# COST FUNCTIONS

## Learning Outcomes:

After having gone through this chapter, the following concepts should be clear:

1. Profits
2. Revenue (Total, average, marginal)
3. Costs (Total, Average, Marginal)
4. Long Run and Short Run costs
5. Cost Curves
6. Cost Functions and their properties
7. Deriving cost functions from production functions
8. Elasticity of input substitution
9. Contingent demand for inputs and Shephard's Lemma

# 1. Firm Behaviour

In the last chapter we learnt to describe the production process through production functions. We can now turn our attention to how firms make their choices regarding two fundamental issues:

1. how much to produce
2. how to produce it, i.e., what combination of inputs to use in production.

The answer to both these questions is related to the objective we ascribe to the firm. We generally believe that the typical firm acts to **maximize profits.** In other words, the typical firm chooses its output level and the combination of factors such that its profits are maximum. Is this a realistic assumption? Do firms always maximize profits? Could firms choose to behave differently? All these are questions that bear thinking about, but profit maximization is a reasonable assumption to start with. Presumably, firms carry out production to make profits. It is then reasonable to assume that they would wish to earn the maximum profits possible. How can they go about doing this? In order to understand this let us first start by defining profits.

**Profits** are the difference between the total revenue earned by the firm by selling its output, and the total cost incurred by the firm to produce this output.

Thus Profit (п) = Total Revenue (TR) – Total Cost (TC)

Before we can analyze firm behaviour, we need to study both the concepts of revenue and cost in some detail.

# 2. Revenues of the Firm.

The **total revenue (TR)** earned by the firm is simply the quantity the firm sells, multiplied by the price the firm receives for each unit sold.

**TR = P × Q**

where P is the price, Q the quantity sold. The **Total revenue curve** plots the total revenue against the quantity. Notice that this will be an upward sloping curve through the origin with slope P. If P is constant, this curve is a straight line.

Question: Usually, given that demand curves are downward sloping, consumers will buy more only if P falls. What shape does the total revenue curve take if P falls as Q increases?

We can then define the **average revenue (AR)** as the per unit total revenue, which is nothing but the price the firm receives for each unit sold.

**AR = TR/Q = (P×Q)/Q = P**

Recall that the price the producer receives per unit will be the price that the consumer pays per unit of output. Since the consumer will be willing to buy each successive unit of output only at a lower price, the average revenue will fall for each successive unit sold. The **Average Revenue curve,** which plots the average revenue against quantity, is nothing but the consumer's demand curve.

Typically we are interested in how the total revenue changes for each *incremental* unit sold. This is called the **Marginal Revenue (MR)** of the firm.

$$\mathbf{MR = \frac{\Delta TR}{\Delta Q}}$$

For infinitesimally small changes in output, the marginal product is the first differential of the total revenue function.

$$\mathbf{MR = \frac{dTR}{dQ}}$$

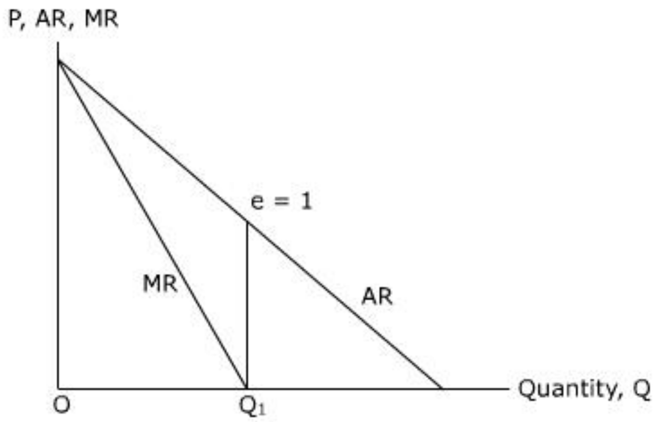
In other words, the slope of the total revenue curve is marginal revenue. If price is constant, note that the Marginal revenue will also be constant. If the price falls as the quantity sold increases, (because consumers will buy an additional unit of output only at a lower price) the marginal revenue will also fall as the quantity increases. In fact we can derive a specific relationship between the marginal revenue, the average revenue and the price elasticity of demand.

$$\mathbf{TR = P \times Q}$$

$$\mathbf{MR = \frac{d\,TR}{dQ} = P + Q\frac{dP}{dQ} = P\left(1 + \frac{Q}{P}\frac{dP}{dQ}\right) = P\left(1 + \frac{1}{e}\right)}$$

where e is the price elasticity of demand. Remember that the price elasticity of demand is negative, so MR is less than P (and therefore the AR) for e>0. Notice that for e=1, the marginal revenue will be zero. Using this relationship, we can easily draw the MR curve corresponding to a straight line demand curve. In the diagram below, we know that the price elasticity of demand equals 1 at the quantity $Q_1$, since it is the centre of the straight line demand (AR) curve. Notice that MR = 0 at $Q_1$.

**FIGURE 1: RELATION BETWEEN AR, MR: EXAMPLE**




Question: If the demand curve for chocolates is given by Q = 280 – 2P, where Q is the number of bars of chocolate sold per day, and P is the price of each bar of chocolate, write the equations for the total revenue and the marginal revenue from the sale of chocolates. Draw the average and marginal Revenue curves. At what quantity does MR become zero?

## 3. Costs of firms

The economic concept of costs includes all costs that the firm incurs to produce output. Some of these costs are incurred to purchase various inputs. The firm must pay wages to the labour it employs. If it hires land or buildings, it must pay rent on those. If it purchases machinery, it must spend some resources. These out-of-pocket expenses are called *explicit* costs (often called **accounting** costs). In addition to these explicit costs, the firm will also incur several *implicit* costs, i.e., the firm may use factors without explicitly paying for them. For example, if the firm owns the land and building in which it operates it does not pay rent. If the firm-owner performs labour in his own firm, he may not pay himself any wages. But the use of these factors does have an opportunity cost. (The firm could have rented out its land and buildings to another firm, the firm-owner could have found employment that did pay him wages etc.) This opportunity cost is also included in the total costs of production. Thus total cost includes both explicit and implicit costs. This is the underlying difference between 'accounting' and 'economic' costs. Accounting costs include only explicit costs, and economic costs include both explicit and implicit costs.

One very important category of implicit costs is the implicit cost of entrepreneurship. This is called the **normal profit.** A useful way to think about normal profits is to think of them as the minimum level of profits required to keep the entrepreneur in the current firm. In other words, if the entrepreneur does not receive at least the normal profits, he will shut the firm down, for alternative activity. Notice that this is the same as the opportunity cost of entrepreneurship. Thus normal profits are also a part of the total cost.

