Paper Name: Discrete Structures
Lesson: Growth Theory-1
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3.1 GRAPH TERMINOLOGY

3.1.1 Introduction

There are many real-life problems that can be abstracted as problems concerning sets of discrete objects and binary relation on them. For example, Consider a series of public-opinion polls conducted to determine the popularity of the Prime ministerial candidates. In each poll, voter's opinions are sought on two candidates, and a favorite is determined.

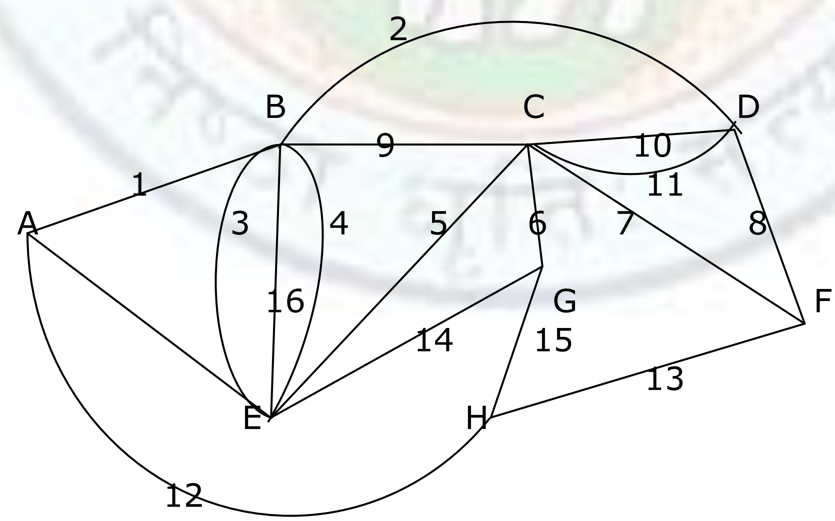
In another example, consider a number of cities connected by highways, and we might want to determine whether there is a highway route between two cities or not. Such problem is formulated and graphically represented in Graph theory. In mathematics and Computer Science, **graph theory** is the study of graphs: mathematical structures used to model pair wise relations between objects from a certain collection.

We will introduce some of the basic terminology of **Graph Theory** in this section. We will use this vocabulary when we solve many different types of problems. Some of the typical problem like, determining whether a graph can be drawn in the plane such that no two of its edges cross. Another example is deciding whether there is one-to-one correspondence between the vertices of two graphs that produces a one-to-one correspondence between the edges of the graphs. We will also introduce several important families of graphs often used as examples and in models.

Value addition:

Graph Theory

- Consider eight metro cities of India, which are connected by highways. Now we want to determine that whether there is a route between two given metros.



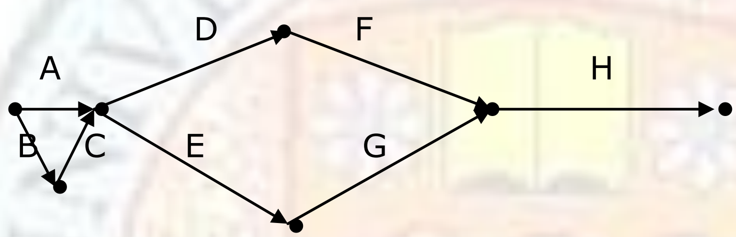
Fig(3.1.1)

Letters-> cities(vertices)
Numerals-> path(edges)
From the given graph we can easily determine all possible routes between two cities, and if the length of the edges is given then we can identify the shortest route.

●Let us consider another problem. Consider a project consisting of eight jobs(A,B,C,D,E,F,G,H) and we have given the sequence of completing jobs.

Jobs	A	B	C	D	E	F	G	H
Predecessors	-	-	B	A,C	A,C	D	E	F,G

Draw the graph (Project network)



Fig(3.1.2)
Nodes-Completion of jobs.
Edges-Jobs

The project network shows that there are four routes to complete the project. The possible routes are, a)A->D->F->H, b)A->E->G->H
c)B->C->D->F->H, and d)B->C->E->G->H.

If we have given the time taken to complete various job, then we can easily say that which route is optimal.

Source: By the author

3.1.2 Basic Terminology

In mathematics/Computer Science, a **graph** is an abstract representation of a set of objects where some pairs of the objects are connected by links. The interconnected objects are represented by mathematical abstractions called ***vertices***, and the links that connect some pairs of vertices are called ***edges***. Typically, a graph is depicted in diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges.

A Graph G consists of two things:

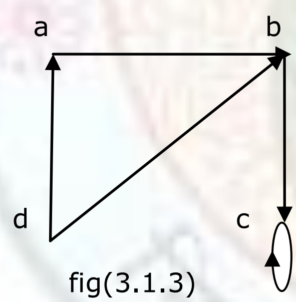
1) A set V=V(G), whose elements are called vertices, points, or nodes of G.
2) A set E=E(G) of ordered pairs of distinct vertices called edges of G.

So, A graph is in general represented by G(V,E)

Directed Graph: A directed graph G is defined as the set of marked points V with a set of arrows E between the points so that there is at most one arrow from one point to another points. So it is an ordered pair (V,E), where V is the set and E is a binary relation on V. Fig (1.3) is a directed graph.

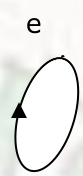
In directed graph, we say an edge (a,b) is incident from a and is incident into b. The vertex a is called **initial vertex** and b is called **terminal/end** vertex of (a,b)
An edge that is incident from and into the same vertex is called a **loop.** Fig(1.4)

A vertex is said to be an isolated vertex if there is no edge incident on it.



fig(3.1.3)

directed graph



fig(3.1.4)

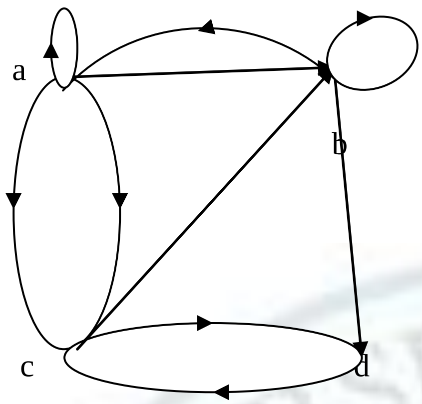
loop

A loop is an edge that connects a vertex to itself.

In directed graph the **in-degree** of a vertex v, is denoted by $deg^-(v)$, which is the number of edges with v as their terminal vertex i.e, number of edges incidence on it. The **out-degree** of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex, i.e, the number of edges incident from it.

A loop at a vertex contributes twice to the degree of that vertex.

Example 1: Find the in-deg and out-deg of each vertex of fig(3.1.5)



fig(3.1.5)

The in-deg of fig (3.1.5) are: $deg^-(a)=2$, $deg^-(b)=3$, $deg^-(c)=3$, $deg^-(d)=2$

The out-deg of fig (3.1.5) are : $deg^+(a)=4$, $deg^+(b)=3$, $deg^+(c)=2$, $deg^+(d)=1$

Theorem 1: Let G=(V,E) be a graph with directed edges, then

$$\sum_{v\in V} deg^-(v) = \sum_{v\in V} deg^+(v) = |E|$$

Proof: Since each edge has an initial and terminal vertex in directed graph, so the sum of the in-degrees and the sum of out-degrees of all vertices of the graph with direct edges are the same. Thus, $deg^-(v)=deg^+(v)$; for all V.

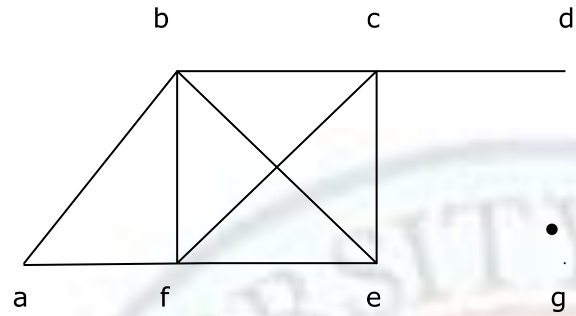
Also, both of these sums are equals to the number of edges in the graph as each edge has initial/terminal vertex.

There are many properties of the directed graph, which do not depends on the direction of its edges. Consequently, it is often useful to ignore these directions. The undirected graph that results from ignoring direction of edges is called the **underlying undirected graph.**

Undirected graph: An undirected graph G is defined abstractly as an ordered pair (V,E), where V is a set of vertices and E is the set of subsets of two elements from V. In other words we can define undirected graph as a set of marked points V with a set of lines E between the points. Fig(3.1.6) is an undirected graph.

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of vertex in undirected graph is denoted by deg(V).

A vertex of degree zero is isolated. It follows that an isolated vertex is not adjacent to any other vertex. Vertex g is isolated in fig(1.6). A vertex is pendant if and only if it has degree one. Vertex d is pendant in fig(1.6)



fig(3.1.6)

Theorem 2: The Handshaking Theorem Let G=(V,E) be an undirected graph with e edges, then

$$2e=\sum deg(V), \text{ for all } v \in V$$

(This is applicable even if multiple edges between two edges and loop are present.)

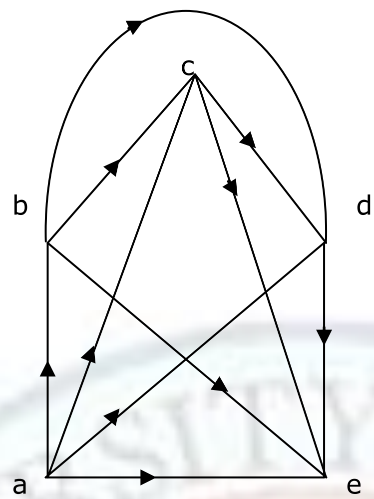
Example 2: How many edges are there in a graph with ten vertices each of degree seven?

Solution: Since the sum of degrees of the vertices is 7.10=70. It follows from handshaking theorem 2e=70

Therefore, e= 35

Example 3: Let V={a,b,c,d,e} be the five Cricket teams in a round-robin tournament. Let E={(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e)} be a binary relation on V, so that (x,y) in E means that x beats y in the match between them. Draw the graphical representation of the tournament.

Solution*:* In the fig (3.1.7) the alphabets(vertex) denote the cricket team and the edges between them denote that there was match between them and the direction describes, who beat whom.



fig(3.1.7)

Theorem 3: An undirected graph has an even number of vertices of odd degree.

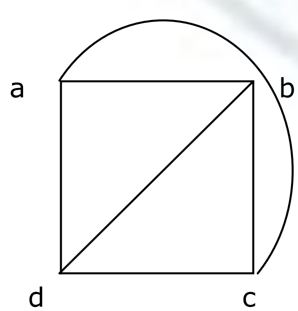
Solution: Let V_1 and V_2 are the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph G=(V,E) then,

$$2e=\sum deg(V) =\sum deg(V_1)+\sum deg(V_2)$$

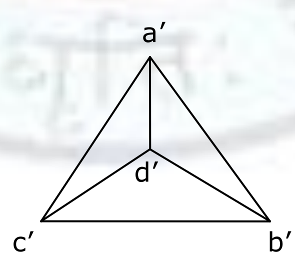
Proof: Since the deg(V) is even for $v \in V_1$, the first term in the right hand side of the last equality is even. Furthermore, the sum of the two terms on the right hand side of the last equality is even, since the sum is 2e. Hence, the second term in the sum is also even.

But according to the question , all the terms in the second term of the last equality are odd, so there must be an even number of such terms. Thus, there are an even number of vertices of odd degree.

Isomorphic Graph: Two Graphs are said to be isomorphic to each other if there is one to one correspondence between their vertices and between their edges such that their incidences are preserved. In other words, there is an edge between two vertices in one graph if and only if there is a corresponding edge between the corresponding vertices in other graph. Fig(3.1.8a) and fig(3.1.8b) are isomorphic.



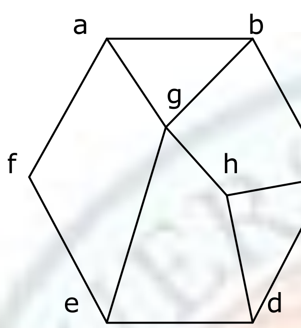
fig(3.1.8a)



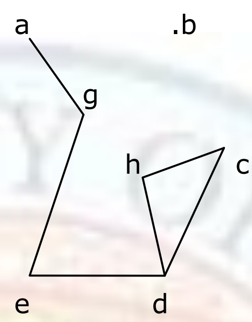
fig(3.1.8b)

Normally we do not distinguish between two isomorphic graph even though their diagram may look different.

Subgraph: Let G=(V,E) be a graph. A graph G′=(V′,E′) is said to be the sub graph of G, if E′ is the subset of E, and V′ be the subset of V, such that the edges of E′ are incident only with the vertices in V′. For example fig(3.1.9b) is the sub graph of fig(3.1.9a)



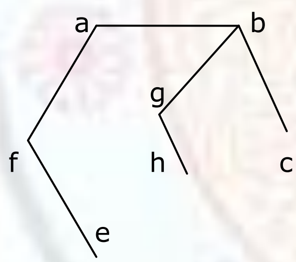
fig(3.1.9a)



fig(3.1.9b)

A Subgraph of G is said to be spanning subgraph if it contains all the vertices of G.

Complement of a Subgraph: Let G=(V,E) be a graph and G′=(V′,E′) be its subgraph. The complement of a subgraph G′=(V′,E′) with respect to the graph G is another subgraph G′′=(V′′,E′′) such that E′′=E-E′ and V′′ contain only the vertices with which the edges in E′′ are incident. Fig(3.1.9c) is the compliment of the fig(3.1.9b)



fig(3.1.9c)

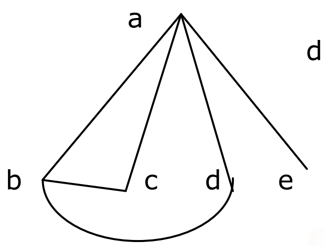
Value addition: Common Misconceptions

Graph Theory

When (u,v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex and v is called the terminal(end) vertex of (u,v)
The initial and terminal vertex of a loop are same.

- An ordered n-tuples ($d_1,d_2,d_3,\ldots\ldots,d_n$) of nonnegative integers is said to be **graphical** if there exist a linear graph with no self loops that has n vertices with the degree of the vertices being $d_1,d_2,\ldots\ldots,d_n$.

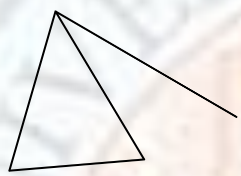
(4,3,2,2,1) is graphical, whereas (3,3,3,1) is not.



deg(a)=4, deg(b)=3, deg(c)=2, deg(d)=2, and deg(e)=1

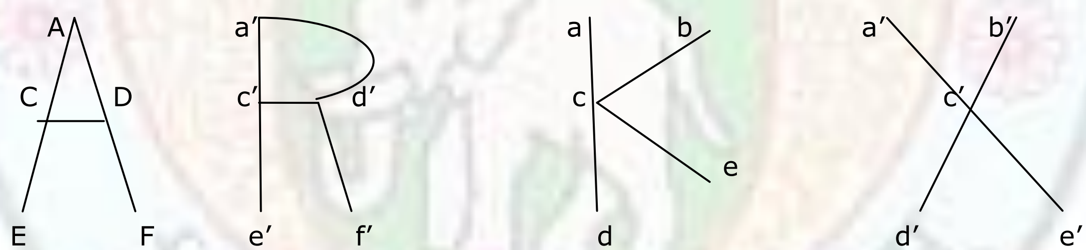
fig(3.1.10)

It is impossible to draw a graph with the degree (3,3,3,1) as one vertex has degree one so it must be pendant, and also we have three vertices with three degree so the maximum number of edges=5 (by theorem 3). The fifth edge cannot be drawn as it contradict the pendant nature of fourth vertex. Hence graph is not possible.



●When the graph is without multiple edges between two vertices and loops, we call it simple graph otherwise multi graph.

● Normally we do not distinguish between two isomorphic graph even though their diagram may look different. Take few examples.



A and R are isomorphic, and K and X are isomorphic.

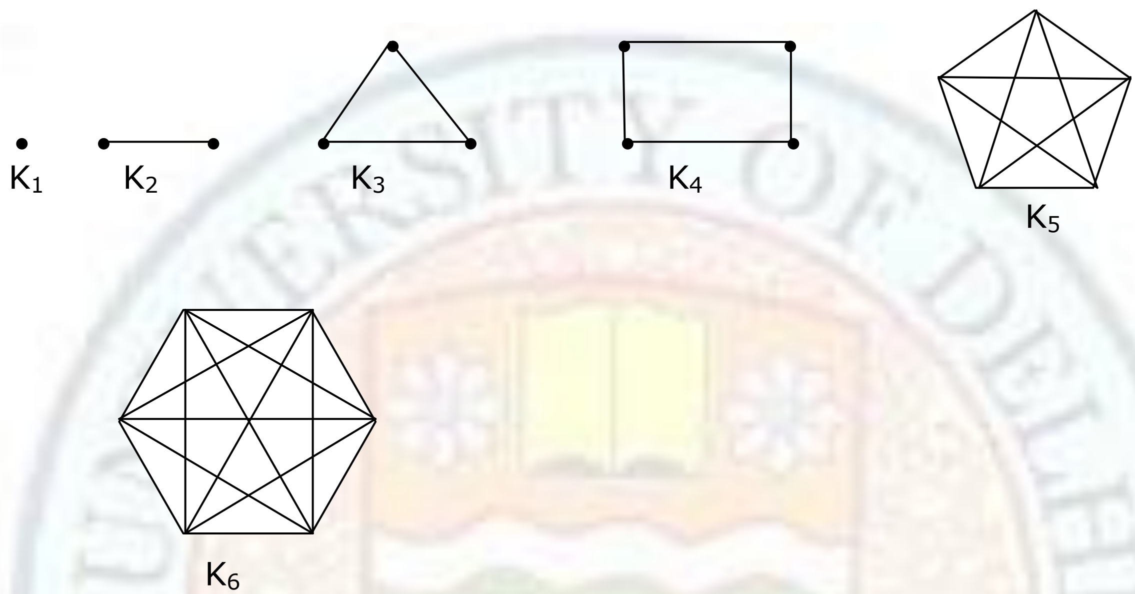
Source: By the author

3.1.3 Some Special Graphs

We will now introduce several classes of simple graphs. These graphs are often used as examples and arise in many applications. Some of the special simple graphs are Complete graphs, regular graphs, bipartite graphs etc.

