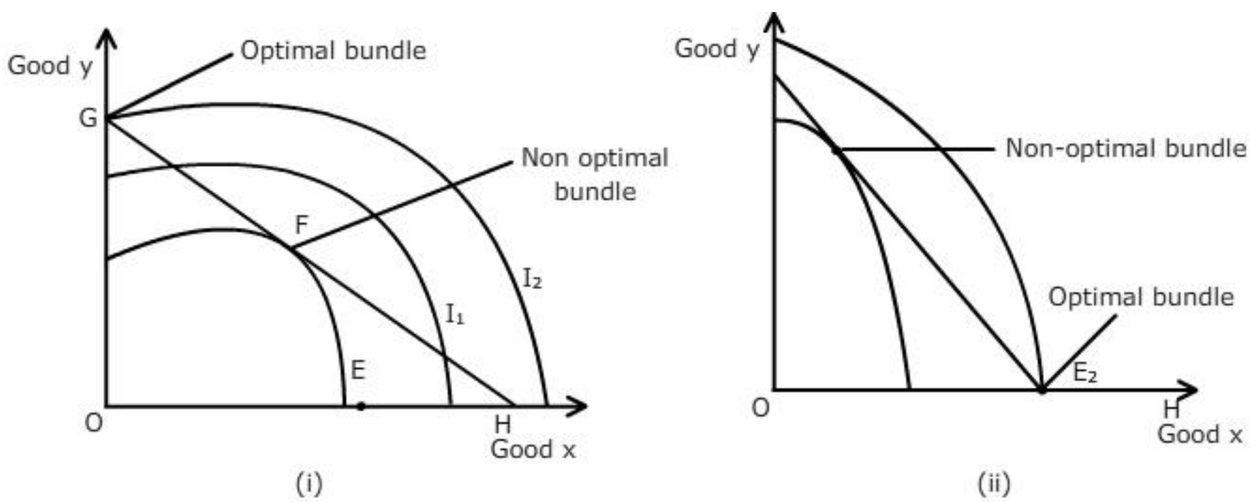


Figure9

2.2.5 Cobb-Douglas preferences

Consider Cobb Douglas preferences where utility function takes the form $U(x,y)=x^c y^d$. Cobb Douglas preferences exhibit well behaved preferences, and henceforth calculus technique can be applied here.

$MUx = \frac{\partial}{\partial x}(x^c y^d)=c\, x^{c-1}y^d$

$MUy = \frac{\partial}{\partial y}(x^c y^d)=d\, x^c\, y^{d-1}$

$MRS_{xy}=\frac{-MU_x}{MU_y} = \frac{cy}{dx}$

For optimal bundle $MRS = \frac{P_x}{P_y}$

$$\frac{cy*}{dx*} = \frac{P_x}{P_y}$$

$$Or, y* = \frac{dP_x}{cP_y} x*$$

Putting this value in budget constraint ,we get:

$P_x x* + P_y \frac{d}{c} \frac{P_x}{P_y} x* = M$

$(\frac{c+d}{c}) P_x x* = M$

$x* = \frac{M}{P_x} (\frac{c}{c+d})$

$y* = \frac{d}{c+d} \frac{M}{P_y}$

On rearranging demand function for x, we get:

$\frac{xP_x}{M} = \frac{c}{c+d}$

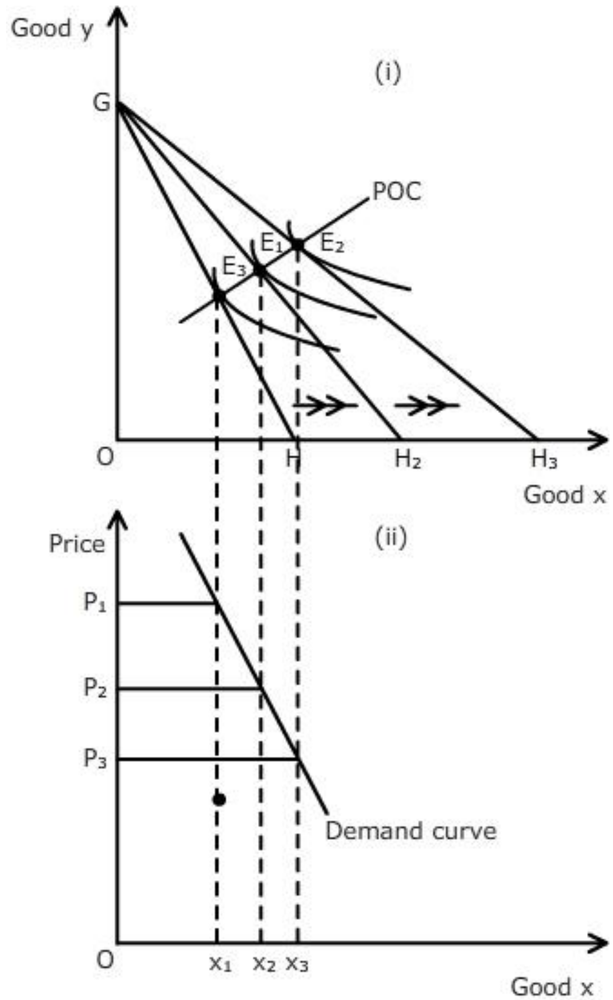
In case of Cobb Douglas preferences, share of income (M) spent on good x ($P_x x$) is equal to the ($\frac{c}{c+d}$). Hence, fraction of income spent on either good is fixed. The size of this fraction is determined by the exponent (of quantity of that good) in Cobb Douglas function. In two goods case, hence, it is better to assume that c+d=1. This assumption makes it clear that income is spent on these goods with some weights given by respective exponents of units of goods in utility function.