In general, a rational producer will not like to incur any avoidable costs. Thus while discussing costs we are actually concerned with the **minimum** levels of cost that the producer must incur to produce a certain level of output, given the prices of inputs.

To summarize then, **total costs are the sum of the minimum explicit and implicit costs (including normal profits) incurred by a firm in producing a specified level of output, given technology and the set of input prices.**

It is reasonable to expect that the total cost increases as the quantity produced increases. The functional relation between the total cost (C) and the quantity produced (Q) is called the **cost function** of the firm.

Analogous to case of total product and total revenue, we can define the average and marginal costs as follows:

Average cost is the per unit cost incurred in production.

**Average cost (AC) = Total cost (TC) / quantity (Q)**

Marginal cost (MC) is the increment in total cost for an increment in quantity.

**Marginal cost (MC) = Change in total cost (Δ TC)/change in quantity (ΔQ)**

For infinitesimally small changes in output, the marginal cost is the first differential of the total cost function.

$$\mathbf{MC = \frac{dTC}{dQ}}$$

## Costs with one input

If we have only one input (say, labour, L), recall that our production function was described as

**Q = f(L)**

Since labour is the only input, the total cost comprises whatever is spent on labour. If each unit of labour has to be paid a wage w, then the total cost (C) of producing is simply w times L. Notice that the relation between C and Q is going to depend on the production function. How much Q is obtained from a particular level of input is defined by the production function, how much it costs to produce that amount of Q depends on wages the firm must pay, and of course, the amount of L used (and hence Q produced). For the moment, we assume that the wages are given to the firm exogenously. Let us return to the second example we explored in the last chapter.

Our production function was:

**Q= (L- ½)/2**

Let us assume that w=10 (Since I am collecting the honey myself, notice that this is an implicit cost. If I were working for someone else, this would be an explicit cost. Either way, our calculation of the total cost does not change)

| Input (Labour in hours, L) | Total Product (Jars of honey, Q) | Total cost C =w× L | Average cost (C/Q) | Marginal Cost (ΔC/ΔQ) |
|---|---|---|---|---|
| 1 | 0.25 | 10 | 40.00 | |
| 2 | 0.75 | 20 | 26.66 | 20 |
| 3 | 1.25 | 30 | 24.00 | 20 |
| 4 | 1.75 | 40 | 22.87 | 20 |

We can derive the total cost function from the production function formally as follows:

$$\mathbf{Q} = \frac{\mathbf{L} - \frac{1}{2}}{2}$$

$$\mathbf{2Q} = \mathbf{L} - \frac{1}{2}$$

$$\mathbf{L} = \mathbf{2Q} + \frac{1}{2}$$

$$\mathbf{C} = \mathbf{w} \times \mathbf{L} = \mathbf{w}\left(\mathbf{2Q} + \frac{1}{2}\right) = \mathbf{20Q} + \mathbf{5}$$

$$\mathbf{AC} = \frac{\mathbf{C}}{\mathbf{Q}} = \mathbf{20} + \frac{\mathbf{5}}{\mathbf{Q}}$$

$$\mathbf{MC} = \frac{\mathbf{dC}}{\mathbf{dQ}} = \mathbf{20}$$

Notice that this cost function has a constant slope (the marginal cost is constant). This is not surprising because the underlying production function had a constant slope. Each increment of labour was yielding a constant increase in output. Since wages are constant, each increment of labour also corresponds to a proportionate increase in costs.

What would happen to the *costs vis-à-vis* output if the production function was not linear? Intuitively it is easy to see that as the marginal productivity of a factor increases, an increment in the factor will lead to a proportionately greater increase in output. As long as the price of the factor remains unchanged, the cost of producing this *incremental* output will fall, suggesting that the marginal cost will fall. Similarly, we can see that when the marginal product falls, the marginal cost is likely to rise.
In general,

$$\mathbf{C} = \mathbf{wL}$$

$$\mathbf{Q} = \mathbf{f(L)}$$

$$\textbf{Let } \mathbf{g(Q)} = \mathbf{f^{-1}(Q)} = \mathbf{L}$$

$$\mathbf{C} = \mathbf{w\,g(Q)}$$

$$\mathbf{AC} = \frac{\mathbf{wL}}{\mathbf{Q}} = \frac{\mathbf{w}}{\mathbf{AP_L}}$$

$$\mathbf{MC} = \frac{\mathbf{dC}}{\mathbf{dQ}} = \mathbf{w}\,\frac{\mathbf{d\,[g(Q)]}}{\mathbf{dQ}}$$

We can see that C will increase as L increases, and by corollary, Q increases. Notice that in this case, average cost is C/Q, and is inversely related to the average product. Notice also that d[g(Q)]/dQ, and hence the marginal cost, will fall when dQ/dL rises and vice versa. So the shape of the cost curve will be derived from the shape of the production

function. This is important to remember. It is easy to see that the shape of the cost curves will 'invert' the shape of the production function.
The relationship between the marginal cost and the average cost is easy to establish. We know that when the average cost is minimum, d(AC)/dQ will be zero. When the average cost rises, d(AC)/dQ must also increase, and vice versa.

**AC = C/Q**

$$\frac{\mathbf{d(AC)}}{\mathbf{dQ}} = \frac{\mathbf{d}}{\mathbf{dQ}}\left(\frac{\mathbf{C}}{\mathbf{Q}}\right)$$

$$= \frac{\mathbf{Q}\left(\frac{\mathbf{dC}}{\mathbf{dQ}}\right) - \mathbf{C}\left(\frac{\mathbf{dQ}}{\mathbf{dQ}}\right)}{\mathbf{Q^2}}$$