3.1.3.1-Complete, Cycles and Wheel graphs:

Complete Graphs: A Graph G is said to be **complete** if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by K_n . The Complete graph K_n for n=1,2,3,4,5,6, is displayed in fig(3.1.11)



Fig(3.1.11)

Properties of Complete graph:

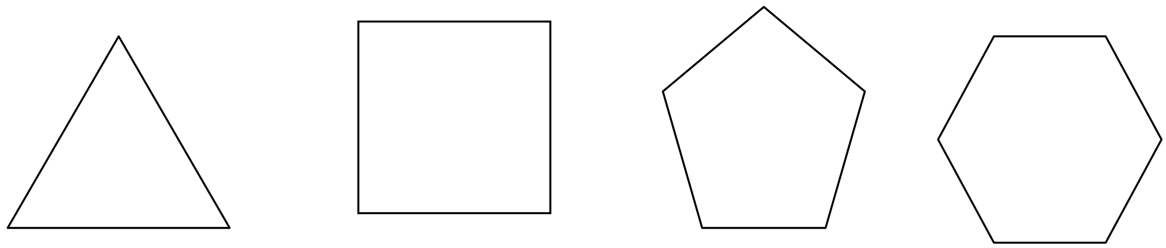
1) Each vertex is connected to every other vertices.
2) Let K_n be a Complete graph, then

   Number of vertices = n

   Number of edges= ${}^{n}C_2$ = n(n-1)/2

   For example K_5, number of vertices=5 and Number of edges=5.4/2=10
3) The degree of each vertex is n-1 in K_n

Cycles: The **Cycle C_n, n≥3**, consists of n vertices and edges $\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\}$, ...,$\{v_{n-1},v_n\}$, and $\{v_n,v_1\}$. This is denoted by C_n. The cycles C_3,C_4,C_5, and C_6 are displayed in fig(3.1.12)



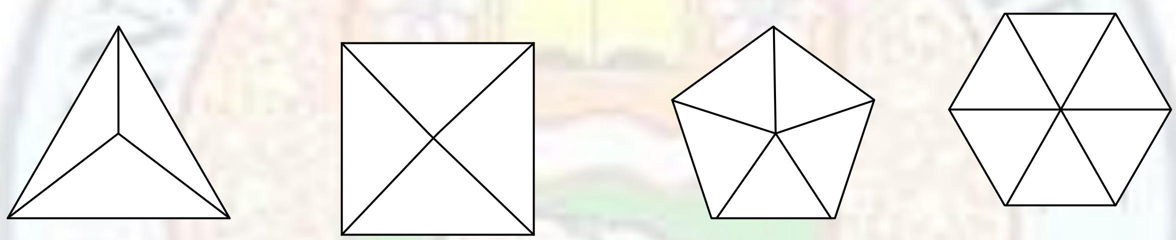
C_3 C_4 C_5 C_6

Fig(3.1.12)

Cycles are also known as 2-regular graph.

Number of vertices=number of edges = n in C_n.

Wheels : We obtain the **wheel W_n** ,when we add an additional vertex to the cycle C_n, for n≥3, and connect this new vertex to each of the n vertices in C_n, by new edges. The Wheels W_3,W_4,W_5, and W_6 are displayed in fig(3.1.13)



W_3 W_4 W_5 W_6

Fig(3.1.13)

Number of vertices = n+1, and

Number of edges=2n in W_n.

3.1.3.2- Regular and Bipartite Graphs :

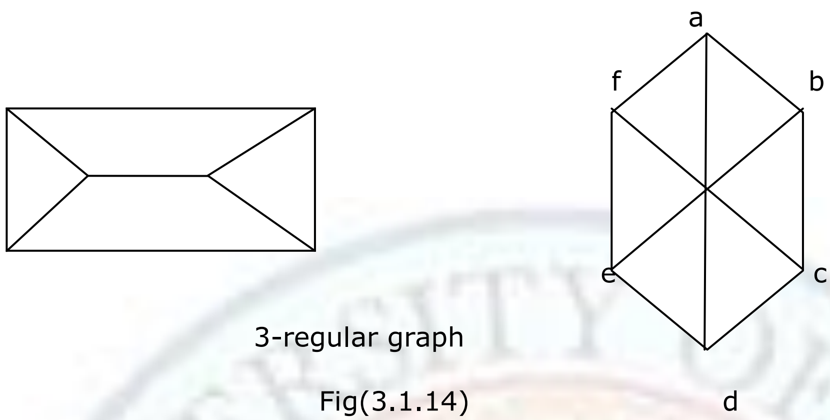
Regular Graph: A Graph G is **regular of degree k** or simply **k-regular** if every vertex has degree k. In other words a graph is regular if every vertex has same degree. The connected regular graph of degree 0,1, or 2 is easily described. See fig(3.1.14)

K_1= 0-regular, K_2=1-regular, C_n= 2-regular, (Symbol has specific meaning)

K_4=3-regular,

In general K_n= (n-1)-regular.

We must note that we have many regular graphs which are not complete.



3-regular graph

Fig(3.1.14)

Bipartite Graph: A Graph is said to be Bipartite if its vertices can be partitioned into two disjoint subsets M and N such that each edge of G connects a vertex of M to a vertex of N. By a **Complete Bipartite Graph,** we mean each vertex of M is connected to each vertex of N. This is denoted by $K_{m,n}$. Fig(3.1.15) display some Complete Bipartite Graph.

$K_{3,3}$

$K_{2,4}$

Fig(3.1.15)

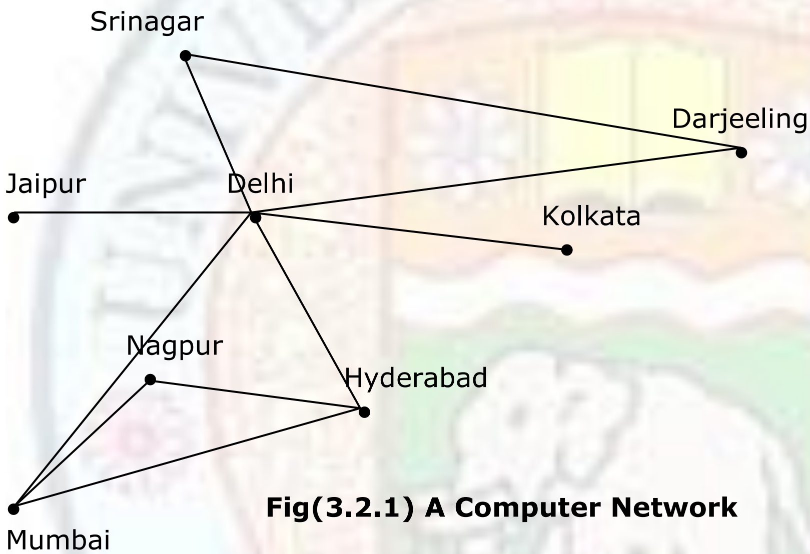
3.2 Models and Types of Graphs

3.2.1 Introduction

In Computer Science/Mathematics, Graphs are discrete structure consisting of vertices and edges that connect these vertices. There are several different types of graphs that differ with respect to the kind and number of edges that connect a pair of vertices. Problems in almost every conceivable discipline can be solved using **Graph Models.** In this section we will give sample examples to show how graphs are used as models in variety of areas. For instance, we will show how graph are used to represent the competition of different species in an ecological environment, how graph is used to represent who influences whom in an organization, how graph is used in defense and Project network etc. Later we will show how graphs can be used to solve many types of problems, such as computing the number of different combinations of flights between two cities in an airline network, application of graph in internet and its networking and many more.

3.2.2 Types of Graphs

We will introduce the different types of graph by showing how each can be used to model a computer network. Suppose that a network is made up of Computers and telephone lines between Computers. We can represent the locations of each computers by a point and each telephone line by an arc, as shown in fig (3.2.1)



Fig(3.2.1) A Computer Network

In this network there is at least one telephone line between two computer centers.
There are basically two categories of graph Directed and Undirected graphs. In both the category there are several types of graphs such as Simple, Multigraph, Pseudo graph etc. There are some special type of graph as well like Complete, Bipartite graph etc.

<u>3.2.2.1 Directed and Undirected Graphs</u>

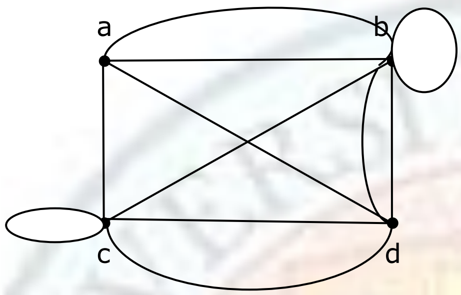
In the above example we have **Undirected edges** that represents the telephone lines. The graphs which have undirected edges are termed as undirected graphs.

An **Undirected Graph** G is defined abstractly as an ordered pair (V,E), where V is the set of points and E is the set of multisets of two elements of V, known as edges.

Example 4: Let us consider an adjacency matrix, in which the non-zero element shows that there is a path between those two elements.

	a	b	c	d
a	0	3	0	1
b	1	2	0	2
c	2	1	1	3
d	0	2	1	0

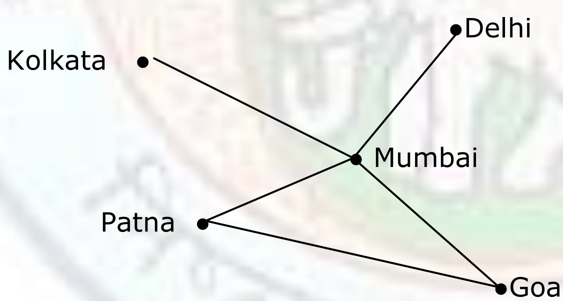
The corresponding directed graph representing the adjacency matrix is given below,



fig(3.2.2) Undirected graph

Example 5: Draw a graph model, to represent airline routes where every day there are four flights from Delhi to Mumbai, two flights from Mumbai to Delhi, three flights from Mumbai to Kolkata, two flights from Kolkata to Mumbai, one flights from Mumbai to Goa, two flights from Goa to Mumbai, three flights from Mumbai to Patna, two flights from Patna to Mumbai, and one flight from Patna to Goa, with an edge between vertices representing cities that have a flight between them (in either direction)

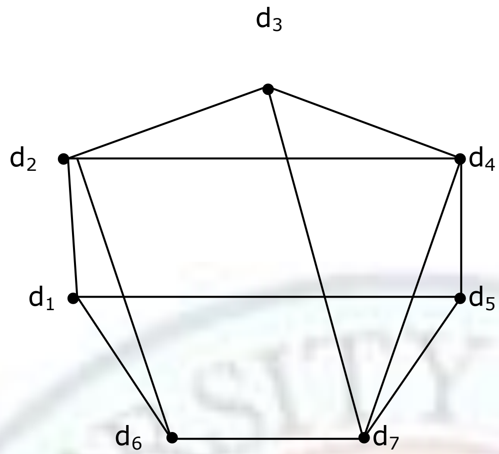
Solution: Let G be a graph with five vertices corresponds to five cities. There is an edge between the two vertices representing the cities that have flight between them



Fig(3.2.3)

Example 6: Consider the Problem of scheduling seven examinations in seven days so that two examinations given by the same instructor are not scheduled on consecutive days. If no instructor gives more than four examinations, show that it is always possible to schedule the examinations.

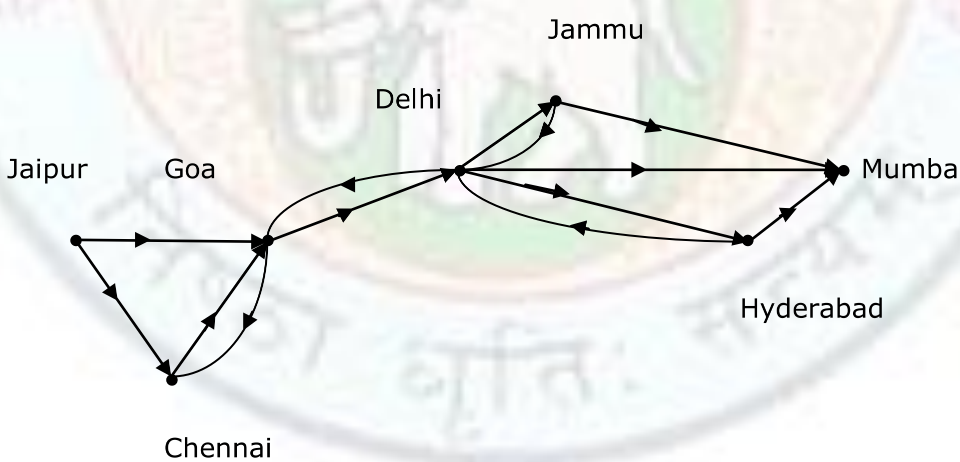
Solution: Let G be a graph with seven vertices corresponding to seven examinations. There is an edge between any two vertices which correspond to two examinations given by different instructor.



fig(3.2.4) Problem of scheduling seven examinations in seven

We can easily observe that the degree of each vertex is at least 3 and hence the sum of degree of any two vertices is at least 6, which means that G contains a **Hamiltonian path**. Existence of Hamiltonian path corresponds to a suitable schedule for seven examinations in seven days with satisfying constraints.(Hamiltonian path and circuit will be discussed in later chapter)

Directed Graph: A directed graph G=(V,E) is defined as the set of marked points V with a set of arrows E between the points so that there is at most one arrow from one point to another point. So it is an ordered pair (V,E), where V is the set and E is a binary relation on V. Fig (3.2.5) is a directed graph.



Fig(3.2.5) A Communications Network with one way telephone lines

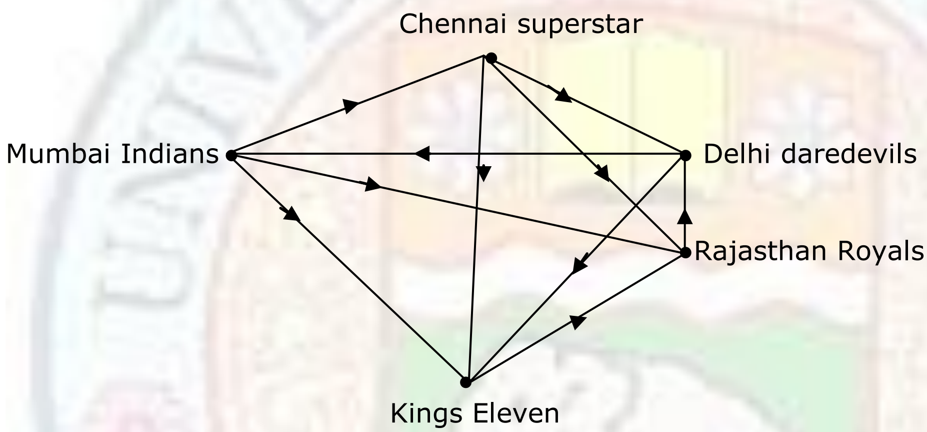
Now , we discuss different type of Directed and Undirected graphs.

Simple Graph: An **Undirected Simple Graph** G=(V,E) consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges. There should not be multiple edges between the vertices nor the loops on any vertex in simple graph. fig(3.2.4) is undirected simple graph.

If E is the set of ordered pair in simple graph then the graph is **Directed Simple Graph**.

Example 7: In a round-robin Cricket tournament, the Mumbai Indians beat Chennai Super star, King's eleven and Rajasthan royals but lost to Delhi daredevils. The Rajasthan royals beat Delhi daredevils but lost to king's eleven, Chennai superstar and Mumbai Indians. The Delhi daredevils able to beat Mumbai Indians and King's eleven only. The Chennai superstar lost to Mumbai Indians only and the kings eleven able to beat Rajasthan royals only. Model this outcome with a directed graph. Whether the obtained model is simple or not?

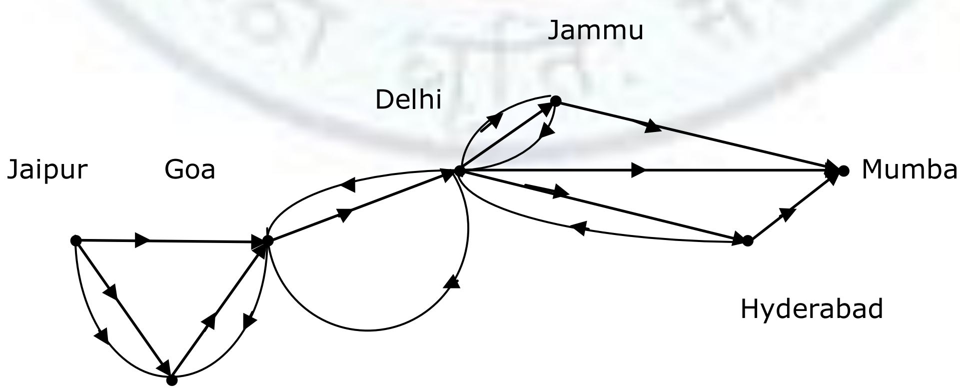
Solution: Let G be a graph with five vertices corresponding to five cricket clubs. The edge between two vertices corresponds to who beat whom, that is (a,b) implies a beat b.



Fig(3.2.6) Graph for round-robin Cricket tournament

As there are no multiple edges between any two vertices nor a single loop in the graph, thus the above **Model is directed Simple Graph**.

Multi Graph: A Multigraph G=(V,E) consists of a set V of vertices, a set E of edges and a function E to {(u,v) ; u,v∈ V, u≠v}. The edges e_1 and e_2 are called multiple or parallel edges if $f(e_1)=f(e_2)$. We should note that multiple edges in a multigraph are associated to the same pair of vertices. Fig(3.2.7) is Directed multigraph.