3. Demand

Demand function shows the relationship between price and quantity demanded. For a normal good, there exists negative relationship between and price and quantity demanded. In this section, we will analyze and derive demand curve in case of consumer’s different preferences.

3.1 Well behaved preferences

When price of good x falls, budget lines pivots (around the y-axis) outward. Let us analyze the path of optimal consumption bundles that is followed when price of good x changes.



In panel (i) of Figure 10, when price of good x falls from P_1 to P_2 and then to P_3 (while holding price of y constant), budget line shifts from GH to GH_1 to GH_2. The optimal bundles are marked E_0, E_1 and E_2, respectively. With the fall in price of good x consumer has enlarged budget set and hence, more of good x can be consumed.[4] The quantity demanded rises from x_1 to x_2 to x_3 which corresponds to prices P_1, P_2 and P_3, respectively. Connecting all the optimal bundles lead to construction of price offer curve. This curve shows bundles that would be demanded at different prices of good x. In panel (ii) of figure 10, we trace down quantities and plot these quantities against their respective prices. We get demand curve which is downward sloping i.e.

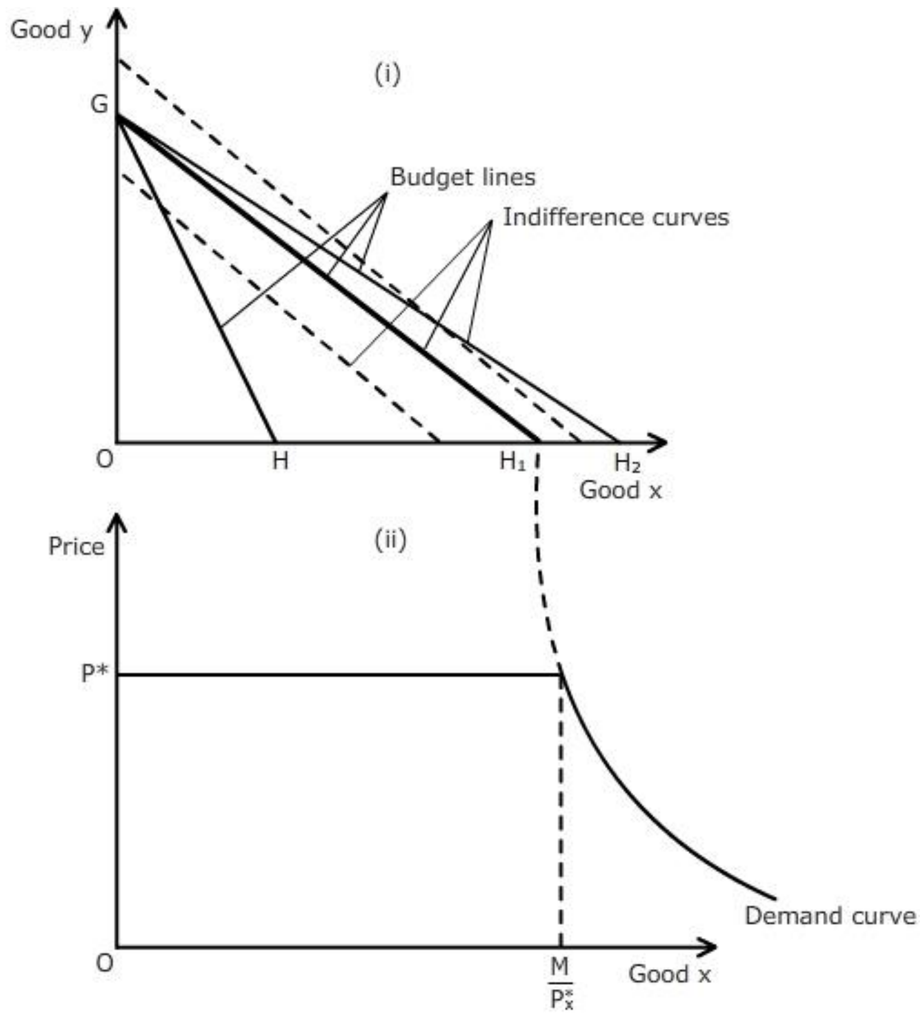
$$\frac{\Delta x}{\Delta P_x} < 0$$

[4] Fall in price leads to two effects. First, purchasing original bundle leaves consumer with some extra income at hand and second, fall in price of x makes it cheaper and sometimes consumer consume more of it in place of good y. This will be discussed in chapters to come.

The price and quantity demanded of that good move in opposite direction cetirus peribus (P_y, M and consumer's preferences are held constant).

3.2 Perfect Substitutes