$$= \frac{\mathbf{Q(MC) - C}}{\mathbf{Q^2}}$$

$$\frac{\mathbf{d(AP)}}{\mathbf{dQ}} = \mathbf{0},$$

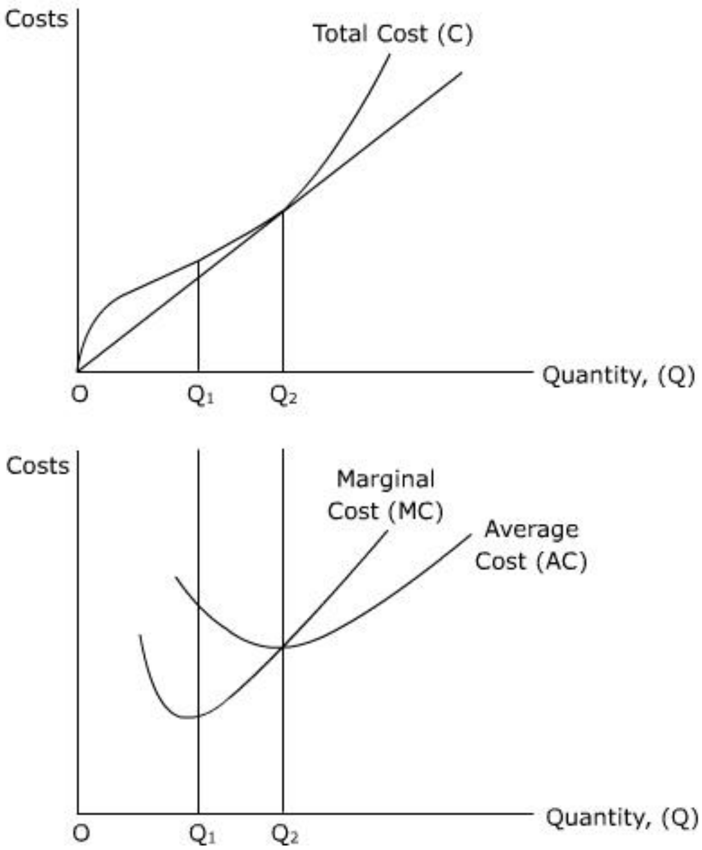
**if** $\mathbf{Q(MC) - C = 0}$

**or, MC = C/Q =AC**

Thus it follows that the marginal cost is equal to the average cost when the average cost is minimum. Notice also, if d(AC)/dQ>0 (AC is increasing), MC>AC and *vice versa.*

**Cost curves with one input**
Cost curves plot cost functions in the C-Q space. It is conventional to depict the quantity produced on the X-axis and the cost on the Y-axis.
The **total cost curve** is upward sloping since costs increase as output increases. We can see from above that total cost curve will become flatter as long as costs rise slower than output. On the other hand, it will become progressively steeper as output increases, if costs are rising faster than output. Correspondingly, the **marginal cost curve** will first fall, and then begin to rise. The **average cost curve** in this case is 'U' shaped, falling initially and then rising, corresponding to the behaviour of total costs *vis-à-vis* output. The relation between the marginal cost and average cost is analogous to the relation between the average product and the marginal product we explored in the last chapter. While the average cost curve is falling, the marginal cost lies below it. While the average cost rises, the marginal cost curve lies above it. The marginal cost curve intersects the average cost curve at the minimum of the average cost curve.
In the diagram below, the total cost rises *slower* than output till $Q_1$, so the total cost curve gets flatter. For this range of output, the marginal cost (which is the slope of the total cost curve) falls. Beyond $Q_1$, the total cost starts rising faster than the output. So the marginal cost increases. The average cost may be represented by the slope of the ray through the origin to the total cost curve. This is minimum at $Q_2$, indicating that the average cost is the minimum at this level of output, as well as equal to the marginal cost..

**FIGURE 2: COSTS WITH ONE INPUT**




## Costs with more than one input

Much of the logic of the discussion above holds in the case of production with more than one input. Let us go back to the production function with two factors. Let L be the amount of labour used, and K the amount of capital used in production. We denote the price that the firm pays for each unit of labour by w, and the price paid for each unit of capital by v. Then the output is defined by

$$\mathbf{Q} = \mathbf{f}(\mathbf{L}, \mathbf{K})$$

and $\mathbf{C} = \mathbf{C}(\mathbf{Q}, \mathbf{w}, \mathbf{v}) = \mathbf{C}[\mathbf{f}(\mathbf{L}, \mathbf{K}), \mathbf{w}, \mathbf{v}] = \mathbf{wL} + \mathbf{vK}$

Before we formalize this discussion it is useful to make the difference between fixed costs and variable costs. As the name suggests, **total fixed costs** are cost that do not change as output increases. These are costs associated with the fixed input. Plant and machinery would be an example. Once a plant has been set up, and machinery installed, output can be increased (up to the capacity of the plant) without incurring any more costs for plant and machinery. On the other hand, **total variable costs** are associated with variable inputs, and change as the quantity produced changes. Labour costs are an example. Given installed machinery, any increase in production would require employment of more labour. Costs would increase as the amount of labour employed increases. Remember that we defined the short run as a scenario in which at least one factor is fixed, and the long run as one in which all factors are variable. So, in the long run, all costs are variable costs.

As before, the production function determines how much output is produced by using a certain quantity of labour and capital. We continue to define the relation between the total cost and the quantity produced as the cost function. Notice that the total variable cost (TVC) is going to be an increasing function of Q, as the total variable cost increases with increased production. The total fixed cost will be independent of Q.
We define the **average fixed cost (AFC)** as the total fixed cost per unit output produced.

**AFC = TFC/Q**

We define the **average variable cost (AVC)** as the total variable cost per unit output produced.

**AVC = TVC/Q**

We define the **marginal cost (MC)** as the increment in total cost per unit *change* in output produced.

$$\mathbf{MC} = \frac{\mathbf{dC}}{\mathbf{dQ}} = \frac{\mathbf{d(TVC + TFC)}}{\mathbf{dQ}} = \frac{\mathbf{d(TVC)}}{\mathbf{dQ}}$$

since by definition, $\frac{\mathbf{d(TFC)}}{\mathbf{dQ}} = \mathbf{0}$.

**COST CURVES WITH TWO INPUTS: SHORT RUN COST CURVES**

**Total cost curves:** What is the difference in costs when we have more than one input as compared to only one? Notice that the basic difference is in terms of the existence of fixed costs. As discussed above, the total fixed cost is constant with respect to the quantity produced. So the total fixed cost curve is a straight line parallel to the quantity axis. What about the total variable cost curve? The shape of this curve will mirror the law of variable returns as they occur in the production function. In the case under discussion, the total variable cost curve will be upward sloping. It will grow flatter while increasing returns to labour exist, and steeper when decreasing returns to labour set in as labour used is increased (so that the quantity produced increases). The total cost curve is the vertical summation of the total fixed and variable cost curves. Notice that a change in the total fixed cost will cause a parallel shift in the variable cost curve.