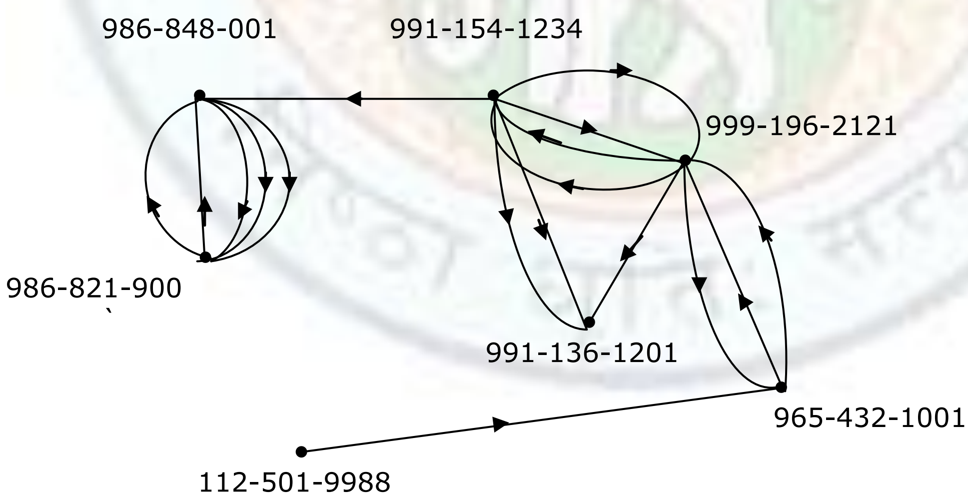
Chennai

Fig(3.2.7) A Communications Network with multiple telephone lines

When the multiple edges are directed, multigraph is **Directed multigraph** otherwise **Undirected multigraph**.

Example 8: Consider an example of telephone calls made in a long distance telephone network. Mishra-family has seven members. Mr. and Mrs. Mishra lives in Delhi and their son and daughter-in-law settled in Mumbai. Their daughter and son-in-law are in Bangalore whereas granddaughter studies in USA. Record of a particular day, form VSNL says three calls have been made from the mobile no. 986-848-0001 to 986-821-900 and two in other direction. One, two and two calls have been made from mobile no. 991-154-1234 to 986-848-0001, 999-196-2121 and 991-136-1201 respectively. Two calls have been made from 999-196-2121 to 991-154-1234 and one call has been made to 991-136-1201 from the same number. Two calls have been made from 965-432-1001 to 999-196-2121 and one in other direction. A Single call has been made from 112-501-9988 to 965-432-1001. It is given that the mobile no. 986-848-0001 belongs to Mr. Mishra, 986-821-900-Mrs Mishra, 991-154-1234- Mr. rajiv (Mishra' son), 999-196-2121-Mrs. rajiv, 991-136-1201- Mrs. Sujata(Mishra's daughter), 965-432-1001- Mr. Rohit(Mishra' son-in-law), and 112-501-9988-Ms. Samiksha (grand daughter of Mr. Mishra).
Present the above calls in graphical model.

Solution: Let G be a graph with seven vertices representing the person's telephone numbers located at different cities. The directed edges between the vertices corresponds to the calls made by someone to someone else.
As there is multiple calls from a person to the other, it implies that it's graphical model is not simple but multigraphical.



Fig(3.2.8) Directed Telephone call Model

Pseudograph: When a Computer Network may contain a telephone line from a Computer to itself, we cannot use multigraph to model such networks, since **loops**, which are edges

from a vertex to itself, are not allowed in multigraph. Instead we use **pseudograph**, which are more general than multigraphs, since in pseudograph may connect a vertex with itself.
" A **pseudograph** g=(V,E) consists of a set V of vertices, a set E of edges and a function f from E to {(u,v); u,vϵV }. An edge is a loop if f(e)={u,u)={u} for some uϵV."

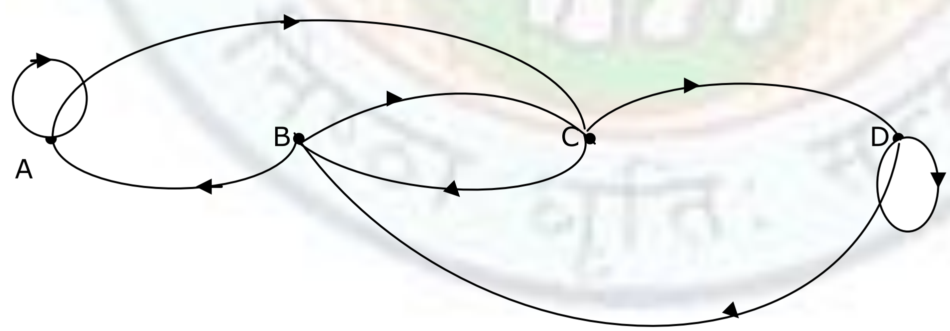
Example 9: Consider a Stochastic process, Suppose that whether or not it rains depends on previous weather conditions through the last two days. Specifically, suppose that it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday then it will rain tomorrow with probability 0.5; if it rained yesterday but not today then it will rain tomorrow with probability 0.4; if it doesnot rain for past two days, then it will rain tomorrow with probability 0.2. The corresponding transition probability matrix is given by

P=

0.7	0	0.3	0
0.5	0	0.5	0
0	0.4	0	0.6
0	0.2	0	0.8

Solution: The corresponding transition diagram (Graphical Model) is drawn by considering four vertices representing the four states (State 1-it rained for past two days, state 2- it rained today but not yesterday, State 3- it rained yesterday but not today, State 4- it does not rain for past two days) and the edges between the vertices represents transition of states.
[The transition matrix is considered similar as adjacency matrix, which we discuss in the later chapter]

p(1,1) means state-1 for past two days and remain same for tomorrow,
p(1,2) means state-1 for past two days and state-2 for tomorrow and so on.



Fig(3.2.9) Pseudograph

We have p(1,2)=p(1,4)=p(2,2)=p(2,4)=p(3,1)=p(3,3)=p(4,1)=p(4,3)=0, it means there will be no edge between (1,2), (1,4),(2,2),(2,4),3,1),(3,3),(4,1) and (4,3)

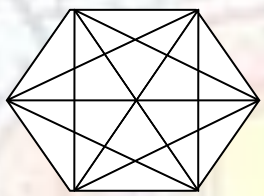
Finite Graph: A multigraph is said to be **finite** if it has finite number of edges.

The finite graph with one vertex and no edges i.e, a single point, is called the **trivial graph**.

3.2.2.2 Some Special Simple Graphs

There are some special classes of simple graphs, which is often used as examples and arise in many applications.

Complete Graph: The Complete graph on n vertices is denoted by $\mathbf{K_n}$, is the simple graph that contains exactly one edge between each pair of distinct vertices.
In other words A Graph G is said to be Complete if each vertex in g is connected to every other vertices of G. Fig(3.2.9) is for n=6



Fig(3.2.10) K_6

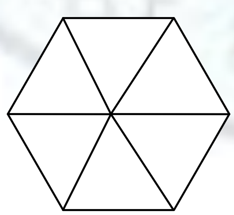
(refer section 3.1.3.1)

Cycles: The Cycle $\mathbf{C_n}$, n≥3, consists of n vertices $V_1, V_2, \ldots V_n$ and edges $\{V_1,V_2\},\{V_2,V_3\},\ldots\{V_{n-1},V_n\}$. Fig(3.2.10) is cycle graph for n=6.

Fig(3.2.11) C_6

(refer section 3.1.3.1)

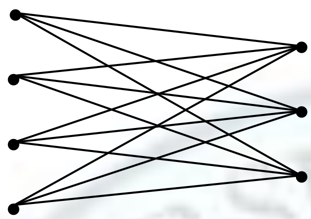
Wheels : When an additional vertex is added to the cycle C_n, n≥3 and connect that vertex to all the vertices of the cycle C_n, we obtain Wheel $\mathbf{W_n}$. It resemble the ordinary wheel so the name is. Fig(3.2.11) is the wheel for n=6



Fig(3.2.12) W_n

(refer section 3.1.3.1)

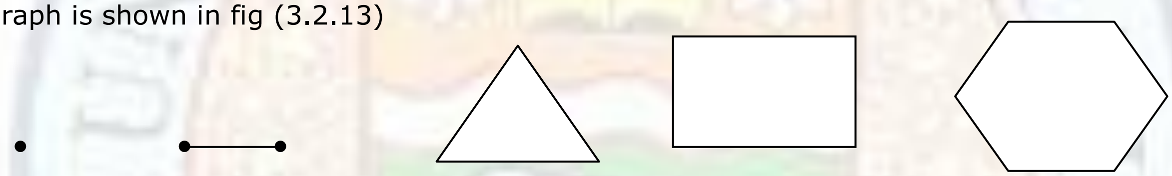
Bipartite Graph: A Graph G is said to be Bipartite if its vertices is partitioned into two parts (subsets) M and N and each edge connects a vertex of M to a vertex of N. By a Complete Bipartite graph we mean that each vertex of M is connected to each vertex of N. By normal Bipartite graph we mean Complete Bipartite graph. Complete Bipartite graph is denoted by $\mathbf{K_{m,n}}$, $m \epsilon M$ and $n \epsilon N$. Fig(3.2.12) is Bipartite Graph for m=4 and n=3



Fig(3.2.13) $k_{4,3}$

(refer section 3.1.3.2)

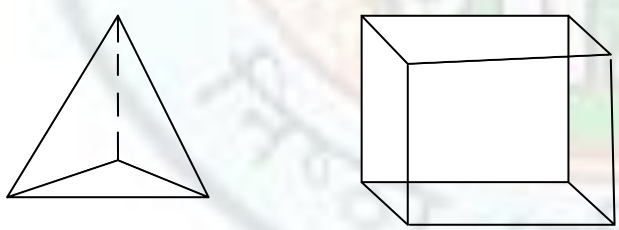
Regular Graph: A Graph is regular of degree K or k-regular if every vertex has degree k. In other words a graph is regular if every vertex has the same degree. Some of the regular graph is shown in fig (3.2.13)



0-regular 1-regular **fig(3.2.14)** 2-regular

(refer section 3.1.3.1)

Platonic Graph: The five-regular polyhydra are known as **Platonic solid**. Their vertex and edges configuration form regular graph called the Platonic Graph.

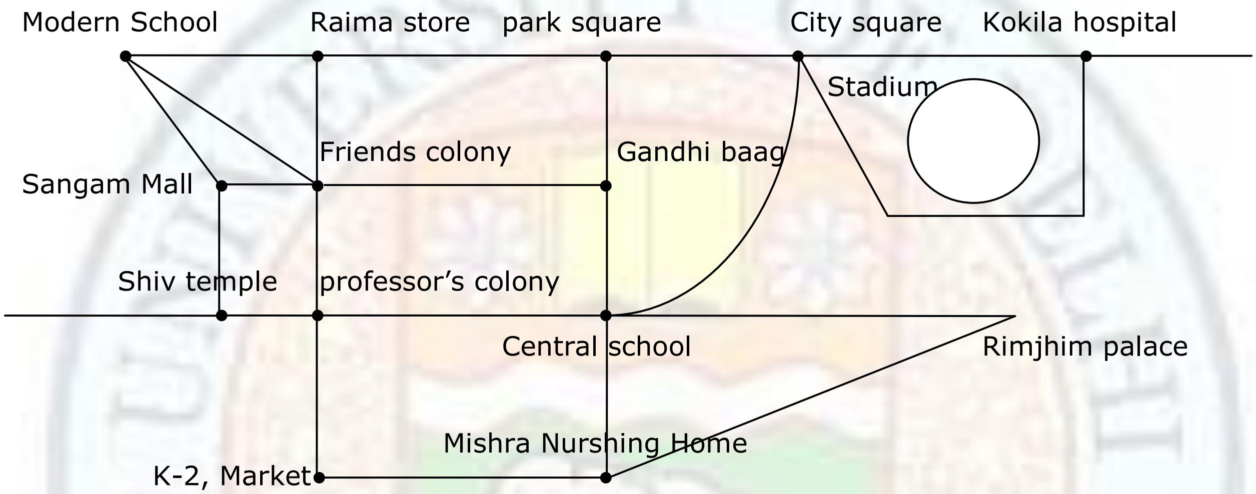


Fig(3.2.15) Platonic Graph

3.2.3 Graph Models

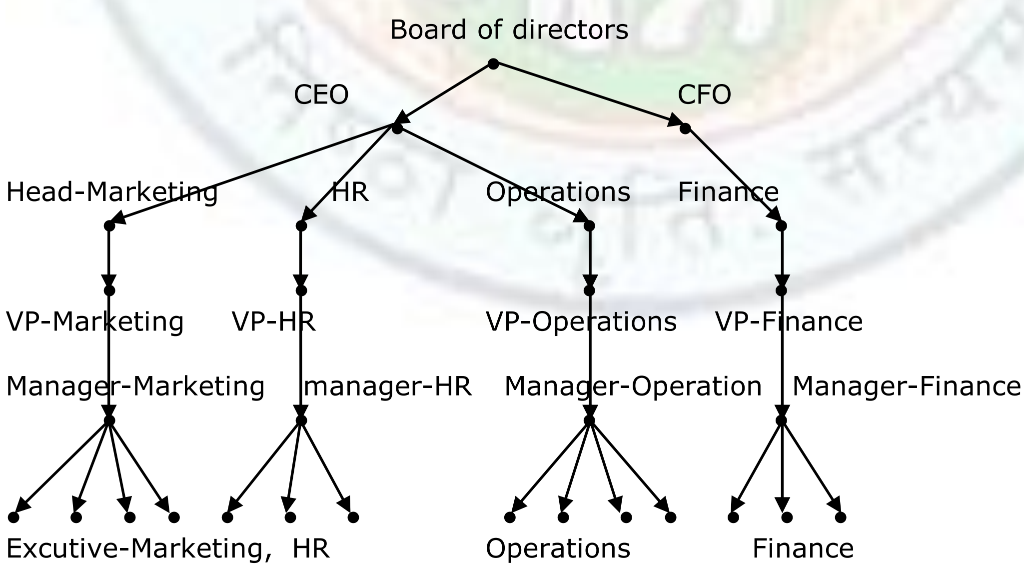
Graphs are used in a wide variety of Models. In this section we will discuss few graph model from the practical world.

Road Map Model: In Geography, we use graph to represent the road/highway, railway track spreaded in the country/state/cities/town etc. Such model helps to identify or locate the specific area or identify the route of a particular place. We can generally observe such map in big town or on important places. The vertices represents the important places and the edges represent the road that connect those places. The road map of a small town is given in fig(3.2.16).



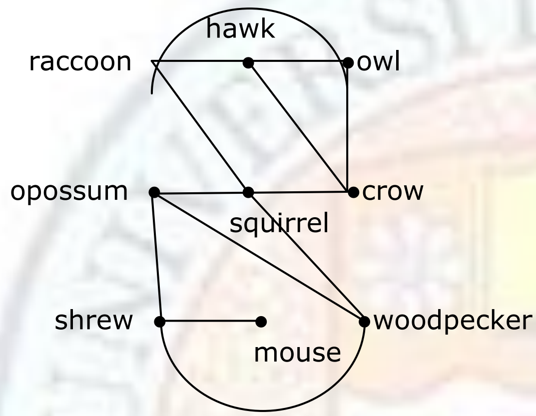
Fig(3.2.16) Road-Map of a small town

Corporate hierarchy Model: A graph called a Corporate hierarchy model which is used to model the Corporate structure of their employees. The vertices denote the designation and the edges represents hierarchy. Fig(3.2.17) gives the Corporate hierarchy model of an FMCG Company.



Fig(3.2.17) Corporate hierarchy Model

Niche Overlap Graphs in Ecology: Graphs are used in many models involving the interactions of different species of animals. For example, the completion between species in an ecosystem can be modeled using a **niche overlap graph**. Each species is represented by a vertex and edge that connect two vertices compete (that is, have some same food resources). Fig(3.2.18) models the ecosystem of forest.

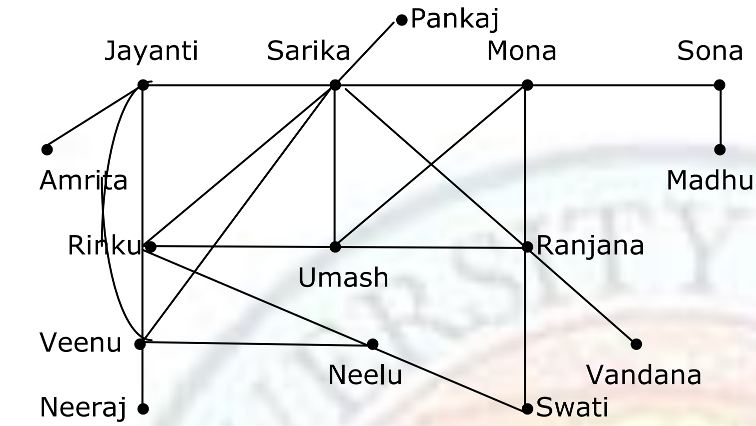


fig(3.2.18) niche overlap graph

Round-Robin tournament: A tournament where each team plays each other team exactly once is called round-robin tournament. Such tournament can be modeled using directed graph, where each team is represented by a vertex. The edge between the vertex represent who beat whom. See fig(3.1.7) of example 3.

Call Graph Model: Graphs can be used to model telephone calls made in a network. In this model telephone numbers are used as vertex and each telephone call is represented by an directed edge. The edge representing a call start at a telephone number from which the call was made ends at the telephone number to which the call was made. See fig(3.2.9) of example 8.