Assume some price of x as P_X^* such that $\frac{P_x^*}{P_y}=\frac{MU_x}{MU_y}$. If price of good x falls below P_x^* , $\frac{P_x}{P_y}<\frac{MU_x}{MU_y}$ which means slope of indifference curve is greater than slope of budget line and hence only good x will be demanded. If price of good x rises above P_x^*, only good y will be consumed and zero quantity of good x is demanded. When price of good x is equal to P_x^* any quantity between zero and $\frac{M}{P_x}$ can be demanded.



Dashed lines in panel (i) of Fig 11 are indifference curves. GH_1 is the budget line parallel to indifference curves and hence have slope $\frac{P_x^*}{P_y}$. GH is steeper than GH_1 and GH_2 is flatter than GH_1.

Price offer curve has two segments:-

i. Budget Line GH_1, since all bundles on this budget line are optimal i.e. when price of good x is P_x^*.
ii. X-axis when price of good x has fallen below P_x^*, only good x is demanded.

Figure 11

Again plotting down the quantities against respective prices yield demand curve in panel (2) of figure 11. Above P_x^*, zero units of good x are demanded; at P_x^* any quantity between zero and M/p*$_x$ can be demanded & if price falls further more of (only) good x is demanded.

3.3 Perfect complements

Perfect complements are consumed in combination and no utility is added if either of the good's consumption increases. So when price of x falls, consumers uses this extra real income to consume both commodities in some fixed proportion. Hence, when price of good x falls ,quantity demanded for x increases.

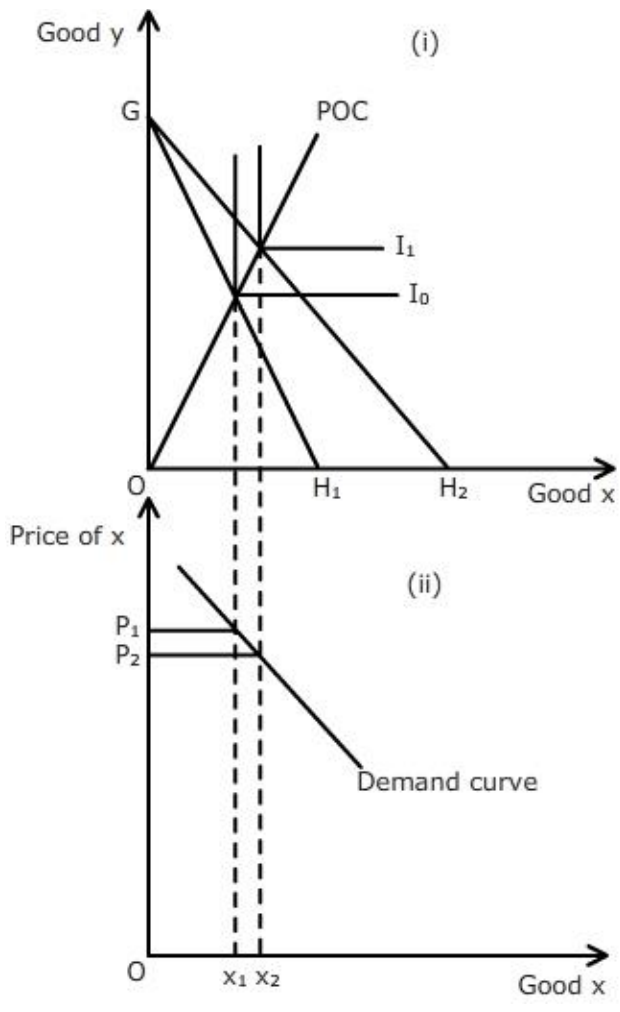


Figure 12

Price offer curve is the line joining all the kinks of indifference curves starting from origin. For 'a' units of x with 'b' units of y example, we computed optimal bundle x*=$\frac{aM}{aP_x+bP_y}$