**FIGURE 3: SHORT RUN TOTAL COST CURVES**

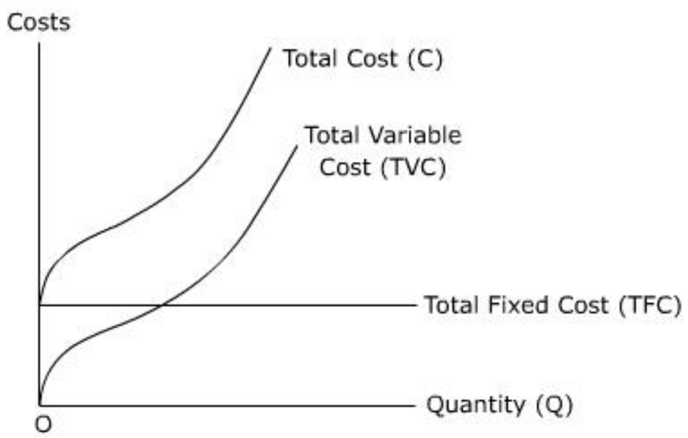
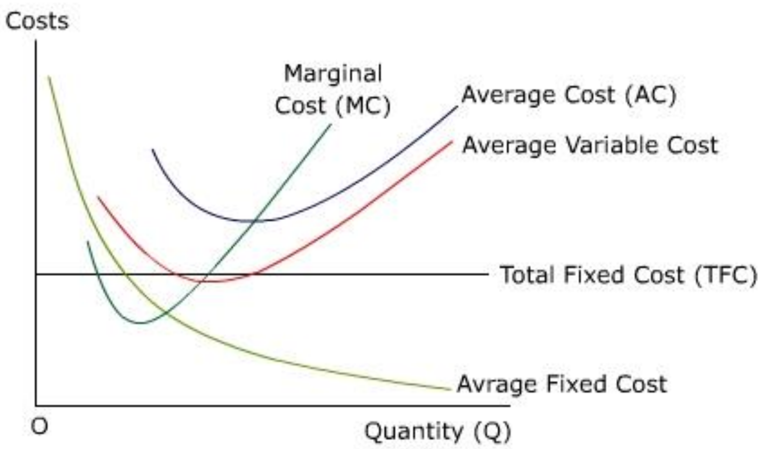


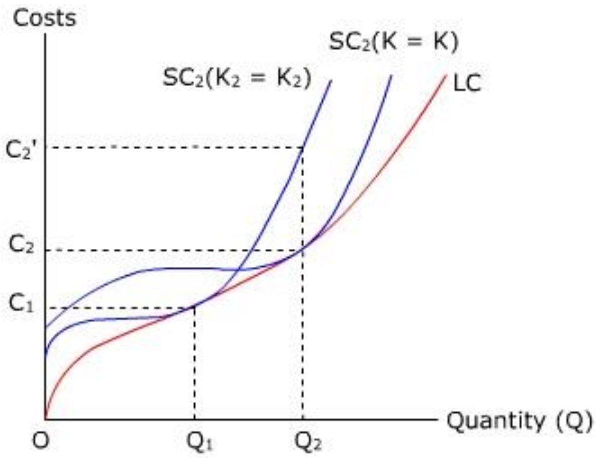
**Figure 4. SHORT RUN AVERAGE AND MARGINAL COST CURVES**




**Average and marginal cost curves:** The shape of the average cost curve from the shape of the corresponding total cost curves. Since the total fixed cost remains unchanged with quantity, notice that the AFC is infinitely high when the quantity produced is zero, or close to zero. As the quantity produced increases, the AFC falls continuously, getting progressively closer to zero for very large quantities of output. The average variable cost curve is 'u' shaped, in keeping with the law of variable returns discussed above. The average cost curve is the vertical summation of the AFC and the AVC curves. As the AFC approaches zero, the AC curve approaches the AVC curve.
The marginal cost curve lies below the AC and AVC curve while the AC and the AVC are falling, and above the AC and AVC curves while they are rising. It intersects both the AC and the AVC curves at their minima. See Figure 3 above.

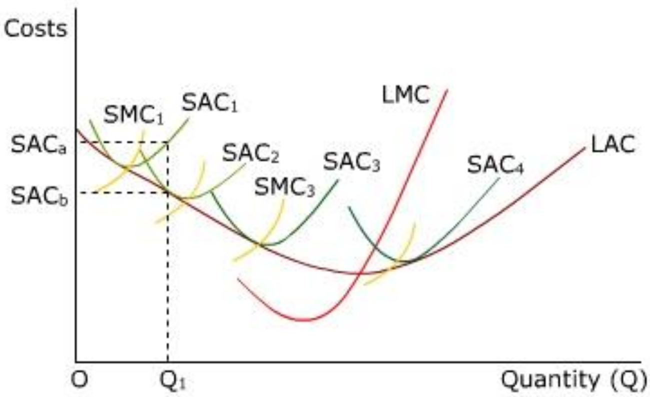
**LONG RUN COST CURVES:**
The long run is distinguished from the short run by the fact that all the inputs are variable in the long run. Thus the long run cost curves can be derived from the short run cost curves by examining the short run cost curves at each level of the fixed input. This is illustrated in the Figure 5 below. $SC_1$ and $Sc_2$ represent two short run total cost curves, corresponding to two different levels of capital (the fixed input) $K_1$ and $K_2$. Suppose the firm is producing $Q_1$ and wishes to increase its output to $Q_2$. Continuing with $K_1$ amount of capital implies that it can produce this output at a total cost of $C_2'$. However, increasing capital to $K_2$ allows it to move to the total cost curve $SC_2$, and incur a much lower cost of $C_2$. **The long run total cost curve (LC) is the 'envelop' of the short run average cost curves.**

**Figure 5: Long Run Total cost Curve**




We can derive the long run average and marginal cost curves by similar analysis. Any firm with a fixed input K continues to produce along its short run average cost (SAC) curve while it is optimal to work with the given level of K. When the total cost can be reduced by shifting to another level of K, the firm will 'jump' onto a new short-run average cost curve. The long run average (LAC) will envelop the short run average cost curves at various levels of K. The long run marginal cost curve (LMC) will relate to the long run average cost curve as discussed above. It will lie below the LAC while the LAC falls, and above the LAC while the LAC rises. It will intersect the LAC at its minimum. Figure 6 below illustrates this. Consider the quantity $Q_1$. The firm could produce it on $SAC_1$ at an average cost of $SAC_a$. However if it changes the amount of fixed input that it uses such that it can move to $SAC_2$, it can produce the same quantity at a lower average cost of $SAC_b$. In the short run the firm is 'tied' to the capital level implicit in $SAC_1$. In the long run it can move to a more efficient combination of inputs, resulting in lower average costs. In this sense the short run may be sub-optimal.