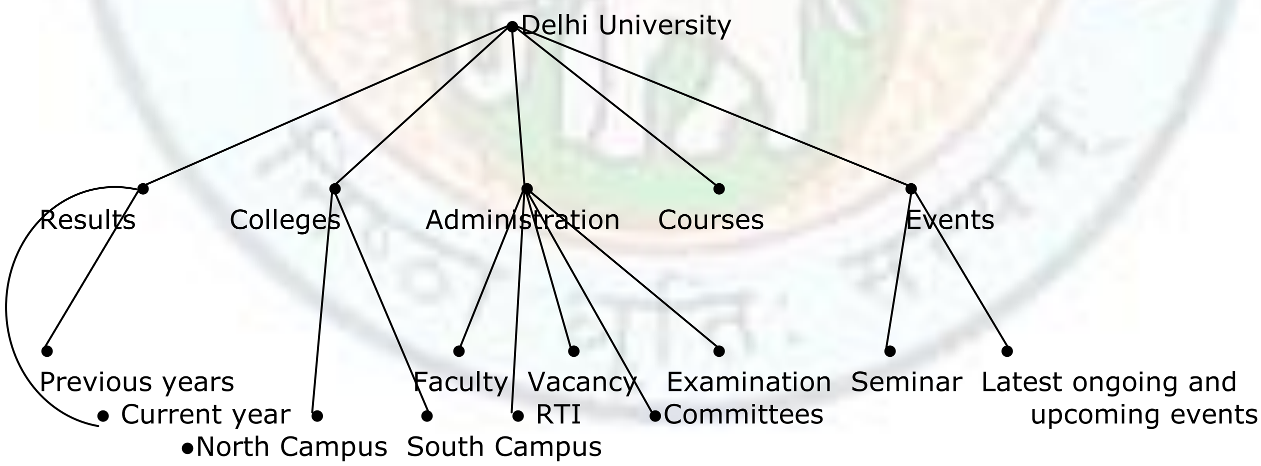
Acquaintanceship Graph Model: We can use graph models to represent various relationships between people. For example, we can use a graph to represent whether two people know each other, that is, whether they are acquainted. Each person in a particular group of people is represented by a vertex. An undirected edge is used to connect two people when these people know each other.



Fig(3.2.19) Acquaintanceship Graph Model

From the above model, we can easily say that Mona and Sona knows each other and Sona knows Madhu, and so on.

The Web Graph: The World Wide Web can be modeled as a directed graph, where each vertex represents a web page and where an edge starts at the web page a and ends at the web page b if there is a link on a pointing to b. Since new web pages are created and others removed somewhere on the web almost every second, the web graph changes on an almost continual basis. Fig(3.2.20) shows the web graph.



Fig(3.2.20)

3.3 MULTIGRAPHS AND WEIGHTED GRAPHS

.1 Introduction 3.1 Introduction

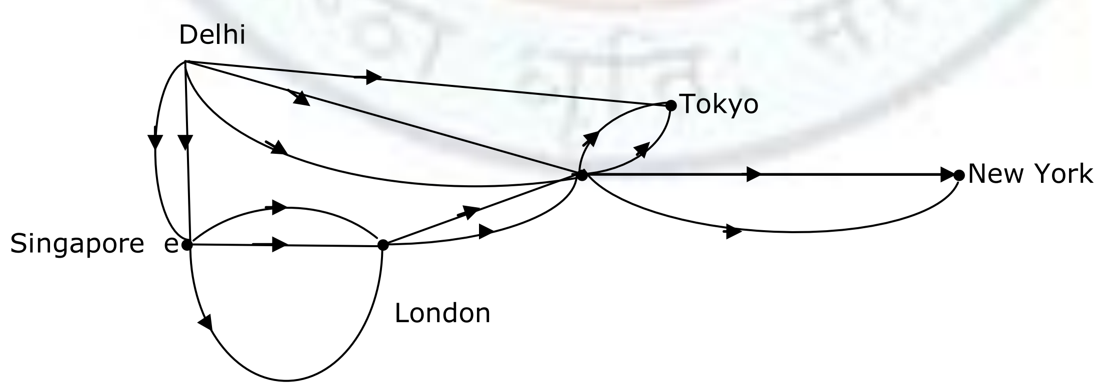
3.3.1 Introduction

In Graph theory, Simple Graph is not sufficient to provide all the solution. In many real life situation, when we try to draw the graph model we find that mutiset of edges connect the same vertices and also we have some loop on the few vertices, so we require to extend the definition of graph, termed as **multi graph**. In modeling a physical situation as an abstract graph, there are many occasions in which we wish to attach additional information to the vertices and/or the edges of the graph. For example in a graph that represents the highway connections among cities, we might wish to assign a number to each edge to indicate the distance between the two cities connected by the edge. We might also wish to assign a number to each vertex to indicate the population of the city represented by the vertex. Thus we require to define such graph which satisfy all the above, termed as **weighted graph**. In this section we introduce the multi graph and weighted graph and some of its application in physical world.

3.3.2 Multi Graph

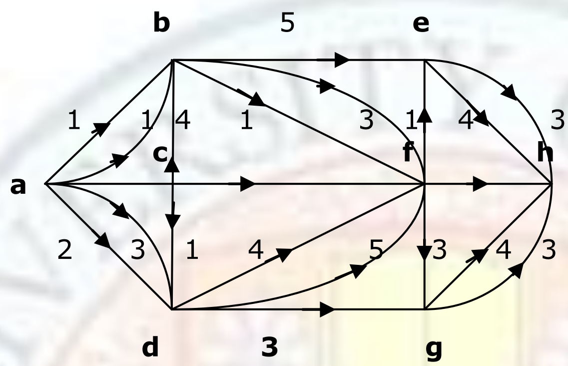
The definition of a graph can be extended in several ways, Let G=(V,E), where V is a set and E is a multiset of ordered pair from V x V. G is called a directed **multigraph** Geometrically, a **directed multigraph** can be represented as a set of marked points V with a set of arrows E between the points, where there is no restrictions on the number of arrows from one points to another point. We can similarly define the **undirected multi graph.** The formal definition of multigraph is,

Multi Graph: A Multigraph G=(V,E) consists of a set V of vertices, a set E of edges and a functionE to {{u,v} ; u,v∈ V, u≠v}. The edges e_1 and e_2 are called multiple or parallel edges if $f(e_1)=f(e_2)$. We should note that multiple edges in a multigraph are associated to the same pair of vertices. Fig(3.3.1) is directed multigraph.



Fig(3.3.1) A Communications Network with multiple telephone lines

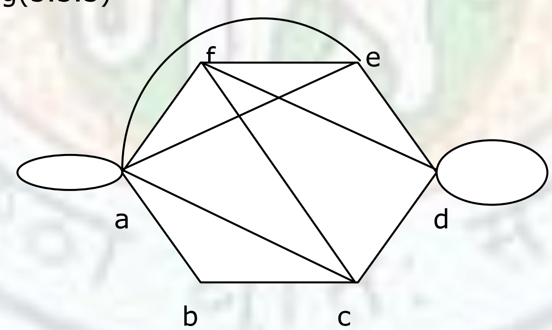
Example 10: We will consider a graph that shows the communication channels and the communication time delays in the channels among eight communication centers by various path. The centers are represented by vertices, the channels are represented by the edges, and the communication time delay(in minutes) in each channel is represented by the weight of the edge.



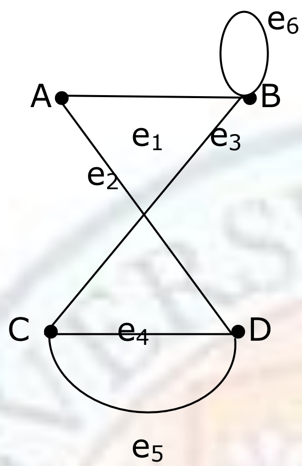
fig(3.3.2) Communication channels among eight centers

Finite Graph and Trivial Graph: A multi graph is said to be finite if it has finite number of edges and finite number of vertices. A finite graph with single vertex and no edges is called trivial graph.

Pseudo graph: A pseudograph G=(V,E) consists of a set of vertices V, a set of edges E, and a function f form E to {{u,v); u,vєV }. An edge is a loop if f(e)={u,u}={u} for some uєV. In other words "A multi graph is said to be a pseudograph if it has at least one loop at one of its vertex." See fig(3.3.3)



fig(3.3.3) Pseudograph

Value addition: Common Misconceptions
Multigraph
Consider a graph in fig (3.3.4)  fig(3.3.4) The edges e_4 and e_5 are called multiple edges. since they connect the same endpoints, the edge e_6 is the loop as its end points are the same. Such a diagram is called multigraph. Few authors have different point of view regarding multigraph, they use the term **pseudograph,** if the multigraph have loops.
Source: By the author

3.3.3 Weighted Graph

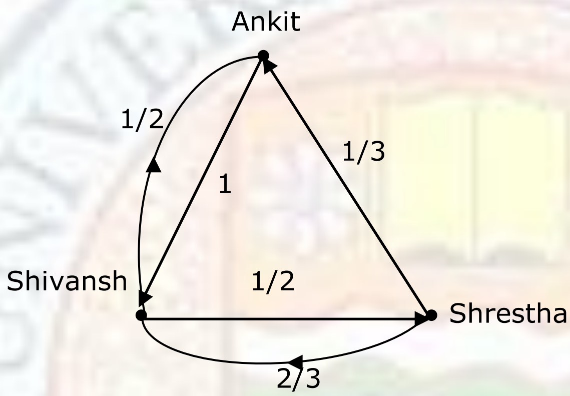
In modeling a physical situation as an abstract graph, there are many occasion in which we wish to attach additional information to the vertices and/or the edges of the graph. For example, In an graph that represents the outcomes of the matches in a tennis tournament , we might wish to label each edge with the score and the date of the match between the players connected by the edge or in a telephone call model we might wish to assign the time to each edge that indicate the duration of the call between the persons. In a formal and general way we define weighted graph as,

" A weighted graph is either ordered quadruple (V,E,f,g) or an ordered triple (V,E,f), or an ordered triple (V,E,g), where V is the set of vertices, E is the set of edges, f is a function whose domain is V and g is a function whose domain is E."

The function f is an assignment of weights to the vertices and the function g is an assignment to the edges. The weight can be a number, or symbol, or whatever quantities that we wish to assign to the vertices and edges.

Example 11: Suppose three boys Ankit, Shivansh and Shrestha are throwing a ball to each other such that Ankit always throwing the ball to Shivansh, but Shivansh and shrestha as likely to throw the ball to Ankit as they are to each other. Shivansh throw the ball to Ankit and shrestha with probability ½ , and Shrestha throws the ball to Ankit and Shivansh with probability 1/3 and 2/3 respectively. Draw the above activity in weighted graph model.

Let G be the weighted graph with three vertices representing three boys, the edges between the vertices represents who throws the ball to whom, and weight is assign to the edge representing the probability of throwing the ball from a boy to another boy



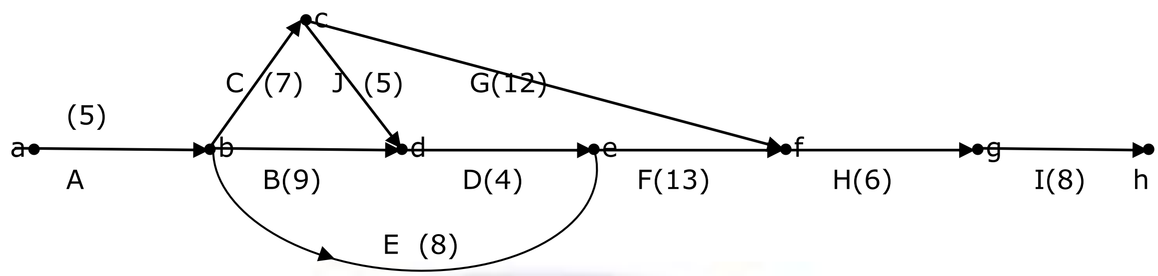
Fig(3.3.5)

Example 12: Consider a Project consisting of nine jobs (A,B,C,D,E,F,G,H,I) with the following precedence relations and time estimates

JOB	PREDECESSOR	ESTIMATED TIME
A	-	5
B	A	9
C	A	7
D	B,C	4
E	A	8
F	D,E	13
G	C	12
H	F,G	6
I	H	8
J	C	5

Draw the project network.

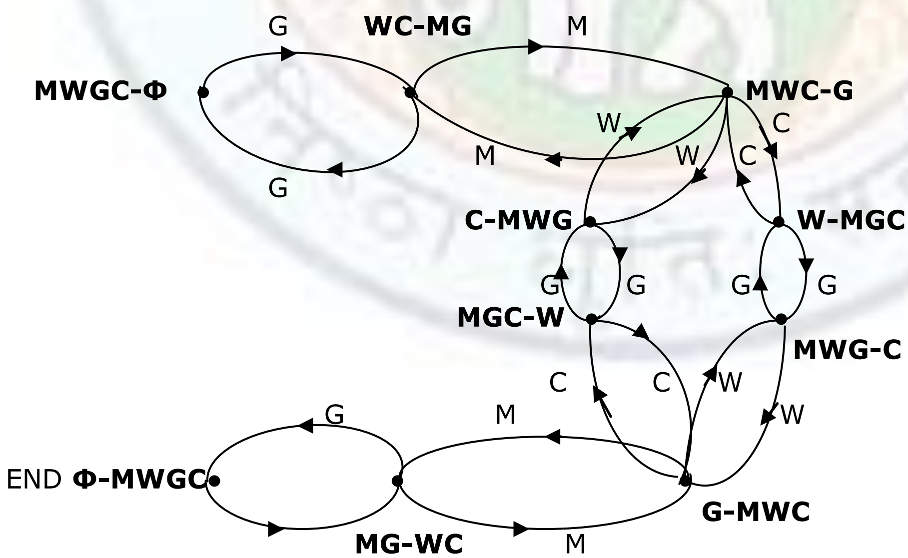
Solution: Let G be a graph with ten vertices, the edges represents " job done". The weight on edge gives the completion time. The estimated time of completing the job is given as the weight on the edges.



Fig(3.3.6) Project network

Example 13: A man with wolf, goat and cabbage is on the left bank of the river. There is a boat large enough to carry the man and one of the other three. The man and his entourage wish to cross the right bank and the man can ferry each a cross, one at a time. However, if the man leaves the wolf and the goat unattended on either shore, the wolf will surely eat the goat. Similarly , if the goat and cabbage are left unattended ,the goat will eat the cabbage. Is it possible to cross the river without the goat and the cabbage being eaten? Draw a graph and determine all possible ways for transport.

Solution: The problem is modeled by observing that pertinent information is the occupants of each bank after a crossing. Set of occupants is ={man, goat, wolf, and cabbage}, which can be coded as{M,G,W,C}. Let us model a transport plan as a digraph. Each vertex in the digraph corresponds to the subset that is on the left bank, so, the number of subsets=2^4**=**16 subsets. The digraph is weighted graph. States are labeled by hyphenated pairs such as: MGC-W, where the symbols on the left of the hyphen denote the subset on the left bank and the symbols on the right of the hyphen denote the subset on the right bank. Edges represent the action the man takes. He may cross alone (labeled as M), with wolf(W), the goat (G) or the cabbage (C).The initial position is MWGC-Φ and the final situation is Φ-MWGC. Figure shows all possible ways of transport.



FIG(3.3.7)

3.4 Graph Representation and Graph Isomorphism

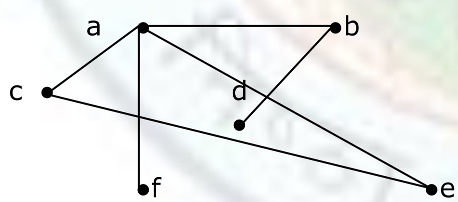
In this section, we will discuss what are the different ways to represent graphs and their isomorphism. There are many useful ways to represent graphs, like using pictorial representation or by list of edges or by adjacency lists or by using incidence matrices etc. It is of great help if we are able to choose its most convenient representation, while working with graphs.

We often need to know whether it is possible to draw graphs in the same way. For example, in chemistry, the graphs are used to model compounds. Different compounds have the same molecular formulas but different structural formulas. Such compounds will be represented by graphs that cannot be drawn in the same way. The graphs representing previously known compounds can be used to determine whether a supposedly new compound has been studied before. Sometimes, two or more graphs have exactly the same form, in the sense that there is one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that two graphs are **isomorphic**. Determine whether two or more graphs are isomorphic is an important problem of graph theory.

3.4.1 Graph Representation

There are many useful ways to represent graphs. One way to represent a graph without multiple edges is to list all the edges of this graph(tabular form). Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph. In this section we will learn some of the common way of representing graphs.

Example 14: Use adjacency list to describe the simple graph given in fig(3.4.1)

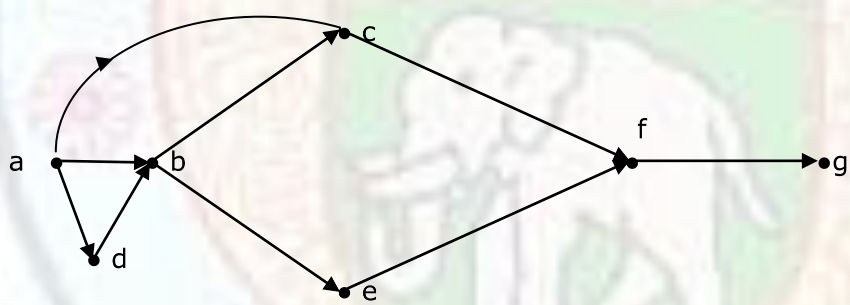


Fig(3.4.1)

Solution: The table gives the lists of those vertices which are adjacent to each of the vertices of the graph.

An edge list for simple graph fig(3.4.1)	
Vertex	Adjacent vertices
a	c,b,e,f
b	a,d
c	a,e
d	b
e	a,c
f	a

Example 15: The graph given below shows the project network of DMRC. Use adjacency list to describe the project.



Fig(3.4.2)

Solution: In the above project, the nodes describes the jobs and the directed arrow gives the predecessors(jobs). The corresponding adjacency list of jobs, which are predecessor to other vertex is given in the table below.

A list of vertices which are predecessor to other vertices	
vertex	predecessors

a	------
b	a,d
c	a,b
d	a
e	b
f	c,e
g	f

3.4.1.1 Adjacency Matrix

When we have multiple edges in the graph, it is convenient to use matrices to represent the graph. Two types of matrices commonly used to represent the graphs, one is based on the adjacency of vertices and the other is based on incidence of vertices and edges.