Differentiating x* with respect to p_x; we get:

$$\frac{\Delta x*}{\Delta P_x} = -\frac{aM}{(aP_x+bP_y)^2} .a = -\frac{a^2M}{(aP_x+bP_y)^2}$$

Or $\frac{\Delta x*}{\Delta P_x}<0$ (since M is always positive and rest all terms are squared)

3.4 Giffen goods

In the nineteenth century, it was found that it is likely that when price of a good falls ,less of that good demanded. Such a good then is not normal and is known as giffen good .This case is depicted in figure 13.

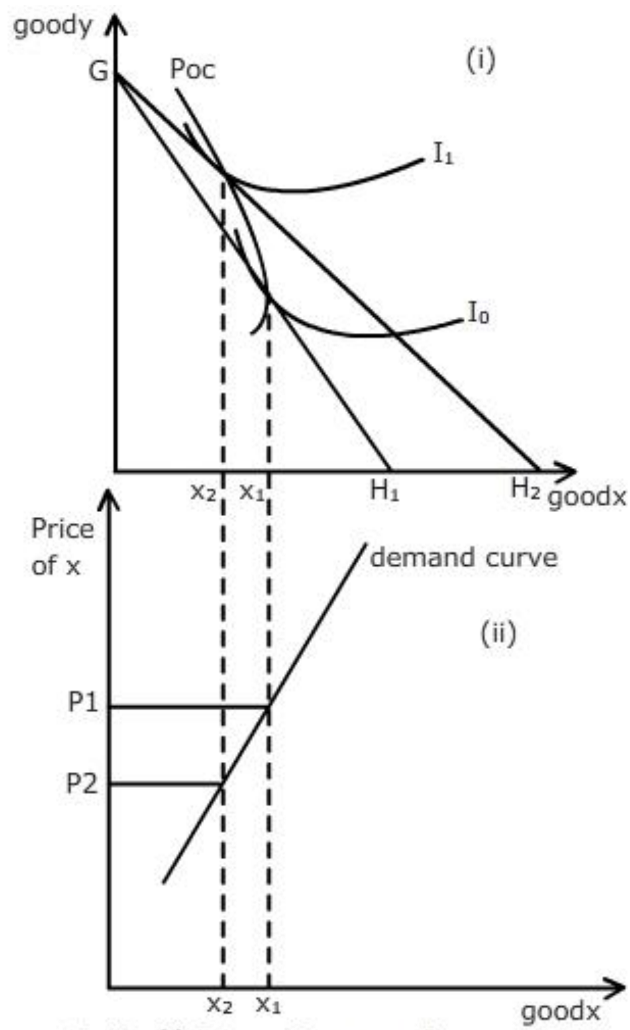
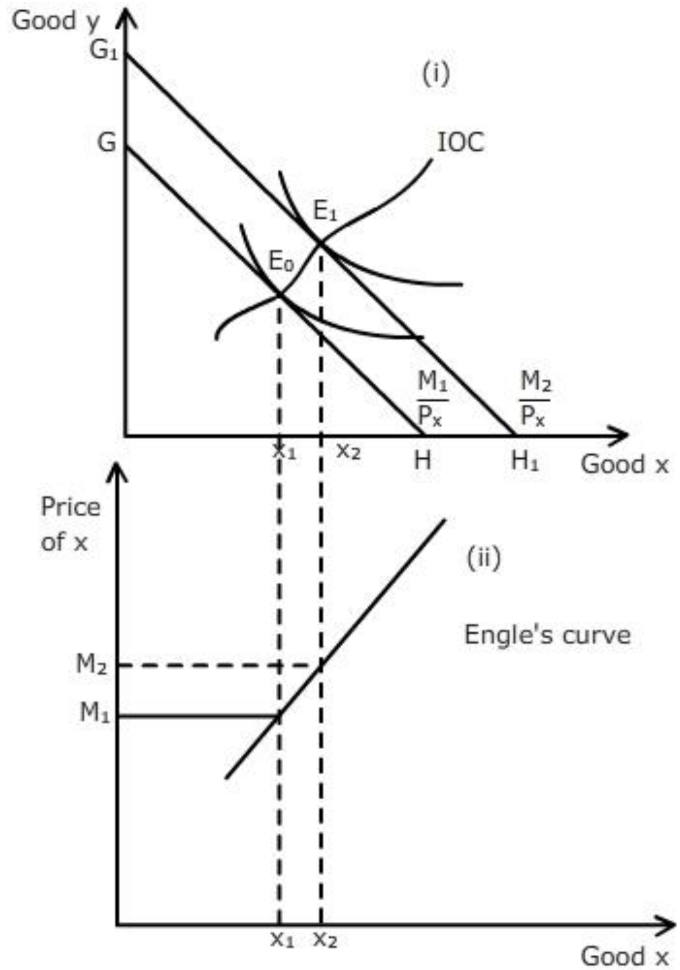