**Figure 6: Long Run Average and Marginal Costs**



# 4. PROFITS AND PROFIT MAXIMIZATION

A firm's economic profits may be defined as the difference between the total revenue and the total cost. To reiterate, we assume that the firm wishes to maximize profits. Thus the objective function that the firm operates with is as follows:

**Max π = Max (TR – TC)**

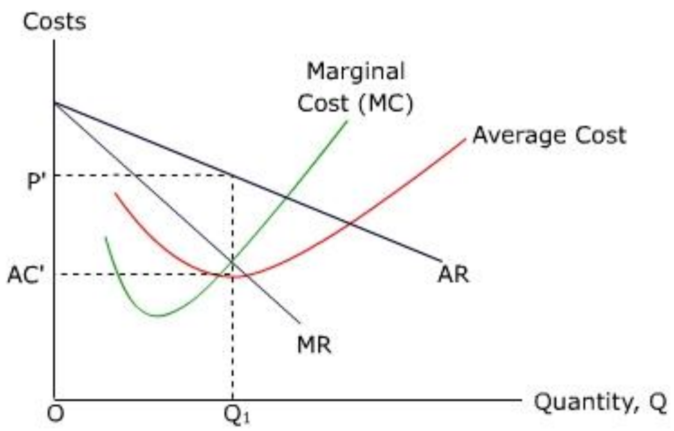
For any quantity Q that the firm produces, the firm will earn revenue of P×Q. The firm must then choose Q such that the profit that it earns is maximum. Using basic calculus we know that for profits to be maximum, the first derivative of the profit function is zero.

$$\frac{d\pi}{dQ} = \frac{d(TR - TC)}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ} = MR - MC$$

$$\frac{d\pi}{dQ} = 0 \Rightarrow MR - MC = 0 \Rightarrow MR = MC$$

**Thus a firm seeking to maximize profits will equate its Marginal Revenue to Marginal Cost**. Figure 4 below illustrates this graphically. A firm's cost curves and revenue curves are depicted in the diagram. the firm chooses to produce $Q^*$ since this if the quantity at which the MR and MC curves intersect. The price at which this quantity will sell can be read off the AR (demand) curve as $P^*$, the average cost of producing each unit is $AC^*$, which is read off the average cost curve. ($P^*$ - $AC^*$) is the profit per unit that the firm earns. The firm has now to produce $Q^*$ at the minimum possible cost. We will analyze the firm's profit maximization behaviour in a later chapter. Here we focus our attention on the cost minimizing behaviour of firms.

**Figure 7: Profit Maximization by firms**




**Cost-Minimizing Input choice:**

We wish to identify the combination of inputs, which given factor prices, and a specified level of output $Q^*$, will minimize costs for a firm. Notice that this is a constrained optimization problem:

Min C = wL + vK, given that Q = f(L,K)= $Q^*$

We can set up the Lagrangian expression for this problem as follows:

$$Z = wL + vK + \lambda(Q^* - f(L,K))$$

The first order conditions for this minimization are as follows:

$$\frac{\partial Z}{\partial L} = w - \lambda f_L = 0$$

$$\frac{\partial Z}{\partial K} = v - \lambda f_K = 0$$

$$\frac{\partial Z}{\partial \lambda} = f(L, K) - Q^* = 0$$

where $f_L$ and $f_K$ are the first derivatives of f(L,K) with respect to L and K respectively, (i.e., the marginal product of the respective input).

These first order conditions yield the equilibrium condition for cost minimization, which is:

$$\frac{f_L}{f_K} = \frac{w}{v} = \frac{MP_L}{MP_K}$$

It follows that a firm that wishes to minimize costs must employ inputs in a manner such that the ratio of their marginal products is equal to the ratio of their factor prices.

Recall that the ratio of the marginal products is the slope of the isoquants of the firm. We can use this property to identify the cost minimizing combination of factors graphically. In Figure 5 below, the optimal level of output is represented by the isoquant $Q^*$. The line $C_1C_1$, represents the locus of combinations of L and K which, given factor prices w, v, cost the same amount $C_1$. Such a line is called an **isocost** line. The expression for this isocost line is

$$C_1 = wL + vK$$

We can rearrange the above equation to write it as

$$K = \frac{C_1}{v} + \frac{w}{v}L$$

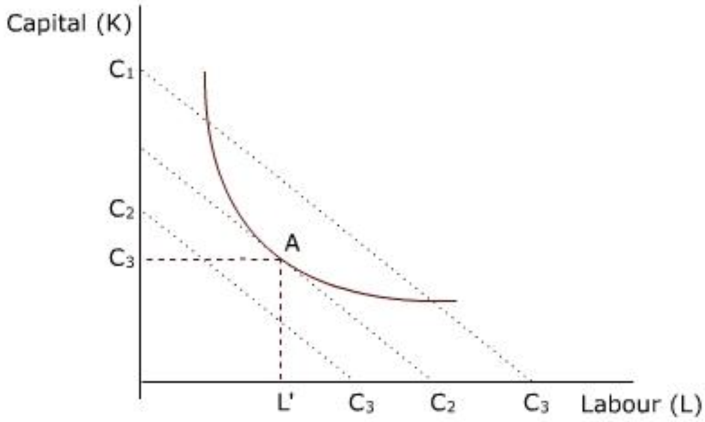
The slope of the isocost line is the ratio of the factor prices. Changing the total cost C will change the vertical intercept of the isocost line, without changing the slope. Thus different levels of total cost may be represented by a set of parallel isocost lines. Notice that the isocost line is analogous to the consumer's budget line with one important difference: in our framework the isocost line does not constrain the firms' choice, the profit maximizing level of output does.

Cost minimization requires us to identify the least cost which will allow us to produce the output $Q^*$. Notice that this is the cost that is represented by the isocost line just tangent to the isoquant (in this case, $C_2C_2$). Any other isocost line either represents a higher level of expenditure (e.g. $C_1C_1$) or is insufficient to allow $Q^*$ to be produced (e.g. $C_3C_3$). This tangency yields $L^*$ and $K^*$ as the optimal combination of inputs. Notice that this combination also satisfies our optimality condition

$$\frac{f_L}{f_K} = \frac{w}{v} = \frac{MP_L}{MP_K}$$

since tangency implies that the slope of the isocost line (w/v) equals the slope of the isoquant ($MP_L/MP_K$)

**FIGURE 8: COST MINIMIZATION: A GRAPHICAL REPRESENTATION**




From the above discussion it is easy to identify the cost-minimizing input combination for each level of output. The locus of all cost-minimizing input combinations with factor prices remaining unchanged is called the firm's **expansion path.** The shape of the firm's expansion path will depend on the shape of the isoquants. Figure 6 below illustrates this. In some cases, the expansion of output may lead to the decrease in the use of one factor yielding 'inferiority' of that factor. For example, at very high levels of output, technology may become capital 'intensive' in that the cost minimizing combination of inputs substitutes away from labour. This can yield a 'backward bending' expansion path such as that described in figure 7 below.