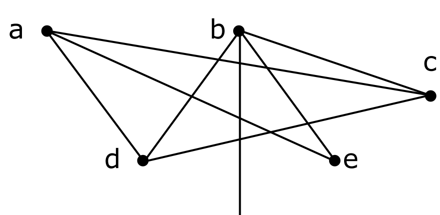
Suppose that G=(V,E) is a simple graph where $|V|=n$. The **adjacency matrix A** with respect to the listing of vertices of G is the nXn zero-one matrix given by,

A=$[a_{ij}]$,

Where a_{ij}={1 ; if {vi,vj} is an edge of G
{0 ; otherwise

> Note: 1) An adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence there are as many as n! different adjacency matrix for the graph with n vertices.
>
> 2) The adjacency matrix of **simple undirected graph** is symmetric, i.e, $a_{ij}=a_{ji}$, also there will be no loop in simple graph so $a_{ii}=0$.
>
> 3) when we have few edges in the simple graph the corresponding adjacency matrix is termed as **Sparse matrix.**

Example 16: Use an adjacency matrix to represent the given graph in fig(3.4.3)

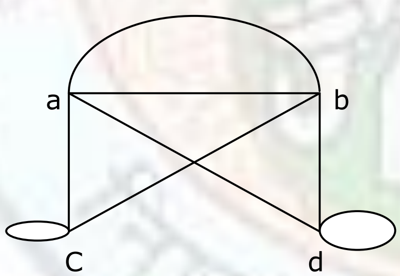


Fig(3.4.3)

Solution: We order the vertices as a,b,c,d,e,f and the corresponding index will be 1,2,3,4,5,6. This means a_{11}=an edge between vertex a to itself, a_{12}=an edge between a to b, similarly a_{56}=an edge between vertex e to f and so on. The corresponding adjacency matrix is

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix} \text{ which implies } \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 17: Use adjacency matrix to represent pseudograph shown in fig(3.4.4)



Fig(3.4.4)

Solution: We order the vertices as a,b,c,d and the corresponding index will be 1,2,3,4.
The corresponding adjacency matrix is

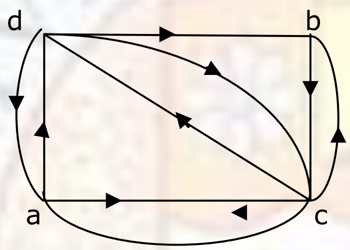
$$A=\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Here, $a_{12}=2$, which means there are two edges between vertex a and b and so is $a_{21}=2$

Example 18: Draw a graph with the adjacency matrix given below, with respect to the ordering of vertices a,b,c,d.

$$A=\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Solution: Since the ordering of the vertices are a,b,c,d, so we have only four vertices, also none of the diagonal element is non-zero, so we do not have any loops as well. The graph corresponding to the give matrix is,



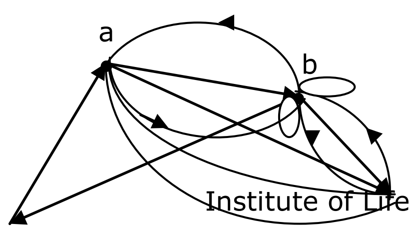
fig(3.4.5)

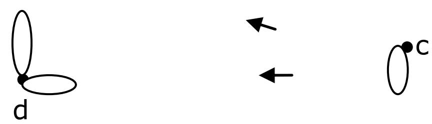
> **Note**:
> we cannot draw the undirected graph from the given adjacency matrix in example 5, as we have $a_{24}=0$ but $a_{42}=1$.
>
> • In case of **undirected graph**, the adjacency matrix must be symmetric, which is not in case of example 5, as $a_{34}\neq a_{43}$
>
> • In case of directed graph, if we have a23=2, it means there are two edges from node 2^{nd} to node 3^{rd}.
> See example 6.

Example 19: Draw a directed graph corresponding to the adjacency matrix A.

$$A=\begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Solution: We order the vertices as a,b,c,d and the corresponding index will be 1,2,3,4.





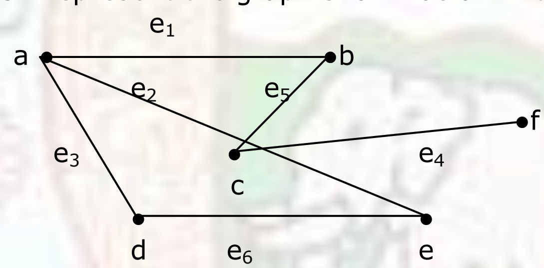
fig(3.4.6)

3.4.1.2 Incidence matrices

Another common way to represent graphs is to use **incidence matrices**. Let G=(V,E) be an undirected graph. Suppose that $v_1,v_2,v_3,.....v_n$ are the vertices and $e_1,e_2,e_3...e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the nXm matrix $M=[m_{ij}]$, where

m_{ij}={1 ; when edge e_j is incident with v_i
{0 ; otherwise.

Example 20: Represent the graph shown below with an incidence matrix.



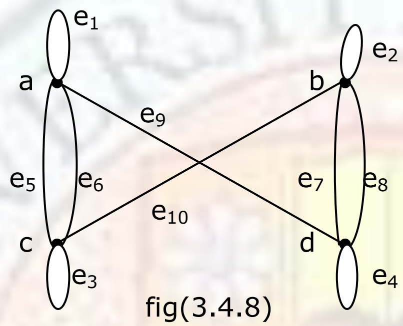
fig(3.4.7)

Solution: We have {e1,e2,e3,e4,e5,e6} and {a,b,c,d,e,f} are set of edges and set of vertices respectively. The correspondence incidence matrix is

$$
\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}
\begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \end{array}
$$

> **Value addition** : **Incidence matrices** can also be used to represent multiple edges and loops. Multiple edges are represented using columns with identical entries, since these edges are incident with the same pair of vertices. Loops are represented using a column with exactly one entry corresponding to the vertex that is incident with this loop.

Example 21: Represent the graph given below using incidence matrix.



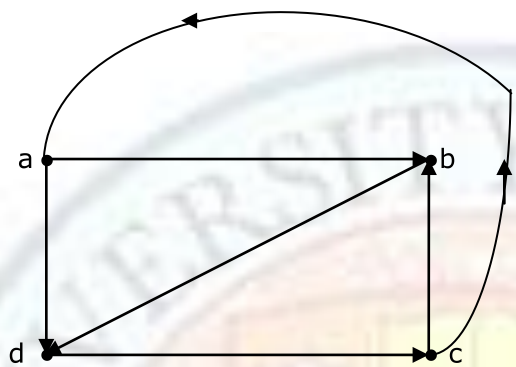
fig(3.4.8)

solution: This is a pseudo graph. He incidence matrix correspondence to the pseudograph is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
a	1	0	0	0	1	1	0	0	1	0
b	0	1	0	0	0	0	1	1	0	1
c	0	0	1	0	1	1	0	0	0	1
d	0	0	0	1	0	0	1	1	1	0

3.4.2 Graph Isomorphism

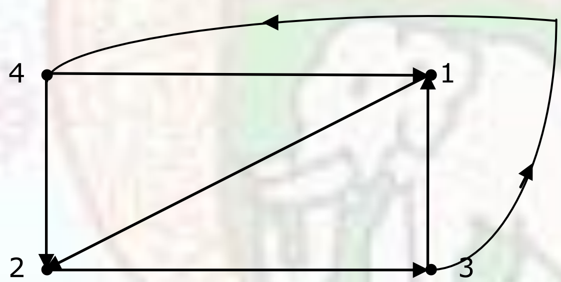
While working with the theory of graphs, sometimes we come across the two graphs which resemble each other. For example, let us consider a graphical representation of Table-Tennis tournament among four teams.



Fig(3.4.9a)

Here arrow sign shows who beat whom.

Consider another example of “Reference chapter” among four chapters of a book.



Fig(3.4.9b)

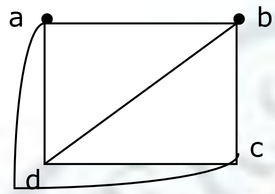
Here arrow sign shows which chapter refer to other chapter, like chapter 4 refer chapter 1, which in turn refer chapter 2.

From the observation we can see fig(3.4.9a) and fig(3.4.9b) resembles each other. Such graphs are known as isomorphic to each other. More precisely “Two graphs are said to be isomorphic if there is a one-to-one correspondence between their vertices and between there edges such as incidences are preserved”. In other words ,there is an edge between the corresponding vertices in the other graph .

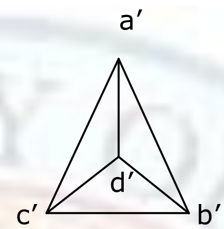
Note: The better way to judge whether two graphs are isomorphic or not is to check

1) Whether they have equal number of vertices and edges or not?
2) Whether same number of vertices have same degree or not?

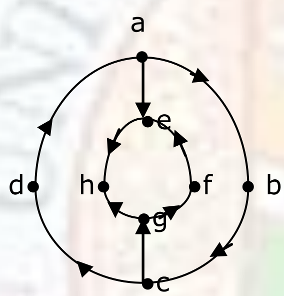
Fig(3.4.10a) and fig(3.4.10b) shows a pair of **isomorphic undirected graphs.** Fig(3.4.11a) and fig(3.4.11b) shows a pair of **isomorphic directed graphs**. In these two figures, corresponding vertices in the two isomorphic graphs are labeled with the same letter, primed and unprimed.



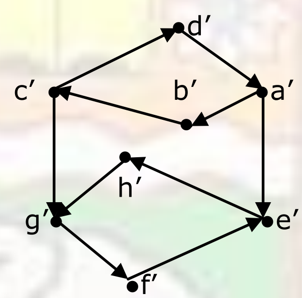
fig(3.4.10a)



fig(3.4.10b)

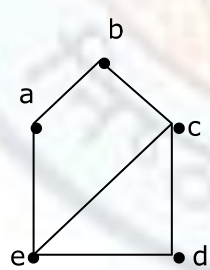


fig(3.4.11a)

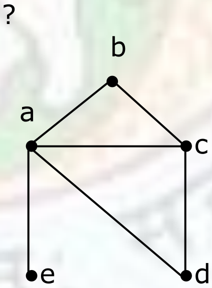


fig(3.4.11b)

Example 22: Are the following graphs isomorphic ?



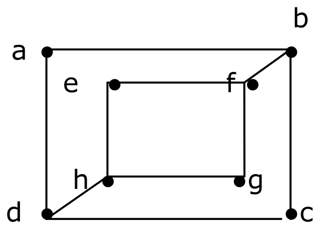
fig(3.4.12a)



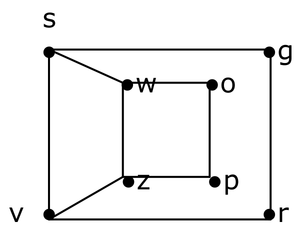
fig(3.4.12b)

Solution : Here both the figure have five vertices and six edges. However, fig(3.4.12b) has a vertex “e” of degree one, whereas fig(3.4.12a) has no vertex of degree one. Hence the two given graphs are not isomorphic.

Example 23: Are the following graphs isomorphic ?



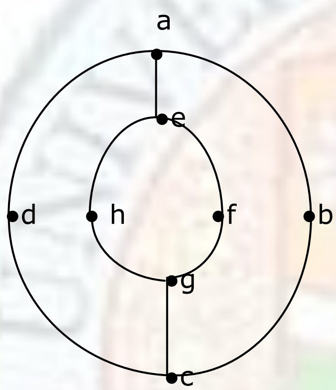
fig(3.4.13a)

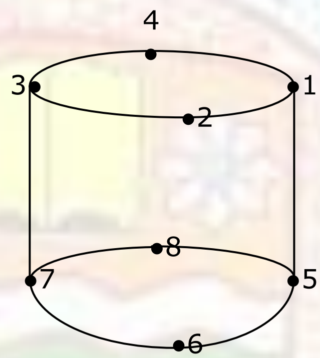


fig(3.4.13b)

Solution : Here both the figures have eight vertices and ten edges. Also both have four vertices of degree two and four of degree three. So, the correspondence between vertices exists. Hence these graphs are isomorphic.

Example 24: Are the following graphs isomorphic ?





Solution :The above two graphs, both have equal number of vertices. Four vertices of both the graphs have degree three and remaining with degree two. The vertices correspondence is as a-1,b-2,c-3,d-4,e-5,f-6,g-7 and h-8. Also an edge indices are preserved. Hence graphs are isomorphic.

3.5 Connectivity

In the Theory of graphs many problem can be modeled with paths formed by travelling along the edges of the graph. For instance, the problem of determining route of travelling salesman or problems of efficiently planning routes for mail delivery, garbage pick-up, diagnostics in Computer networks, and so on, can be solved using models that involve paths in graphs.

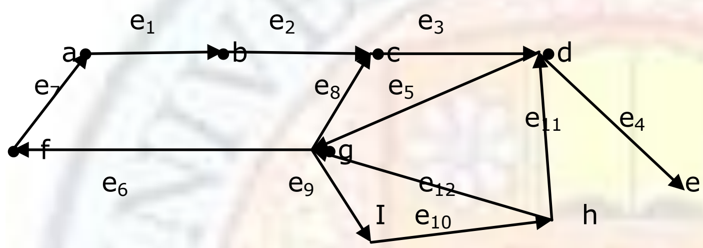
In this section we will discuss about paths and circuits in directed and undirected graph model and its connectedness.

3.5.1 Paths and Circuits

Informally, a path is a sequence of edges that begins at a vertex and travels along edges of the graph, through connecting pair of adjacent vertices. The formal definition of path and its related terminology is given below.

Path: In a directed graph, a path is a sequence of edges ($e_1,e_2,e_3,....,e_k$) such that the terminal vertex of e_i coincides with the initial vertex of e_{i+1} for $1\le i\le k-1$. A path is said to be **simple** if does not include the same edge twice. A path is said to be **elementary** if it does not meet the same vertex twice.

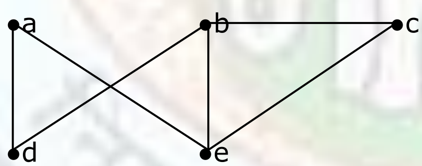
In figure(3.5.1), (e_1,e_2,e_3,e_4) is a path; ($e_1,e_2,e_3,e_5,e_8,e_3,e_4$) is a path but not simple. ($e_1,e_2,e_3,e_5,e_9,e_{10},e_{11},e_4$) is a simple path but not elementary one.



fig(3.5.1)

Length of path=number of edges crossed (one edge is considered only once)
In the above figure length of path (e_1,e_2,e_3,e_4)=4, and of ($e_1,e_2,e_3,e_5,e_8,e_3,e_4$)=6 and of ($e_1,e_2,e_3,e_5,e_9,e_{10},e_{11},e_4$)=8

Example 25: Does each of these lists of vertices form a path in the given graph? Which paths are simple? What are its length?



a)a,e,b,c,b
b)a,e,a,d,b,c,a
c)e,b,a,d,b,e
d)c,b,d,a,e,c

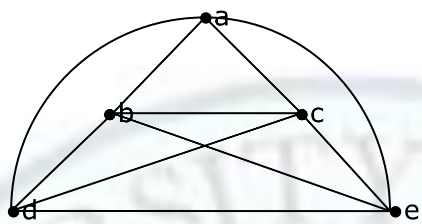
Fig(3.5.2)

Solution: a) yes, it is a path, but not simple. Length of path=4
b) no, it is not a path as we do not have an edge connecting vertex "c" to "a"
c) It is also not a path as we donot have an edge ehich connect the vertex "b" to "a"
d) yes, it is a path(circuit) and simple aswell. Length=5

Circuit: A circuit is a path ($e_1,e_2,e_3,.....,e_k$), in which the terminal vertexof e_i, coincides with the initial vertex of e_i. A circuit is simple if it doesnot include the same edge twice. A circuit is said to be elementary if it doesnot include the same vertex twice.

In figure (3.5.1) ($e_1,e_2,e_3,e_5,e_9,,e_{10},e_{12},e_6,e_7$) is a **simple circuit** and (e_1,e_2,e_3,e_5,e_6,e_7) is an **elementary circuit**

Example 26: Consider all possible paths adopted by a salesman in a town, shown in fig(3.5.3)



Fig(3.5.3)

Here there are many possible paths for the salesman, for instance (a,b,c,d,e,a) or (a,b,d,c,e,a) or (a,d,b,c,e,a) or (a,d,c,b,e,a) etc. All possible paths are the example of circuit.

The notations of paths and circuits in an undirected graph can be defined in the similar way.

3.5.2 Connectedness in Undirected Graphs

In case of LAN, WAN, and INTERNET a graph is used to represent Computer network, where vertices represents computer and edges represents the communication links, then computers share the information, which means each computer is connected and in graph theory we say that the path is connected.

Undirected connected graph: An undirected graph is said to be **connected** if there is a path between every two distinct vertices, and is said to be **disconnected** otherwise.

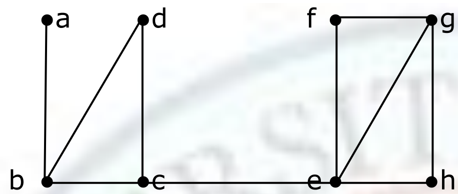
A graph that is not connected is the union of two or more connected sub graphs, each pair of which has no vertex in common. These disjoint connected sub graphs are called the **connected components** of the graph.

Theorem 4: In a (directed or undirected) graph with n vertices, if there is a path from vertex v_1 to vertex v_2 ,then there is a path of no more than n-1 edges from vertex v_1 to vertex v_2 .

Proof: Suppose there is a path from v_1 to v_2 .Let(v_1 ,...,v_i ,...,v_2)be the sequence of vertices that the path meets when it is traced from v_1 to v_2 .If there are l edges in the path,there are l+1vertices in the sequence . For l larger than n-1,there must be a vertex v_k ,that appears more than once in the sequence ,that is ,(v_1 ...,v_i ,...,v_k ,...,v_k ,...v_2).Deleting the edges in the path that leads v_k back to v_k ,we have a path from v_1 to v_2 that has fewer edges than the original one. This argument can be repeated until we have a path that has n-1 or fewer edges.

Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components than in the original graph. Such vertices are called **cut vertices**. The removal of a cut vertex from a connected graph produces a subgraph that is not connected. Analogously, an edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**.