fig13: (i) Price offer curve &
(ii) demand curve for giffen goods.

The demand curve then for giffen good is positively sloped. These type of goods are exception to law of demand.

4 Engel curves

Engel curve shows the relationship between demand for a good and income of the consumer. For a normal good, one can argue that there exists positive relationship between the two. But there are goods whose demand falls when income of the consumer goes up. Such goods are known as inferior goods.



4.1 Well behaved preferences: normal good case

When income of a consumer rises, budget line parallelly shifts outward, like shown in panel (i) of figure 14. This changes optimal bundle from E_0 to E_1. Joining all optimal bundles one can construct income offer curve. Repeating same exercise of plotting different quantity demanded of good x for varying levels of income in panel (ii),we get Engel curve.

4.2 Inferior goods

Let consider that good x is inferior so when income rises then optimal bundle changes from E_0 to E_1 in such a fashion that optimal quantity of x falls x_1 to x_2.In panel (ii) of figure 15 Engel's curve is constructed joining (x_1 ,M_1) and (x_2,M_2). Engel curve for inferior good is negatively sloped.

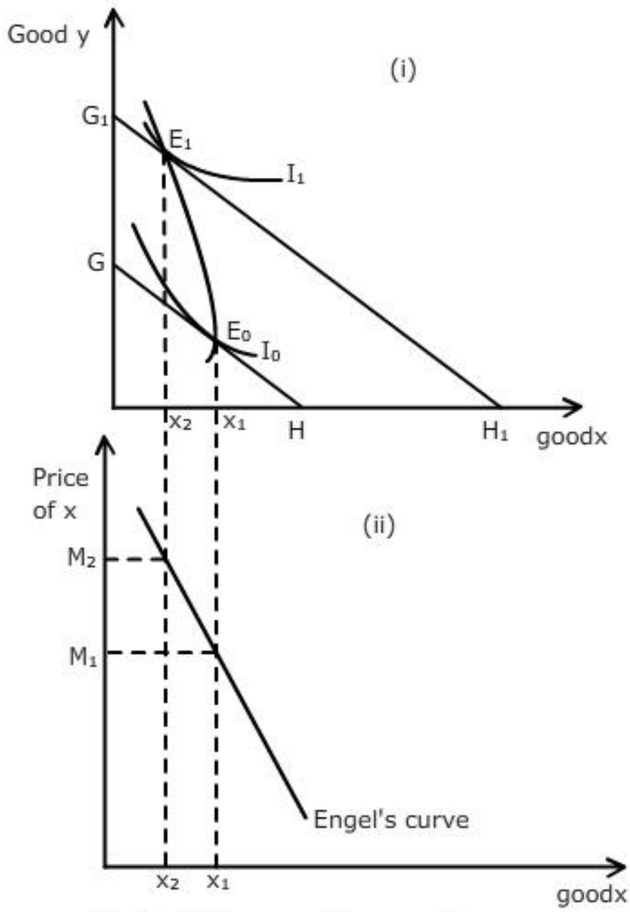


Fig15: (i) Income offer curve &
(ii) Engel curve for inferior good

4.3 Perfect substitute

Good x which is perfect substitute to goody y is demanded only when $\frac{P_x}{P_y} < \frac{MU_x}{MU_y}$. So when income increases, the entire addition to income is used up to consume good x. So when income is M_1 then $x_1 = \frac{M_1}{P_x}$ and then income increased to M_2; $x_2 = \frac{M_2}{P_x}$.