**FIGURE 9: A FIRM'S EXPANSION PATH**

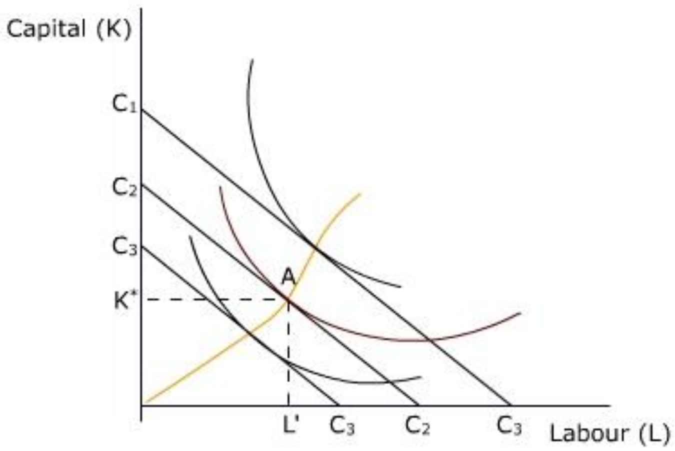
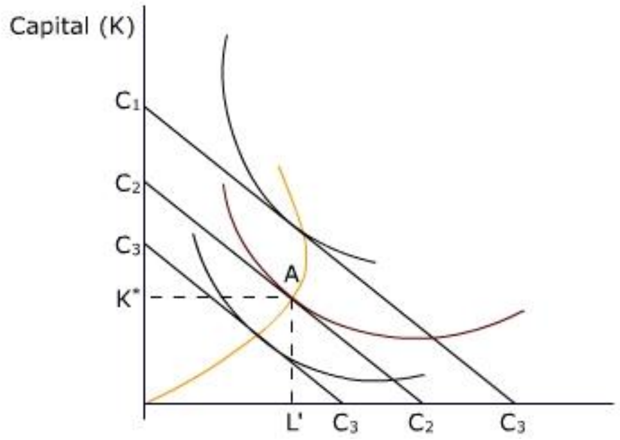


**FIGURE 10: THE FIRM'S EXPANSION PATH WITH LABOUR AS AN INFERIOR INPUT**



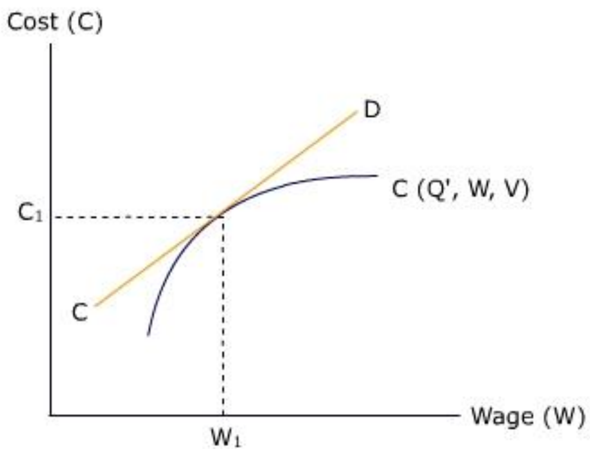

## PROPERTIES OF TOTAL COST FUNCTIONS

Once we have described the firm's optimum in terms of profit maximization and cost minimization, we can proceed to discuss some interesting properties of total cost functions.

1. **Total cost functions are non-decreasing in Q, w, v.** This follows from the fact that the total cost is the minimum cost of producing a particular level of output. Consider two levels of output $Q_1 < Q_2$, produced at the respective total costs $C_1$ and $C_2$. If $C_2 < C_1$, the firm could not have been minimizing costs at $Q_1$. (It could easily have produced $Q_2$ at a lower cost and discarded $Q_2 - Q_1$, thus obtaining $Q_1$ at $C_2 < C_1$.
   Similarly, consider a situation where a firm is using factors in the combination $L_1$, $K_1$. Now w increases from $w_1$ to $w_2$, but the total cost decreases from $C_1$ to $C_2$. An increase in wages should induce the firm to substitute away from labour to a new combination of factors $L_2$, $K_2$ . This costs $C_2 < C_1$. By definition then, $L_1$, $K_1$ cannot have been a cost minimizing input combination ($L_2$, $K_2$ costs less). So $C_2$ cannot be less than $C_1$. A similar argument can be made for increases in v.
   A formal proof of this property uses the envelop theorem.
2. **Total cost functions are homogenous in input prices.** This is easy to see from Figure 5 above. The total cost is $C_1 = wL^* + vK^*$. If both w,v are changed in the same proportion, the slope of the isocost line does not change, and factors continue to be used in the proportion $L^*$, $K^*$. If w and v have gone up by the proportion $\lambda$, the new total cost $C_2$ must be
   $$C_2 = \lambda wL^* + \lambda vK^* = \lambda(wL^* + vK^*) = \lambda C_1$$
3. **Total cost functions are concave in input prices:** Let us return to Figure 5. Suppose we increase one of the input prices, say w, while holding v and $Q^*$ constant. Notice that this will cause the isocost line to swivel inwards along the labour axis. The firm will produce $Q^*$ using less labour and more capital. What implication does this have for the total cost vis-à-vis w? Figure 8 below describes this. $C_1$ is the cost corresponding to the initial wage $w_1$. If the firm did not change the amount of labour it employs in response to a change in w, its total cost would increase in proportion to the change in wage, along the line CD. However, the cost minimizing firm *does* change the amount of labour it uses, so that the actual

cost incurred lies below this line, yielding concavity of costs with respect to wages.