Example 27: Find the cut vertices and cut edges in the graph.



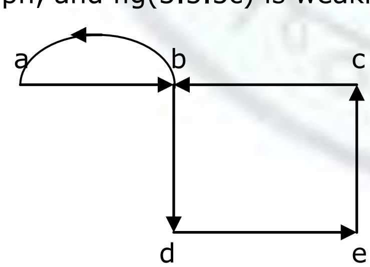
fig(3.5.4)

Solution: The cut vertices of the graph are b,c, and e. The removal of one of these vertices and its adjacent edges disconnects the graph. The cut edges are {a,b} and {c,e}. Removing either one of these edges disconnects the graph.

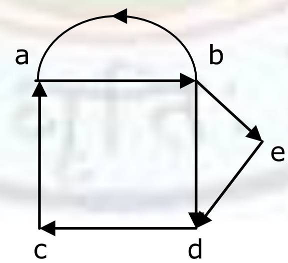
3.5.3 Connectedness in directed Graphs

Directed connected graph: A directed graph is said to be connected if the undirected graph derived from it by ignoring the directions of the edges is **connected** and is said to be **disconnected** otherwise.

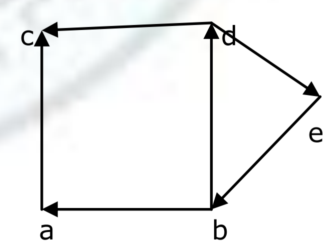
A directed graph is said to be **strongly connected** if for every two vertices a and b in the graph, there is a path from a to b as well as a path from b to a. For a directed graph to be strongly connected there must be a sequence of directed edges from any vertex in the graph to any other vertex. A directed graph can fail to be strongly connected but still be in "one piece". A directed graph is **weakly connected** if there is a path between every two vertices in the underlying undirected graph. That is, a directed graph is weakly connected if and only if there is always a path between two vertices when the directions of the edges are disregarded. Clearly, any strongly connected directed graph is also weakly connected. Figure(3.5.5a) is a directed connected graph, where as figure(3.5.5b) is strongly connected graph, and fig(3.5.5c) is weakly connected.



fig(3.5.5a)



fig(3.5.5b)



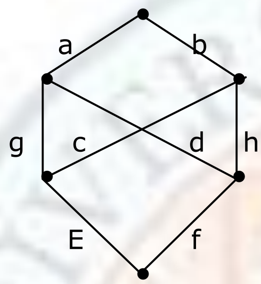
fig(3.5.5c)

3.5.3.1 Paths and Isomorphism

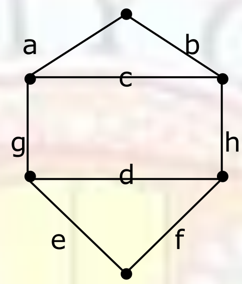
There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition paths can be used to construct mappings that may be isomorphic.

A useful isomorphic invariant for simple graph is the existence of a simple circuit of length k, where k is a positive integer greater than 2.

Example 28: Determine whether the fig(3.5.6a) and fig(3.5.6b) are isomorphic?



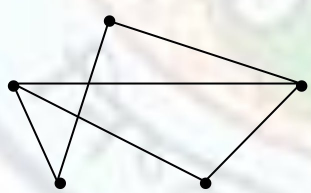
Fig(3.5.6a)



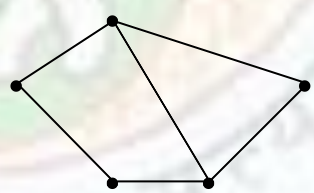
fig(3.5.6b)

Solution: From the given figure, it is clear that both the graphs have six edges and six vertices, as well as two vertices have degree 2 and four vertices have degree 3. So from the previous concept we can say that both the graphs are isomorphic to each other. However, fig (3.5.6b) has a simple circuit of length 3 (a,b,c) but fig (3.5.6a) has no simple circuit of length 3. Since the existence of simple circuit of length 3 is an isomorphic invariant, hence both the graphs are not isomorphic.

Example 29: Determine whether the fig(3.5.7a) and fig(3.5.7b) are isomorphic?



Fig(3.5.6a)



fig(3.5.6b)

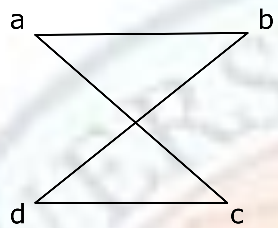
Solution: Both the fig(3.5.6a) and fig(3.5.6b) have five vertices and six edges also three vertices have degree 2, and two vertices have degree 3. Also both the graphs have a simple circuit of length three, a simple circuit of length four and a simple circuit of length five. Since all these isomorphic invariant agree so they are isomorphic.

3.5.3.2 Counting Paths between vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

> **Theorem 5**: Let G be a graph with adjacency matrix **A** with respect to the ordering $v_1, v_2, \ldots, v_n$. the number of different paths of length r from v_i to v_j, where r is a positive integer, equals the $(i,j)^{th}$ entry of $\mathbf{A}^r$

Example 30: How many paths of length 4 are there from a to d in the simple graph G in the figure (3.5.8)



fig(3.5.8)

Solution: The corresponding adjacency matrix A, with respect to the ordering a,b,c,d is,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Hence, the number of paths of length 4 from a to d is the (1,4)th entry of A^4. Since

$$A^4 = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

So, there are eight such paths.

3.6 Euler and Hamilton Paths and Circuits

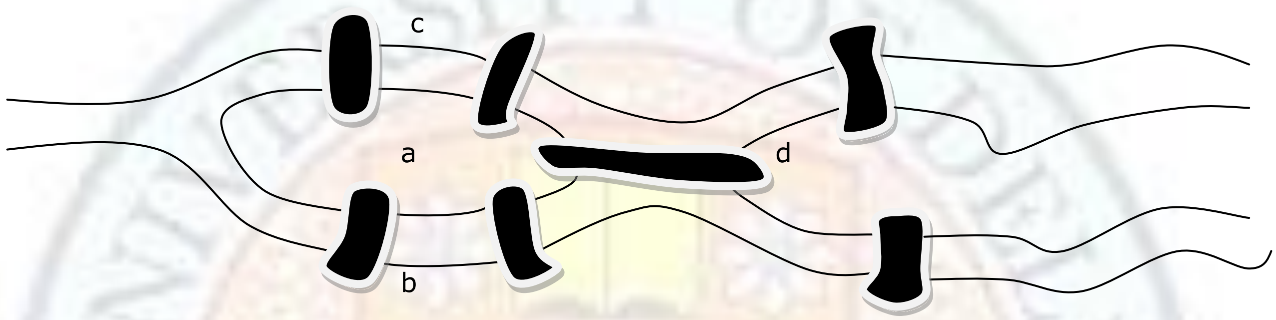
3.6.1 Introduction

Can we travel along the edges of a graph starting at a vertex and returning to it by traversing each edge exactly once? Similarly, can we travel along the edges of a graph starting at a vertex and returning to it while visiting each vertex of the graph exactly once? The first question leads to Eulerian paths and circuits and second question leads to Hamiltonian paths and circuits.

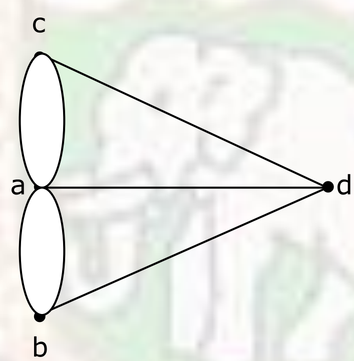
In this section we will discuss about the Eulerian and Hamiltonian paths and circuits, their importance and difficulty in solving them.

3.6.2 Eulerian paths and Circuit

The Swiss mathematician **Leonhard Euler** became the father of the theory of graphs when he proved that it is not possible to cross each of the seven bridges on the river Pregel in Kőnigsberg, Germany, once and only once in a walking tour. A map of the Kőnigsberg bridges is shown in figure(3.6.1) and graphical representation is shown in figure(3.6.2). The edges represent the bridges and the vertices represent the islands and the two banks of the river. It is clear that the problem of crossing each of the bridges once and only once is equivalent to finding a path in the graph in fig(3.6.2) that traverses each of the edges once and only once.



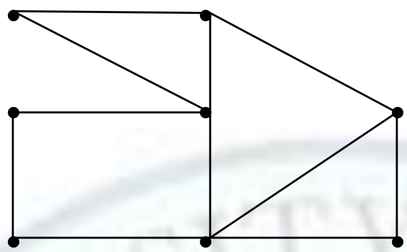
Fig(3.6.1)



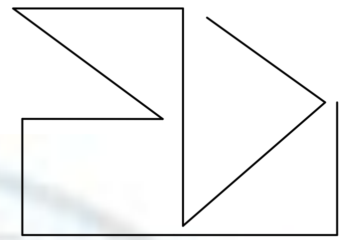
fig(3.6.2)

Euler discovered a very simple criterion for determining whether there is a path in a graph that traverses each of the edge exactly once.

Eulerian path: An Eulerian path in a graph is a path that traverses each edge in the graph once and only once. Figure(3.6.3a) has an Eulerian path, which is shown in fig(3.6.3b)



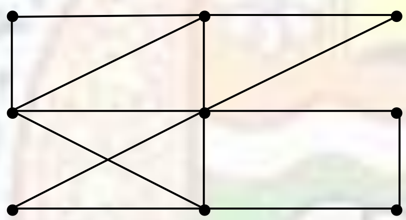
Fig(3.6.3a)



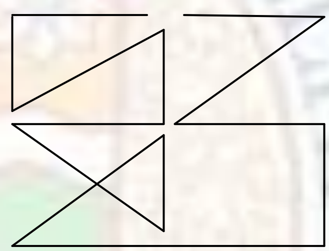
fig(3.6.3b)

Similarly we can define Eulerian circuit,

Eulerian Circuit: An Eulerian circuit in a graph is a simple circuit that traverses each edge in the graph exactly once. Figure (3.6.4a) has an Eulerian circuit, which is shown in figure (3.6.4b)



Fig(3.6.4a)



fig(3.6.4b)

Theorem 6: An undirected graph possesses an Eulerian path if and only if it is connected and has either zero or two vertices of odd degree.

Necessary part: Given that the graph possesses an Eulerian path and we have to prove that it has either zero or two vertices of odd degree.

Proof: Suppose that the graph possesses an Eulerian path. That the graph must be connected is obvious. When the Eulerian path is traced, we observe that every time the path meets the vertex, it goes through two edges which are incident with the vertex and have not been traced before. Thus, except for the two vertices at the two ends of the path, the degree of any vertex in the graph must be even. If the two vertices at the two ends of the Eulerian path are distinct, they are the only two vertices with odd degree. If they coincide, all vertices have even degree, and the Eulerian path becomes an Eulerian circuit.

Sufficient part: Given that a graph has either zero or two vertices of odd degree and we have to prove that the graph has Eulerian path/circuit.

Proof: We construct an Eulerian path by starting at one of the two vertices that are of odd degree and going through the edges of the graph in such a way that no edge will be traced more than once. For a vertex of even degree, whenever the path "enters" the vertex through an edge, it can always "leave" the vertex through another edge that has not been traced before. Therefore when the construction eventually comes to an end, we must have reached the other vertex of odd degree.

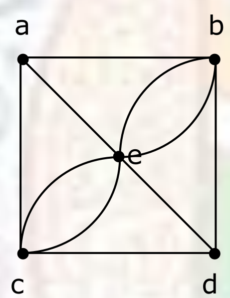
We can verify this from fig(3.6.4b)

Corollary 1: An undirected graph possesses an Eulerian circuit if and only if it is connected and its vertices are all of even degree

Corollary 2: A directed graph possesses an Eulerian circuit if and only if it is connected and the incoming degree of every vertex is equal to its outgoing degree. A directed graph possesses an Eulerian path if and only if it is connected and the incoming degree of every vertex is equal to its outgoing degree with the possible exception of two vertices. For these two vertices, the incoming degree one is one larger than its outgoing degree, and the incoming degree of the other is one less than its outgoing degree.

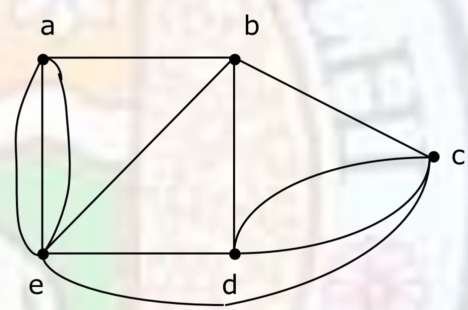
Example 31: Check whether the following graph has Eulerian path or Eulerian circuit or neither of them.

a)



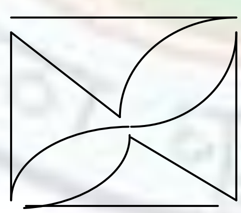
fig(3.6.5)

b)



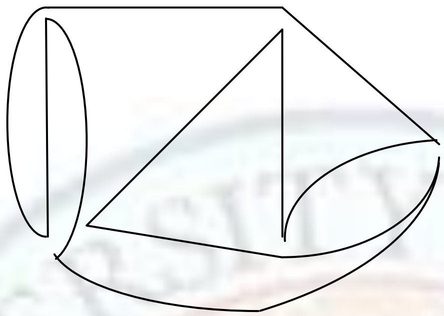
fig(3.6.6)

Solution:a) The degree of a is 3, b is 4, c is 4, d is 3 and of e is 6. As the degree of two vertices is odd, so by theorem 6 the graph has **eulerian path**. The break-up is given in fig(3.6.7)



Fig(3.6.7)

b) The degree of vertex a is 4, b is 4, c is 4, d is 4 and of e is 6. As the degree of each vertices is even, so by Corollary 1 of theorem 6 the graph has E**ulerian circuit**. The break-up is given in fig(3.6.8)



Fig(3.6.8)

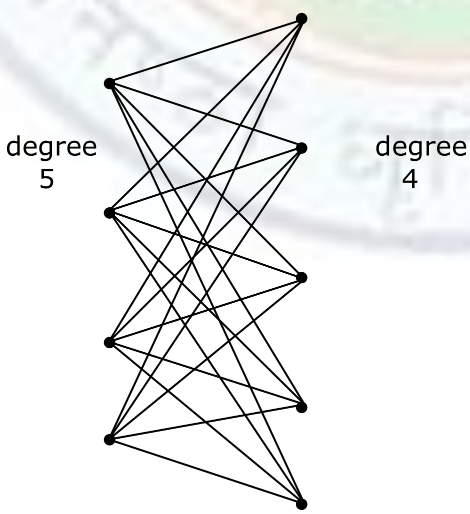
Example 32:Does the Complete graph K_{13} has an Eulerien circuit?

Solution: We know that in Complete graph each vertex is attached with rest all the vertices. So the degree of each vertex is (m-1) for the Complete graph K_m
Thus in K_{13} the degree of each vertex will be 12, which is even. Hence by corollary 1 of theorem 6, the graph has an Eulerian circuit.

Example 33: Is there an Eulerian path or Eulerian circuit in the Bipartite graph $K_{4,5}$?

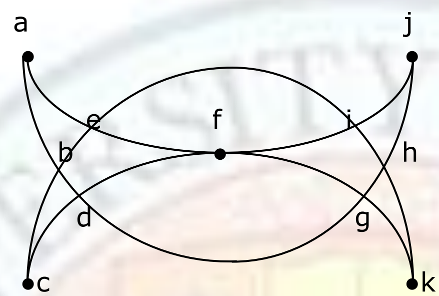
Solution: We know that in bipartite graph $K_{m,n}$, each of the left m vertices are attached with each of the right n vertices and vice-versa. So the m vertices have degree n and right n vertices have degree m.

Thus in $K_{4,5}$; four vertices have degree 5 and five vertices have degree 4. So it do not satisfy the condition for either Eulerian path or Eulerian n circuit. Hence the graph $K_{4,5}$ does not have either Eulerian path or Eulerian circuit.



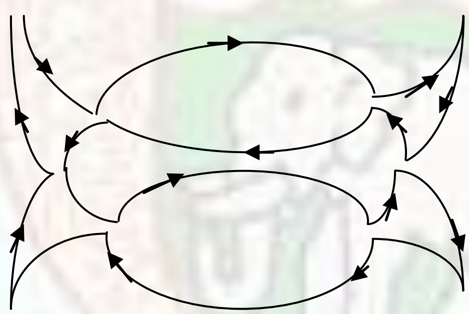
Fig(3.6.9)

Example 34: Many puzzles ask you to draw a picture in a continuous motion without lifting a pencil so that no part of the picture is retraced. We can solve such puzzles using Euler circuits and paths. For example Can **Mohammed's scimitars**, shown below be drawn in this way, where the drawing begins and end at the same point?



fig(3.6.10)

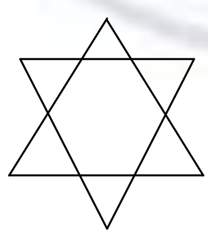
Solution: Since the degree of each vertex is even as the degree of a=2, b=4, c=2, d=4, e=4, f=4, g=4, h=4, i=4, j=2 and k=2. So, the graph **Mohammed's scimitars** is actually an Eulerian circuit by the corollary 1 of theorem 6. Thus we can trace the entire graph without lifting a pencil. The breakup figure is given below.



Fig(3.6.11)

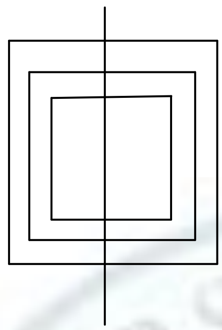
Example 35: Whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.

a)



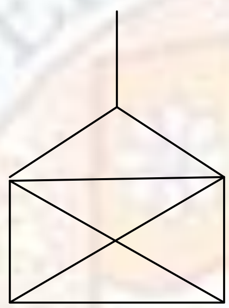
fig(3.6.12)

b)



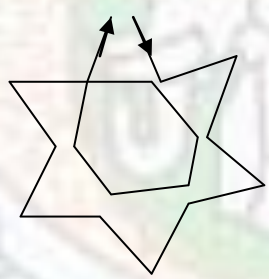
fig(3.6.13)

c)



fig(3.6.14)

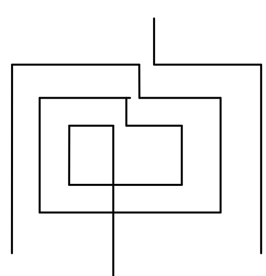
Solution: a) Since the degree of each vertex is even , so the graph is an Eulerian circuit. Thus we can trace the entire graph without lifting the pencil. The Eulerian circuit is given in fig(3.6.15)



Fig(3.6.15)

b) In this graph all vertices are of even degrees except two vertices, which are of odd degree. Hence we can still draw the graph without lifting the pencil as it is an Eulerian path.