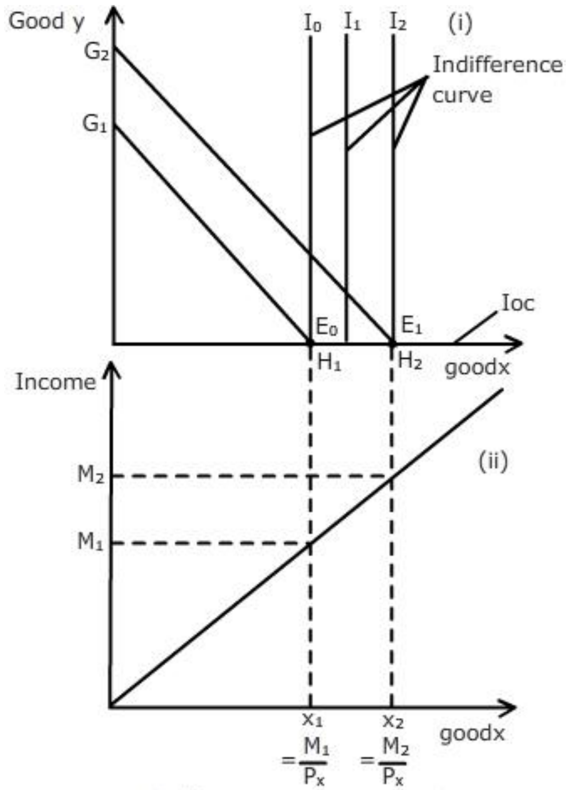


fig16: (i) Income offer curve &
(ii) Engel curve for perfect substitution

x_1 and x_2 are boundary optimum shown as bundles E_0 and E_1 in panel (i) of figure 16. Panel (ii) of figure 16 depicts Engels curve. Slope of Engel ´s curve is calculated as follows:

$$\frac{\Delta M}{\Delta x} = \frac{M_2 - M_1}{x_2 - x_1} = \frac{M_2 - M_1}{M_2 - M_1 / P_x} = P_x$$

For x to change by 1 unit (exactly), income must change by P_x.

4.4 Perfect complements

Demand pattern for good x which is used in some fixed proportion with y is depicted in figure 17. Optimal quantity of x is given by

$$x * = \frac{aM}{aP_x + bP_y}$$

Differentiating above equation with respect to x*, we get:

$$\frac{\Delta x*}{\Delta x*} = \frac{a\frac{\Delta M}{\Delta x*}}{aP_x + bP_y}$$

Or $\frac{aP_x + bP_y}{a} = \frac{\Delta M}{\Delta x*}$ (slope of Engel curve)