**Figure 11: Concavity of costs with respect to input prices**




**PROPERTIES OF AVERAGE AND MARGINAL COST FUNCTIONS:**

1. Both average and marginal costs are homogenous in input prices. This follows from the homogeneity of total costs in input prices.

$$\mathbf{AC(\ Q, \lambda w, \lambda v) = \frac{C(\ Q, \lambda w, \lambda v)}{Q} = \frac{\lambda\ C(Q, w, v)}{Q} = \lambda\ AC(\ Q, w, v)}$$

$$\mathbf{MC\ (\ Q, \lambda w, \lambda v) = \frac{dC(\ Q, \lambda w, \lambda v)}{dQ} = \frac{\lambda\ dC(Q, w, v)}{dQ} = \lambda\ MC(\ Q, w, v)}$$

2. In general, neither the AC nor the MC are non-decreasing in Q. We have already seen that they have negatively sloped portions. However, if the input prices rise, total cost cannot decrease, so average cost cannot decrease either. Thus average costs are non-decreasing in input prices. By and large, marginal costs are also non-decreasing in input prices. The important, though rare, exception to this is when an input is inferior. In that case, the increase in the price of the inferior good may lead to an reduction of the marginal cost

**INPUT SUBSTITUTION:**

Any change in the ratio of input prices w/v will cause the slope of the isocost line to change, and hence the ratio in which the inputs are used (i.e., K/L) will also change. The change in K/L relative to a change in w/v *along an isoquant* is called the **elasticity of input substitution (s)**

$$\mathbf{s = \frac{\%\ \Delta(K/L)}{\%\Delta(w/v)}}$$

$$\Rightarrow \mathbf{s = \frac{\partial(K/L)}{\partial(w/v)} \cdot \frac{w/v}{K/L} = \frac{\partial \ln(K/L)}{\partial \ln(w/v)}}$$

A large elasticity of substitution implies that a small change in w/v causes a large change in K/L. Obviously if w rises relative to v, we expect the firm to use more capital (labour has become relative more expensive.) s will therefore have a positive sign. A smaller s implies the firm is unable to 'shift away' from the use of an input whose price is rising with ease. In the limiting case, s = 0 implies the firm continues to use labour and capital in the same ratio regardless of the relative price of the two.

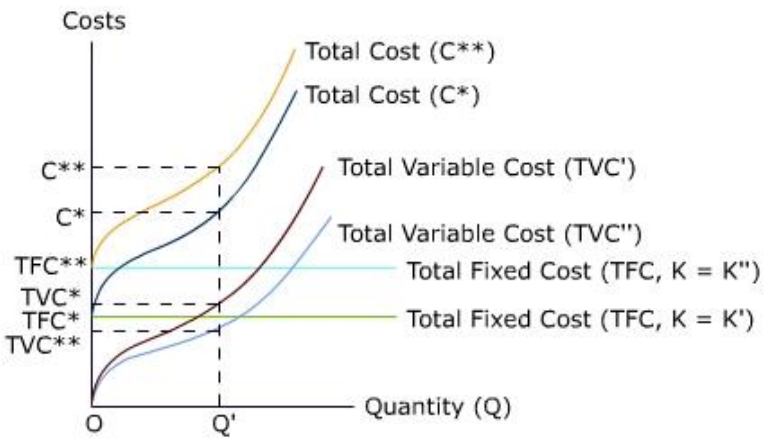
Question: What kind of isoquants are likely to have s=0? s=∞?

Why is the elasticity of input substitution important to us? Let us return to Figure 5. Remember the total cost of producing the quantity $Q^*$ is the cost of using $L^*$ and $K^*$. In terms of our cost curves, (as shown in figure 2), this translates into a total fixed cost of $vK^*$ and a variable cost of $wL^*$ for an output $Q^*$ Now assume that w goes up to w', while v remains unchanged, changing the slope of the isocost line, and thereby the cost minimizing input combination to $K^{**}$ (> $K^*$), $L^{**}$(<$L^*$). In terms of our cost curves, this means a higher fixed $vK^{**}$ to produce the same output; the impact on the variable cost is more difficult to guess a priori. w has gone up, L has gone down. The net impact in any case, is a shift in the total cost curve. The magnitude of this shift clearly depends on the ability of the firm to substitute now relatively cheaper capital for the now relatively more expensive labour.

Figure 9 below illustrates one such possibility. In this case total cost of producing $Q^*$ has gone up from $C^*$ to $C^{**}$ as a result of an increase in wages from w to w'. The firm responds to the change in input price by increasing the amount of capital it uses, resulting in an increase in the total fixed cost from $TFC^*$ to $TFC^{**}$. In our diagram, the total variable cost at the new wage rate is $TVC^{**}$, which is lower than the original $TVC^*$. However, the decrease in TVC is clearly not enough to offset the increase in TFC, resulting in higher total cost of producing $Q^*$. The important thing to remember is that with the change in relative prices, $Q^*$ is no longer going to be the profit maximizing level of output anyway. A change in w will have generated a change in MC, and therefore a new level of profit maximizing output.

These concepts will become clearer after going through the examples below.

**FIGURE 12**



**Example 1:**

Consider a firm that produces footballs using two inputs L and K according to the production function $Q = 2L^{1/2}K^{1/2}$. The firm has a fixed amount of K in the short run at a level K = 100. The price of L is given by w = 4, and the price of K is given by v = 1.

**With K = 100, the short run costs of the firm may be derived from its production function as follows:**

$Q = 2L^{1/2} (100)^{1/2}$

$= 20L^{1/2}$

It follows that $L = Q^2/400$

The Short Run Total Cost for this firm is:

$C = wL + vK$

$= 4L + 1(100)$

$= 4L + 100$

The Short Run Total Cost Function can be written as:

$= 4(Q^2/400) + 100$

$= Q^2/100 + 100$

where the Total fixed cost is 100, the total variable cost is $Q^2/100$

The Short Run Average Cost may be described by C/Q

SAC = Q/100 + 100/Q

SAFC = 100/Q

SAVC = Q/100

The marginal cost in this case is

$MC = \partial C/\partial Q = Q/50$

For K = 100, we could write the short run cost function for this firm generally in terms of w and v as

$C = wQ^2/400 + v100$

**In the long run,** K may vary**.** Long run costs must be such that for any K, total costs are minimum.

Since $Q = 2L^{1/2}K^{1/2}$ it follows that $L = Q^2/4K$

The total cost may be written in terms of K as

$C = wL + vK = w(Q^2/4K) + vK$

A firm must choose K so that the total cost for any level of output is minimum. To identify this level of K we set $\partial C/\partial K = 0$

$\partial C/\partial K = - [(wQ^2)/(4K^2)] + v = 0$

It follows that $K = (Q/2)(w/v)^{1/2}$

Substituting this value of K into our total cost we get the long run total cost function for this firm:

$C = Qw^{1/2}v^{1/2}$

Once again, Notice that any increase in Q, v, w will increase the total cost. Assure yourself that C is concave in w as well as v.