The path is given in fig(3.6.16)

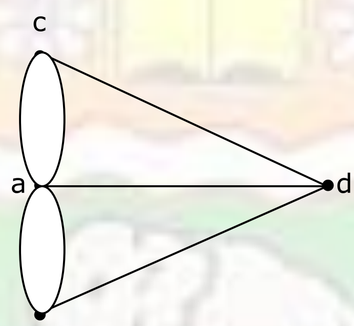


Fig(3.6.16)

c) This graph has one vertex with degree one, three vertices have degree three, and three vertices have degree four. Hence it has neither Eulerian path nor Eulerian circuit. So we cannot trace the graph in continuous motion without lifting pencil.

Kőnigsberg bridges Problem: Is it possible to start at some point in the town, travel across all the bridges, and end up at some other point in town?

Solution: The graphical representation of the **Kőnigsberg bridges** is given below. The degree of vertex a is 5, b is 3, c is 3 and of d is 3. Thus by the theorem 6 and its corollary we can say that it does not possesses either Eulerian path or Eulerian circuit. Hence such trip is impossible.



fig(3.6.17)

Value addition: Common Misconceptions

Eulerian paths and circuit

Euler paths and circuit can be used to solve many practical problems. For example, many applications ask for a path or circuit that traverses each street in a neighborhood, each road in transportation network, each connection in utility grid or each link in communication network exactly once. Finding Euler path or circuit in the appropriate graphical model can solve such problem. Consider a specific case,

If a postman can find an Euler path in the graph that represents the street the postman needs to cover, this path produces a route that traverses each street of the route exactly once. If no Euler path exists, some streets will have to be traversed more than once. This problem is known as **Chinese postman problem**

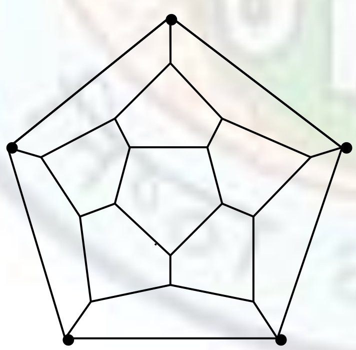
Among the other areas, where Euler circuits and paths are applied is in layout of circuits, in network multicasting, and in molecular biology, where Euler paths are used in the sequencing of DNA.
Source: Discrete Mathematics by K.H. Rosen

3.6.3 Hamiltonian Paths and Circuits

Finding Hamiltonian paths or circuit is a problem similar to the determination of an Eulerian path or an Eulerian circuits is to determine a path or a circuit that passes through each vertex in a graph once and only once. **Sir William Hamiltonian** invented the game "**All around the world**" in which the player is asked to determine a route along the dodecahedron that will pass through each angular point once and only once.

Hamilton's **A Voyage Round the World** Puzzle

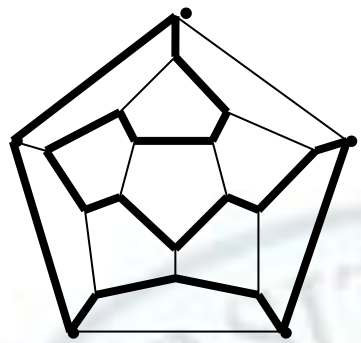
In 1857, Irish mathematician Sir William Rowan Hamilton, invented a puzzle **A Voyage round the world**. It consisted of a wodden dodecahedron, with a peg at each vertex of the dodecahedron, and string. The twenty vertices of the dodecahedron were labeled with different cities in the world.



Fig(3.6.18)

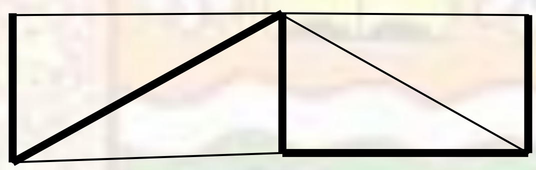
The object of the puzzle was to start at a city and travel along the edges of the dodecahedron, visiting each of the other 19 cities exactly once, and end up at the first city.

The solution of Hamilton's puzzle is shown below with dark lines.



Fig(3.6.19)

Hamilton's paths: A path in the graph is said to be Hamiltonian path if it traverses each of the vertices once and only once. Path in fig(3.6.20) with dark lines is an Hamiltonian path.



Fig(3.6.20)

Hamilton's circuit: A circuit is said to be Hamilton's circuit if it passes through each vertices in a graph exactly once. Fig(3.6.19) is an Hamiltonian circuit.

Example 36: Consider the problem of seating a group of people at a round table. If we let the vertices of an undirected graph denote the people and the edges represent the relation that two people are friends, a Hamiltonian circuit corresponds to a way of seating them so that everyone has a friend on each side.

Example 37: Let us consider the problem of ranking players in a round-robin tennis tournament such that player **a** will be ranked higher than player **b,** if **a** beats **b** ,or **a** beats a player who beats **b**, or **a** beats a player who beats another player who beats **b**, and so on. Since the outcomes of the matches can be represented as a directed complete graph,the existence of a Hamiltonian path in the graph means that it is always possible to rank the players linearly.

Example 38: We consider the problem of printing and then binding n books . There is one printing machine and one binding machine . Let p_i and b_i denote the printing time and binding time of book i, respectively. If it is known for any two books i and j that either $b_i = p_j$ or $b_j = p_i$, show that it is possible to specify an order in which the books are printed (and then bound) so that the binding machine will be kept busy until all books are bound, once the first book is printed . (Thus the total time it takes for completing the whole task is $pk+\sum^{n}_{i=1}b_i$ for some k.) Let us construct a directed graph of n vertices corresponding to the n books . There is an edge from vertex i to vertex j if and only if $b_i \geq p_j$. We note that this is a directed complete graph, and a Hamiltonian path in the directed complete graph will be an ordering of the books satisfying the condition stated above.

Although the problem of determining the existence of Hamiltonian paths or circuits has the same flavor as that of determining the existence of Eulerian paths or circuits, no simple necessary and sufficient condition is known. However, many theorems are known that give the sufficient conditions for the existence of Hamiltonian circuit. Also there are some properties can be used to show that a graph has no Hamiltonian circuit. **Some of the common properties are**

1. A Hamilton's circuit cannot contain a smaller circuit within it.
2. A graph with a vertex of degree one cannot have an Hamiltonian circuit.
3. If a vertex in a graph has degree two, then both edges that are incident to this vertex must be part of any Hamilton's circuit.
4. A complete bipartite graph $k_{m,n}$ has an Hamilton's circuit if m=n≥2

Theorem 7(Dirac's Theorem): If G is a simple graph with n vertices, n≥3, such that the degree of every vertex in G is at least n/2, then G has a Hamilton circuit.

Theorem 8: Let G be a linear graph of n vertices. If the sum of the degree for each pair of vertices in G is n-1 or larger, then there exist a Hamiltonian path in G.

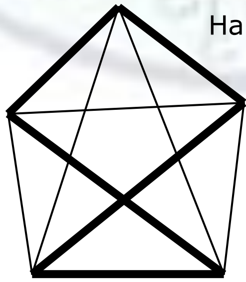
This theorem provides a sufficient condition, but not a necessary condition, for a simple connected graph. Or

Ore's Theorem: If G is a Simple graph with n vertices, n≥3, such that deg(u)+deg(v)≥n for every pair of non-adjacent vertices u and v in G, then G has an Hamilton's circuit.

Theorem 9: There is always a Hamiltonian path in a directed complete graph for n≥3.

Proof: We can form a Hamiltonian circuit in complete graph beginning at any vertex. Such a circuit can be built by visiting vertices in order we choose, as long as the path begins and ends at the same vertex and visit each other vertex exactly once. This is possible since there are edges in complete graph between any two vertices. Fig(3.6.21) shows the Hamiltonian circuit in K_5.

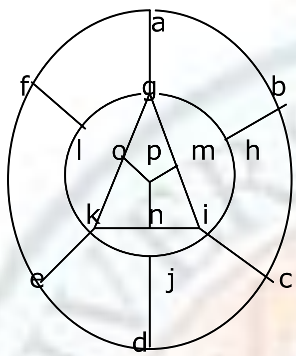
dark lines shows the Hamilton's circuit.



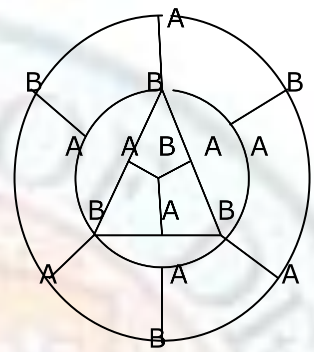
fig(3.6.21)

Example 39: There is no general method of solution to the problem of proving the non-existence of a Hamiltonian path in a graph. We are going to discuss an important example, which helps us in determining non existence of Hamiltonian path.

We want to show that the graph in Fig.(3.6.22) has no Hamiltonian path.



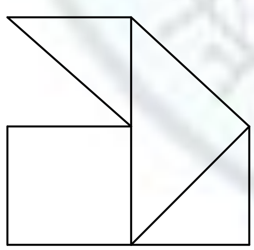
Fig(3.6.22)



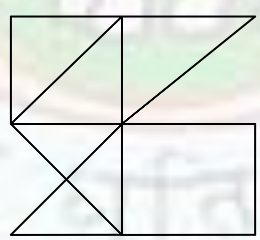
fig(3.6.23)

We label the vertex a by A and label all the vertices that are adjacent to it by B, which is shown in fig(3.6.23). Continuing, we label all the vertices that are adjacent to B vertex by A and label all the vertices that are adjacent to A vertex by B, until all vertices are labeled. The labeled graph is shown in graph is shown in Fig(3.6.23). If there is a Hamiltonian path in the graph, then it must pass through the A vertices and the B vertices alternately. However, since there are nine A vertices and seven B vertices, the existence of a Hamiltonian path is impossible.

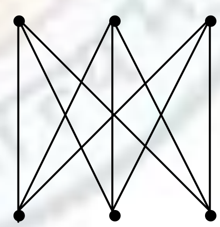
Example 40: Which of the graphs have Hamilton's circuit? If not, why?



Fig(3.6.24)



fig(3.6.25)

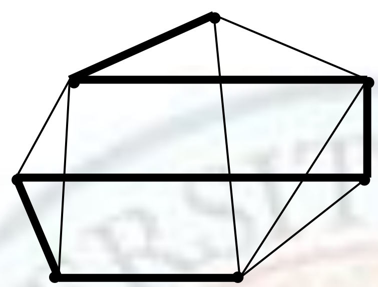


fig(3.6.26)

Solution: The graph fig(3.6.24) and fig(3.6.26) have Hamilton's circuit, whereas fig(3.6.25) has no Hamilton's circuit. We know that "A complete bipartite graph $k_{m,n}$ has an Hamilton's circuit if $m=n \geq 2$".

Example 41: While scheduling examination, the care is taken so that no instructor is assigned more than four examinations and no two examinations given by the same instructor are scheduled in consecutive days. Can seven examinations be scheduled suitably in seven days?

Solution: Let G represent a graph with seven vertices corresponding to seven examinations as shown in fig



Fig(3.6.27)

Let G=(V,E), V is set of examinations. Let E represent relation in V, that is, edge representing that the two examinations are not given by the same examiner, that is, edge between any two vertices represents that two examinations are given by two different examiners.

As every examiner is given at the most four examinations and edge represent that examination is given by two different examiners, degree of each vertex is at least 3. Sum of degree of any two vertices is at least six, which means that G contains a Hamiltonian path. Existence of Hamiltonian path corresponds to a suitable schedule for seven examinations in seven-day period satisfying the given constraints.

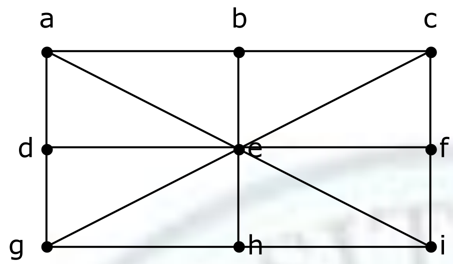
Example42: Does the graph in fig(3.6.28) have an Hamilton's path? If so, find such a path. If it does not, then give argument to show why no such path exists.

Fig(3.6.28)

Solution: Yes this graph has Hamilton's path. a,b,c,f,d,e is an Hamilton's path, which is also shown in the fig(3.6.29) by dark lines.

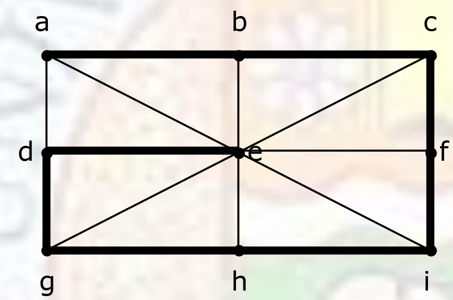
Fig(3.6.29)

Example 43: Does the graph in fig(3.6.30) have an Hamilton's path? If so, find such a path. If it does not, then give argument to show why no such path exists.



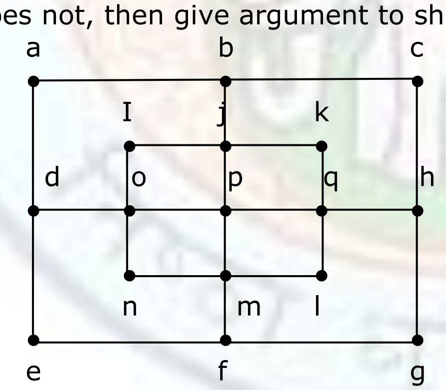
Fig(3.6.30)

Solution: yes, this graph has Hamilton's path. a,b,c,f,i,h,g,d,e is an Hamilton's path, which is shown by dark lines in the fig(3.6.31)



fig(3.6.31)

Example 44 : Does the graph in fig(3.6.32) have an Hamilton's path? If so, find such a path. If it does not, then give argument to show why no such path exists.



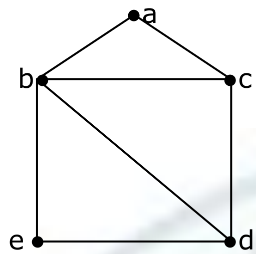
Fig(3.6.32)

Solution: No, in the graph given above has no Hamilton's path. Since, there are eight vertices of degree 2, and only two of them can be end vertices of a path. For each of other six vertices, their two incident edges must be in the path. It is not hard to see that if there is to be an Hamilton's path, exactly one of the inside corner vertices must be an end point, and that is impossible.

Example 45 : Does the graph in fig(3.6.33) and fig(3.6.34) have an Hamilton's circuit? Whether Dirac's theorem can be used to show that the graph has Hamilton's circuit?

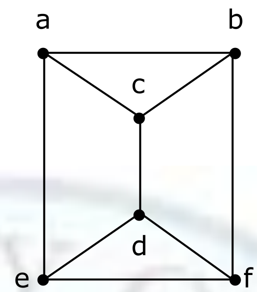
Whether Ore's theorem can be used to show that the graph has Hamilton's circuit?.If it does not, then give argument to show why no such path exists?

1)



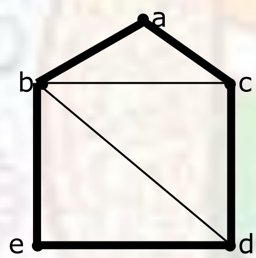
Fig(3.6.33)

2)



fig(3.6.34)

Solution: 1) For fig (3.6.33), there exist an Hamiltonian circuit, which is shown below by dark lines. So far Dirac's and Ore's theorem is concerned, these two theorem cannot be used to show that the graph has Hamiltonian circuit, as if we apply these two we find no Hamiltonian circuit exist in the graph. [Since the degree of vertex a is 2<3 (n/2), so by Dirac's theorem no Hamiltonian circuit exist also, the sum of degree of vertex a and vertex e is 4 <5 so, by Ore's theorem no Hamiltonian circuit exist.]

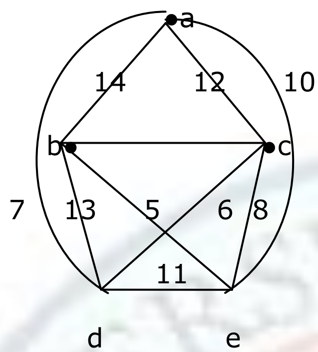


fig(3.6.33a)

2)For fig(3.6.) Dirac's as well as Ore's theorem is applicable as the degree of each vertex is 3=n/2=6/2=3 and the sum of two non-adjacent vertices is 6=n. Thus by both the theorem there exist an Hamiltonian circuit in the graph.

THE TRAVELING SALESPERSON PROBLEM

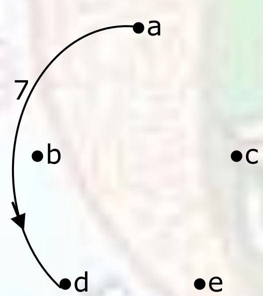
Hamilton's paths and circuits can be used to solve many practical problems. For example, many applications ask for paths or circuits that visits each road intersection in a city, each place pipelines intersect in a utility grid, or each node in a communication network exactly once. Finding an Hamiltonian paths or circuit in the appropriate graph model can solve such problems. The famous **Travelling salesman problem** asks for finding shortest route a travelling salesman should take to visit a sets of places in the city. Though it require a weighted graph and problem is related to shortest path, which we discuss in the later section. Here we are considering one such problem and correlate it with Hamilton's circuit.