Or $\frac{\Delta M}{\Delta x*} > 0$

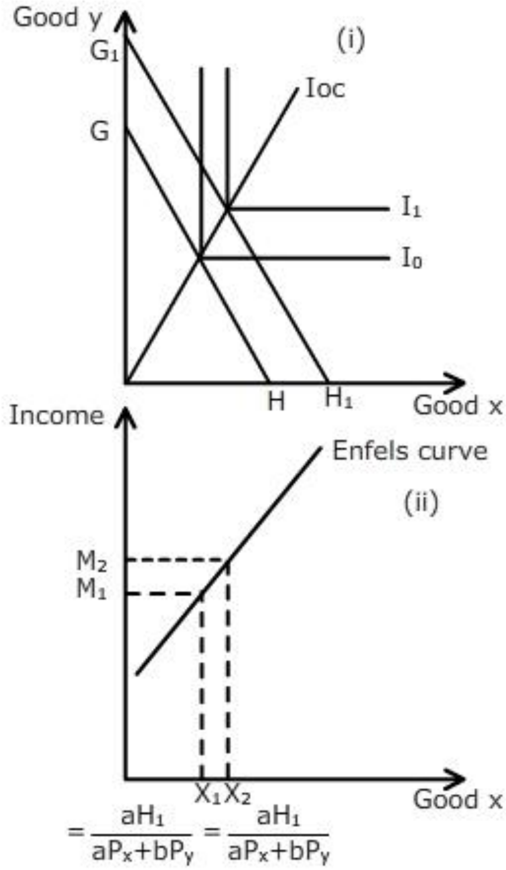


Figure 17

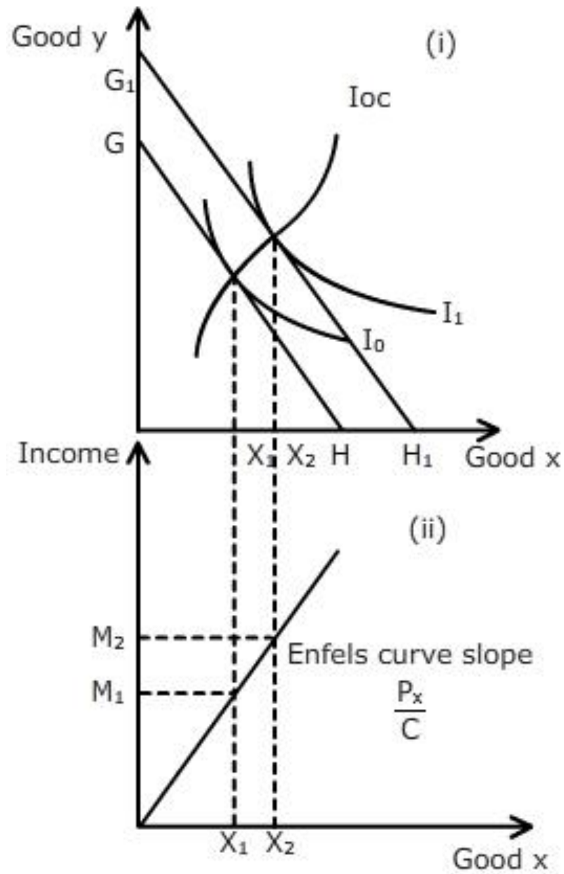


Figure 18

4.5 Cobb Douglas preferences

Optimal value of good x is linearly dependent on money income of the consumer, given by the following equation:

$x^* = \frac{c}{c+d} \cdot \frac{M}{P_x}$

Again, differentiating this equation with respect to Δx* and upon rearranging, we get:

$P_x \frac{c+d}{c} = \frac{\Delta M}{\Delta x*}$

Assuming c+d =1 the reason for which was explained earlier, $\frac{\Delta M}{\Delta x*} = \frac{P_x}{c}$; which is slope of Engel's curve.

4.6 Homothetic preferences

A consumer's preferences homothetic if marginal rate of substitution depends upon ratio of units of two goods and not on total quantities of goods. If one look at MRS of Cobb Douglas, perfect complements and perfect substitutes; their MRS are dependent on ratio y/x. Indifference curve for homothetic preferences appear like copy of one pasted at different levels. Slope of curves depend only ratio y/x, not on how far the curve is from the origin.

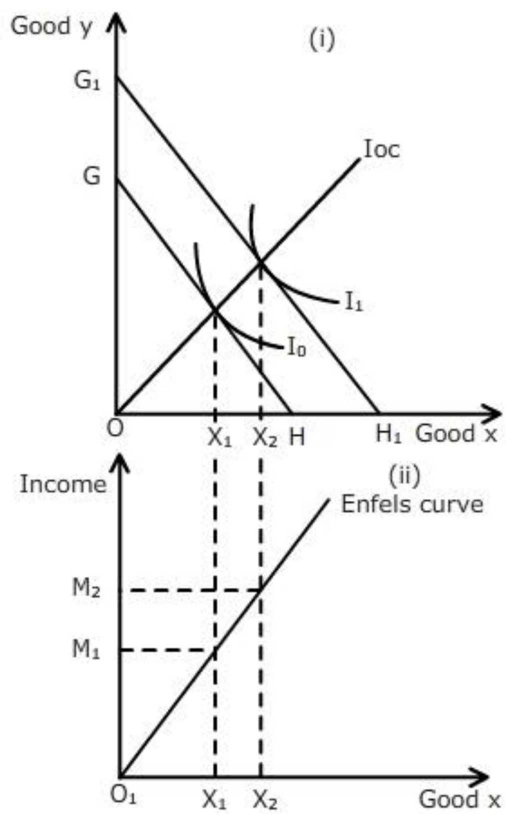


Figure19

This would mean Income offer curve is a straight line joining (x_1,y_1),$(2x_1,2y_1)$,$(3x_1,3y_1)$ and so on. Where (x_1,y_1) is optimal bundle when income is M and $(2x_1,2y_1)$ is when income doubles is optimal and likewise.

4.7 Quasi linear preferences

Quasi linear preferences are not homothetic. Assume U(x,y)=MRS=$\frac{MU_x}{MU_y}=\frac{1}{x}$. So MRS depends on how much a consumer x and not on ratio (y/x).If consumer's income is M_1 and optimal bundle is (x_1,y_1) and now if income increases his optimal bundle becomes (x_1,y_1+k) for any constant k.