Notice that the Long Run Average cost for this firm is:

$LAC = C/Q = w^{1/2}v^{1/2}$ which is constant for given values of w, v

The long run marginal cost is:

$LMC = \partial C/\partial Q = w^{1/2}v^{1/2}$ which is also constant for given values of w, v

**Example 2:**

This example explores a more direct derivation of the cost function from the production function. Suppose a firm has a production function described by $Q = 2L^{1/4}K^{1/4}$. As before, the input prices of L and K are w and v respectively. The firm uses L and K to minimize the cost of producing any quantity $Q_0$. This firm's cost function can then be derived from its constrained optimization exercise. The Lagrangian corresponding to this problem may be written as:

$Z = wL + vK + \lambda(Q_0 - 2L^{1/4}K^{1/4})$

The first order conditions to minimize Z are

$\partial Z/\partial L = w - \lambda 2L^{-3/4}K^{1/4} = w - \lambda Q/L = 0$

$\partial Z/\partial K = w - \lambda 2L^{1/4}K^{-3/4} = w - \lambda Q/K = 0$

$\partial Z/\partial \lambda = Q_0 - 2L^{1/4}K^{1/4} = 0$

Solving these yields cost-minimizing values of L and K as follows:

$w/v = K^*/L^*$ (from the first two equations; the stars indicate that these are cost minimising values of K and L.)

- ⇨ $wL^* = vK^*$
- ⇨ $L^* = vK^*/w$

Substituting back into the production function yields L and K in terms of w,v and Q:

$L^* = (¼)Q^2v^{1/2}w^{-1/2}$

$K^* = (¼)Q^2v^{-1/2}w^{1/2}$

- ⇨ $C = wL + vK = (1/2)Q^2v^{1/2}w^{1/2}$

The corresponding average cost and marginal costs may easily be computed as:

$AC = C/Q = (1/2)Qv^{1/2}w^{1/2}$

$MC = \partial C/\partial Q = Qv^{1/2}w^{1/2}$

**CONTINGENT DEMAND FOR INPUTS**

From the above discussion it is clear that the typical profit maximizing firm will choose the amount of input it employs in order to minimize its costs at the profit maximizing level of output. It will vary the amount of each input it employs in response to a change in the price of that input depending on the possibility for substitution allowed by the production function. The contingent demand for inputs captures this concept. The contingent demand for an input relates the demand for that input to the price of the input, while keeping the quantity at the profit maximizing level (hence the demand is 'contingent' on the quantity produced.)

Let us see how the cost function may capture this information. Let us go back to **Figure 8.** The question is how does the total cost change if the price of labour goes up? Let the initial price of labour and capital be $w_1, v_1$ respectively. As before, let the (profit-maximizing) level of output be $Q^*$ and the cost minimizing level of inputs be $L^*$ and $K^*$. Then the total cost is $C_1 = w_1L^* + v_1K^*$. If the price of labour now goes up to $w_2$ (>$w_1$), then, assuming that nothing else changes, i.e., Q,L,K,v all remain unchanged, the new cost will now be $C_2 = w_2L^* + v_1K^*$. Notice that in this case, $\partial C/\partial w = 1$. But $L^*$ and $K^*$ are no longer the cost minimizing input combination. What is the new level of L demanded? We can answer this question by using the **envelope theorem.** This theorem states that in the case of constrained optimization, the first differential of the Lagrangian expression with respect to each parameter yields the maximal values for the corresponding variable. In our case, we wish to minimize $C = wL + vK$ subject to $Q= f(L,K) = Q^*$. The Lagrangian expression for this is:

$$\mathbf{Z = wL + vK + \lambda(Q^* - f(L,K))}$$

By the envelope theorem, then

$$\frac{\partial \mathbf{C}}{\partial \mathbf{w}} = \frac{\partial \mathbf{Z}}{\partial \mathbf{w}} = \mathbf{L^c(w, v, Q)}$$

$$\frac{\partial \mathbf{C}}{\partial \mathbf{v}} = \frac{\partial \mathbf{Z}}{\partial \mathbf{v}} = \mathbf{K^c}(\mathbf{w}, \mathbf{v}, \mathbf{Q})$$

where $L^c$ and $K^c$ indicate the contingent demand for labour and capital.
This result is known as **Shephard's Lemma**. To sum up, the Shephard's Lemma states that the contingent demand function for any input is given by the partial derivative of the cost function with respect to the price of that input. Example 3 below illustrates this.

**Example 3:**
Consider a firm whose cost function is given by $C = Qw^{2/3}v^{1/3}$. We can use Sheppard's lemma to compute the contingent demand for L and for K as below:
$L^c = \partial C/\partial w = (2/3)Qv^{1/3}w^{-1/3}$
$K^c = \partial C/\partial v = (1/3)Qw^{2/3}v^{-2/3}$
It is also possible to construct the firm's production function from here. Since $L^c$ and $K^c$ are contingent demands for the inputs, $(K^c, L^c)$ must lie on an isoquant for our firm.
$L^c = (2/3)Qv^{1/3}w^{-1/3} \Rightarrow v/w = (3/2)^3(L/Q)^3$
$K^c = (1/3)Qw^{2/3}v^{-2/3} \Rightarrow v/w = (3)^3(Q/K)^{3/2}$

$\Rightarrow (3)^3(Q/K)^{3/2} = (3/2)^3(L/Q)^3$
$\Rightarrow Q = 3^{1/3}2^{-2/3}L^{2/3} K^{1/3}$

# Exercises:

1. Suppose that a firm's production function is given by Q = min(5L, 10K)
    a. Calculate its (long run) total, average, marginal cost functions.
    b. If K is fixed at 100 in the short run, calculate its short run total, average, marginal cost functions.
    c. If w= 2 and v = 3, calculate the firm's short run and long run total, average, marginal cost curves.
2. Suppose that a firm's production function is given by $Q = \{\min(L, K)]^{1/3}$
    a. Calculate its (long run) total, average, marginal cost functions.
    b. If w=1 and v = 3, calculate the firm's short run and long run total, average, marginal cost curves.
3. Use Shephard's lemma to calculate the contingent input demand functions for a firm with the following cost function:

    $C = Q(w + 2w^{1/2}v^{1/2} + v)$
4. Use the result in Question 3 above to calculate the underlying production function.