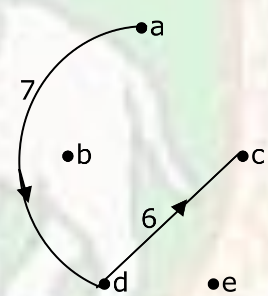
Fig(3.6.35)

Here the vertices represent the different places in the city, where the salesman has to visit and the edges represent the possible path between the two different places in the city. The weight on the edge represent the length of the path.

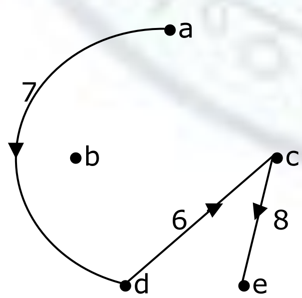
Salesman will try to select that possible path which is shortest from the place ,where he is. Here we have started with the vertex a.



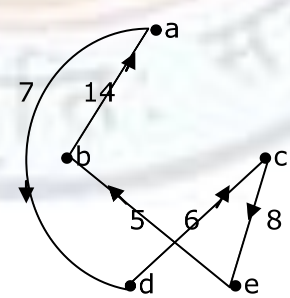
Fig(3.6.35a)



fig(3.6.35b)

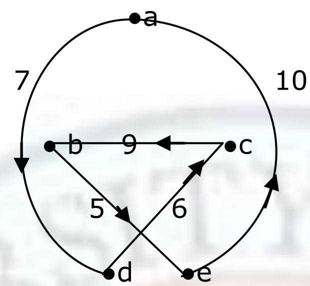


Fig(3.6.35c)



fig(3.6.35d)

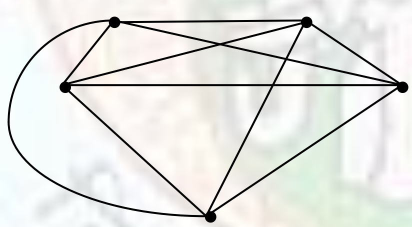
Fig(3.6.d) is the Hamilton's circuit. Length of route is 40 unit. In shortest path problem we put an extra effort to find is there an alternative route, which is shorter than we found. That extra effort resultant in fig(3.6.e), which is the shortest and also an Hamiltonian circuit. The length of shortest route is 37 unit.



Fig(3.6.35e)

Example 46: Show a graph that has both an Eulerian circuit and a Hamiltonian circuit.

Solution: Since all the vertices have even degree, i.e, 4, so by the corollary 1 of theorem 6 the graph shown in fig(3.6.36) has an Eulerian circuit. We can easily observe that the given graph is isomorphic to complete graph K_5, and we know that the complete graph has an Hamiltonian graph.

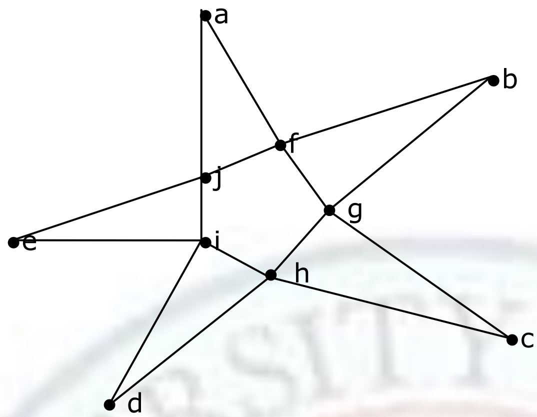


Fig(3.6.36)

Example 47: Show a graph that has an Eulerian circuit but has no Hamiltonian circuit.

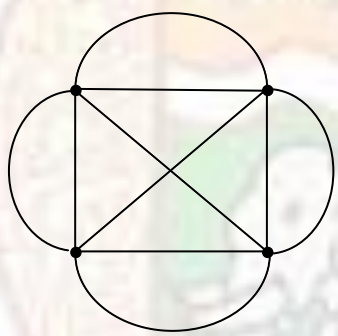
Solution: Since all the vertices have even degree, five vertices have degree 2 and five have degree 4, so it must have an Eulerian circuit by corollary 1 of theorem 6.

We know that, from Ore's theorem "If G is a Simple graph with n vertices, n≥3, such that deg(u)+deg(v)≥n for every pair of nonadjacent vertices u and v in G, then G has an Hamilton's circuit". Now, sum of degrees of vertices a and b is 4≤10(number of vertices in fig(3.6.37). hence it does not have an Hamilton's circuit.



Fig(3.6.37)

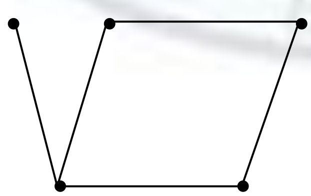
Example 48: Show a graph that has no Eulerian circuit but has a Hamiltonian circuit.
Solution: Since all the vertices have odd degree, that is 5, so by the corollary of theorem 6 it does not have Eulerian circuit, but clearly it has Hamiltonian circuit (refer example 38)



Fig(3.6.38)

Example 49: Show a graph that has neither an Eulerian circuit nor a Hamiltonian circuit.

Solution: Since the degree of one vertex is 1, three vertices have degree 2 and one vertex has degree 3. So by the theorem 6 it has Eulerian path but not Eulerian circuit. Also we know that "A graph with a vertex of degree one cannot have an Hamiltonian circuit". Hence it does not have Hamilton's circuit.



Fig(3.6.39)

Summary

- A **graph** is an abstract representation of a set of objects where some pairs of the objects are connected by links
- The interconnected objects are represented by mathematical abstractions called ***vertices***, and the links that connect some pairs of vertices are called ***edges***.
- A vertex is said to be an isolated vertex if there is no edge incident on it
- A loop is an edge that connects a vertex to itself
- A loop at a vertex contributes twice to the degree of that vertex
- The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at which a vertex contributes twice to the degree of that vertex.
- **The Handshaking Theorem** Let G=(V,E) be an undirected graph with e edges, then $2e=\sum deg(V)$, for all $v \in V$
- An undirected graph has an even number of vertices of odd degree
- Two Graphs are said to be isomorphic to each other if there is one to one correspondence between their vertices and between their edges such that their incidences are preserved
- A Graph G is said to be **complete** if every vertex in G is connected to every other vertex in G.
- Let K_n be a Complete graph, then number of vertices = n number of edges= ${}^nC_2$ = n(n-1)/2
- A Graph G is **regular of degree k** or simply **k-regular** if every vertex has degree k. In other words a graph is regular if every vertex has same degree.
- The five-regular polyhedral are known as **Platonic solid**
- A multi graph is said to be a pseudo graph if it has at least one loop at one of its vertex.
- When we have multiple edges in the graph, it is convenient to use matrices to represent the graph.
- In a directed graph, a path is a sequence of edges $(e_1,e_2,e_3,....,e_k)$ such that the terminal vertex of e_i coincides with the initial vertex e_{i+1} for $1\le i\le k-1$. A path is said to be **simple** if does not include the same edge twice
- A circuit is a path $(e_1,e_2,e_3,.....,e_k)$, in which the terminal vertex of e_i, coincides with the initial vertex of e_i. A circuit is simple if it does not include the same edge twice
- In a (directed or undirected) graph with n vertices, if there is a path from vertex v_1 to vertex v_2 ,then there is a path of no more than n-1 edges from vertex v_1 to vertex v_2 .
- A useful isomorphic invariant for simple graph is the existence of a simple circuit of length k, where k is a positive integer greater than 2.
- An Eulerian path in a graph is a path that traverses each edge in the graph once and only once
- An Eulerian circuit in a graph is a simple circuit that traverses each edge in the graph exactly once
- An undirected graph possesses an Eulerian path if and only if it is connected and has either zero or two vertices of odd degree.
- An undirected graph possesses an Eulerian circuit if and only if it is connected and its vertices are all of even degree

- A path in the graph is said to be Hamiltonian path if it traverses each of the vertices once and only once
- A circuit is said to be Hamilton's circuit if it passes through each vertices in a graph exactly once
- If G is a simple graph with n vertices, $n \geq 3$, such that the degree of every vertex in G is at least n/2, then G has a Hamilton circuit
- Let G be a linear graph of n vertices. If the sum of the degree for each pair of vertices in G is n-1 or larger, then there exist a Hamiltonian path in G
- If G is a Simple graph with n vertices, $n \geq 3$, such that $deg(u)+deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has an Hamilton's circuit.
- There is always a Hamiltonian path in a directed complete graph for $n \geq 3$.

Exercises

3.1 Prove that (4,3,2,2,1) is graphical. Where numerals in the bracket represent the degree of vertices.

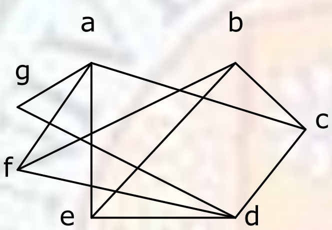
3.2 What does the degree of a vertex represent in a collaboration graph? What do isolated and pendant vertices represent?

3.3 Prove that 3-regular graphs must have even number of vertices.

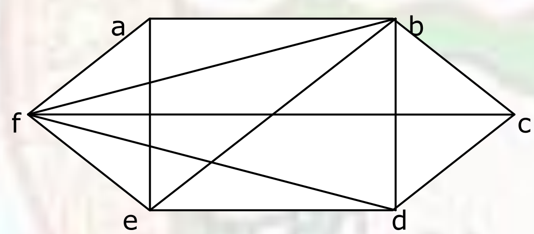
3.4 Show that C_6 (cycle-6 graph) is Bipartite.

3.5 Are the following graphs Bipartite? Give reasoning in your support.

a)



b)



3.6 State and prove Handshaking theorem .

3.7 For which value of n are these graphs bipartite and for which value of n these are regular?

a) K_n b) C_n c) W_n

3.8 How many vertices does a graph have if it has vertices of degree 5,2,2,2,2,1? Draw such graph.

3.9 Draw a graph model, to represent airline routes where every day there are four flights from Delhi to Mumbai, two flights from Mumbai to Delhi, three flights from Mumbai to Kolkata, two flights from Kolkata to Mumbai, one flights from Mumbai to Goa, two flights from Goa to Mumbai, three flights from Mumbai to Patna, two flights from Patna to Mumbai, and one flight from Patna to Goa, with

A) An edge between vertices representing cities that have a flight between them(in either directions)

B) An edge between vertices representing cities for each flight that operates between them(in either direction)

C) An edge for each flight from a vertex representing a city the flight begins to the vertex representing the city where the flight ends.

3.10 What are the different types of graph? Differentiate between Simple, Multi-graph and Pseudo graph.

3.11 Identify the Special graph whose symbols are $\mathbf{W_n}$, $\mathbf{K_n}$, $\mathbf{K_{m,n}}$, $\mathbf{C_n}$. Give the general formula for their number of vertices and number of edges.

3.12 Construct an influence graph model for the board members of a company if the president can influence the director of operation, director of finance, the director of marketing and director of human resources; the director of HR can influence the director of R&D, and the director of finance; the director of operation can influence the director of marketing and director of R&D. The CFO can influence the director of finance only.

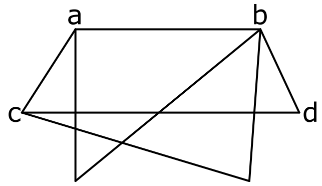
3.13 Draw a Bollywood graph model by representing actors by vertices and the edge represents the two actors have worked together.

3.14 Explain how can a graph can be used to model electronic mail massages in a network?

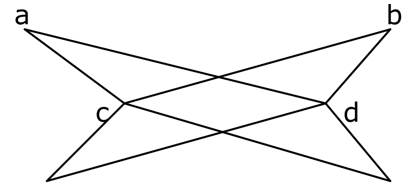
3.15 Three married couples on a journey come to a river where they find a boat, which cannot carry more than two persons at a time. The crossing of river is complicated by the fact that the husbands are all very jealous and will not permit their wives to be left without them in a company where there are other men present. The situation becomes more complicated as wives are not allowed even to simply cross the river in a boat, drop a person and come back, when there are other men present on the bank and the respective husband is not present. Construct a graph to show the transfer can be made.

3.16 Determine whether the given graphs are bipartite

a)



c)



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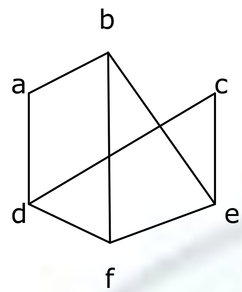
b)

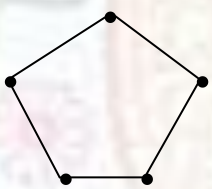


3.17 Draw the graph corresponding to the adjacency matrix

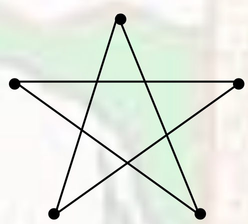
a) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

3.18 Determine whether the given pair of graphs is isomorphic.

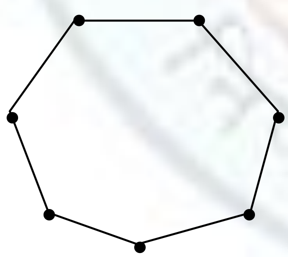
a)



a’)

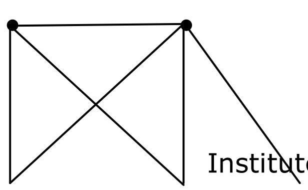


b)

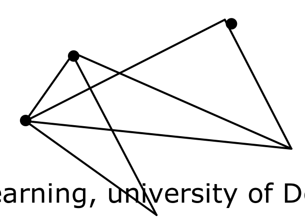


b’)

c)



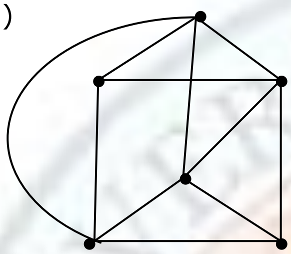
c’)



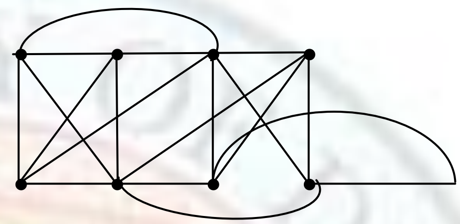
3.19 Represent these graph with an adjacency matrix
a) K_6 b) $K_{3,4}$ c) W_6 d) Q_3

3.20 Determine whether the given graphs has an Eulerian paths or circuit.

a)



b)

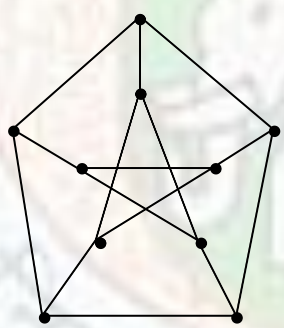


3.21 For which value of m and n does the complete bipartite graph Km,n have
a) Eulerian Path b) Eulerian Circuit

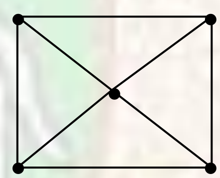
3.22 Show that the complete graph K_n, $n \geq 3$, has Hamiltonian circuit.

3.23 Determine whether the given graphs has an Hamiltonian circuit

a)



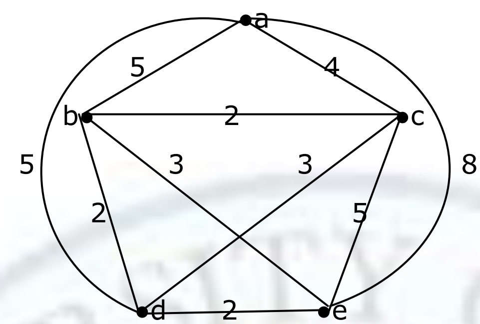
b)



3.24 Is it possible to move a knight on a 8X8 chessboard so that it completes every possible move exactly once? A move between two squares of the chessboard is completed when it is made in either direction.

3.25 For what value of m and n a complete bipartite graph $k_{m,n}$ has an Hamilton's circuit?

3.26 Use nearest neighbor method to determine a Hamiltonian circuit for the graph given below, starting with vertex a.


a
5
4
b
2
c
5
3
3
8
2
5
d
2
e

Glossary

Arranged in alphabetical sequence. First word as bold and its description in regular font.

1. **Adjacency Matrix:** A matrix whose elements are the numerals 1 or 0 depending on the edges which are adjacent or not to the given vertex.
2. **Adjacent :** Two vertices are adjacent if they are connected by an edge.
3. **Arc :** A synonym for edge
4. **Bipartite Graph :** A graph is **bipartite** if its vertices can be partitioned into two disjoint subsets U and V such that each edge connects a vertex from U to one from V
5. **Complete graph :** A complete graph with *n* vertices (denoted K*n*) is a graph with *n* vertices in which each vertex is connected to each of the others
6. **Connected graph :** A graph is connected if there is a **path** connecting every pair of vertices
7. **Directed graph:** A graph, whose edges are an arrow which shows the direction is directed graph.
8. **Graph :** Informally, a graph is a finite set of dots called **vertices** (or **nodes**) connected by links called **edges**
9. **Incidence Matrix:** A matrix whose elements are the numeral 1 or 0 depending upon whether an edge is incidence on the given vertex or not.
10. **In-degree :** The in-degree of a vertex *v* is the number of edges with *v* as their terminal vertex.
11. **Isolated :** A vertex of degree zero (with no edges connected) is isolated
12. **Isomorphic graphs:** Two graphs are said to be isomorphic if there is one-to-one correspondence between their vertices and between edges such that incidences are preserved.
13. **Loop ;** A loop is an edge that connects a vertex to itself
14. **Out-degree :** The out-degree of a vertex *v* is the number of edges with *v* as their initial vertex.
15. **Pendant :** A vertex of degree (with only one edge connected) is a pendant edge.
16. **Pseudo graph:** A multi graph is said to be a pseudo graph if it has at least one loop at one of its vertex.
17. **Regular graph**: A Graph G is **regular of degree k** or simply **k-regular** if every vertex has degree k.

References

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3. Web Links

Unit no. . no. of web link

1.1 http://en.wikipedia.org/wiki/
1.2 http://www.itl.nist.gov/div897/sqg/dads/#A

5. Any Other (Please Specify)