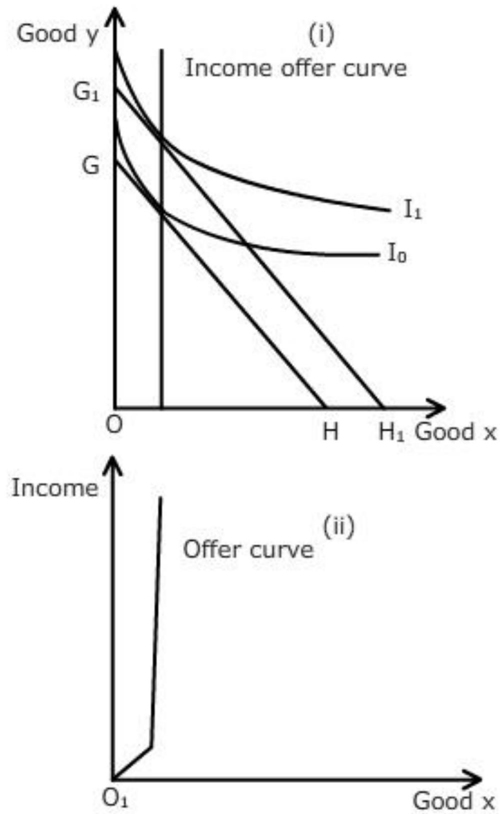


Figure 20

The example of such a good is salt. Even when income is added there is no increase in the quantity of salt demanded. You spend addition to income on all goods but salt. Hence there is ‘zero income effect’.

Summary

- For solution to utility maximization problem, it requires that indifference curve is tangent to budget line or equivalently slope of the two are equal. When indifference curves have kink, the kinked point should touch budget line for optimal solution.
- There are boundary optimums when a consumer consumes a) perfect substitutes, b) a neutral good and, c) bad good or else if he has concave preferences.
- For a normal good, law of demand operates and quantity demanded moves in opposite direction of –in response to- prince change. Demand curves are negatively sloped in all cases but Giffen goods.
- For a normal good, change in quantity demanded is positively related to the change in income of the consumer and hence, Engel curve is positively sloped in all cases but inferior goods.

Exercises

Q1. a) If a consumer has a utility function $U(x,y)= x^1y^4$, what fraction of his income will he spend on good y?

b) If prices are P_x and P_y and income, M; what will be consumer's optimal choice bundle?

Q2. Suppose that a consumer always consumes 2 spoons of sugar with 1 cup of tea and their respective prices are Ps and Pt and consumer has m rupees to spend on sugar and tea. How much will he demand?

Q3. Suppose a consumer's utility function is $U(x,y) = x^2+y^2$.

a) Calculate his MRS.

b) Is his MRS diminishing?

c) Solve for optimal choice bundle if prices are P_x and P_y and consumer's income is M.

Q4. Henry is currently consuming only Coke and Pizza. At his current consumption bundle marginal utility of Coke is 10 and that of Pizza is 5. Each Coke costs Rs.2 and each Pizza costs Rs.10. Is he maximizing his utility? Explain. If he is not, how can he increase his utility while keeping his expenditure constant?

Q5. Assume good x is inferior. Draw income offer curve. Is it possible, even good y is inferior? Explain.

Q6. Madhu views Pepsi and Coca-cola as perfect substitutes. The price of 750 ml bottle of Pepsi is Rs. 10 and price of 750 ml bottle of Coca-cola is Rs.12. what does Madhu's Engel curve for Pepsi look alike? By how much her Budget should increase so that she can consume one more unit of Pepsi?

Glossary

- **Optimal choice:** It is optimum when it is the best state of affairs and choice which is optimum is called optimal choice.
- **Price offer curve:** The locus of all consumer equilibria when price changes is known as price offer curve.
- **Demand curve:** Demand curve is curve showing the negative(for normal good) relationship between price and quantity demanded by consumer.
- **Giffen good:** In case consumer violates law of demand, and for a good positive relationship between price and quantity demanded by consumer is observed then that good is called giffen good.
- **Income offer curve:** The locus of all consumer equilibria when income of consumer changes is known as income offer curve.

- **Engel’s curve:** Engel’s curve is curve showing the positive (for normal good) relationship between income and quantity demanded by consumer.
- **Inferior good: :** If for a good negative relationship between income and quantity demanded by consumer is observed then that good is called inferior good.
- **Homothetic Preferences:** A consumer’s preferences homothetic if marginal rate of substitution depends upon ratio of units of two goods and not on total quantities of goods.
- **Quasi-linear Preferences:** Quasi-linear preferences are non-homothetic preferences and have zero income effect.

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