Paper Name: Discrete Structures
Lesson : Planar Graphs and Graph Coloring
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2.1 Introduction

In this chapter we will study two important concepts of Graph theory. First, we will study the question of whether a graph can be drawn in the plane without edges crossing and if it is possible to find at least one way to represent a graph in a plane without any edges crossing. Second, we will look at ways of coloring vertices in a graph in such a way that adjacent vertices are colored differently.

2.2 Planar Graphs

Consider the problem of joining three houses to each of three separate utilities (Gas, Water and Electricity), as shown in Figure 2.1. Is it possible to join these houses and utilities so that none of the connections cross? This problem can be modeled using the complete bipartite graph $K_{3.3}$, as shown in Figure 2.2. The original question can be rephrased as: Can $K_{3.3}$ be drawn in the plane so that no two of its edges cross?

Figure 2.1 – Three Houses and three Utilities (Source: Web Link 1)

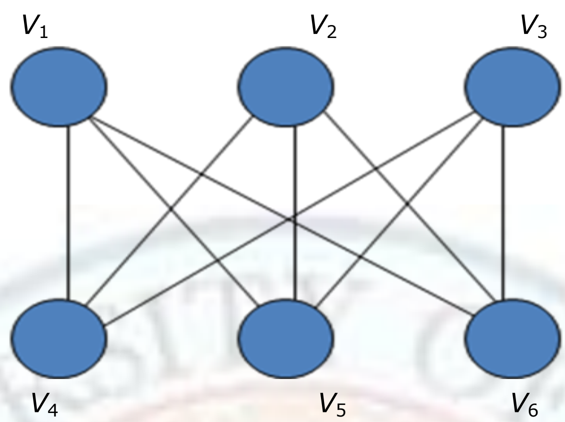


Figure 2.2: The Graph $K_{3,3}$

Here, we will study the question of whether a graph can be drawn in the plane without edges crossing. In particular, we will answer the house and utilities problem.

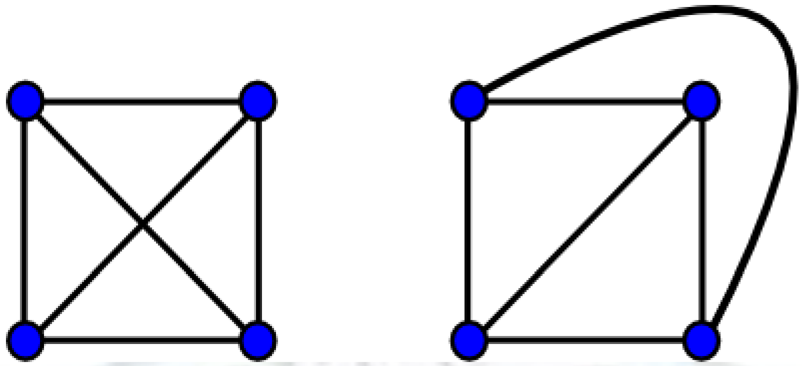
There are always many ways to represent a graph. When is it possible to find at least one way to represent this graph in a plane without any edges crossing?

DEFINITION: A graph is ***planar*** if it can be drawn in two-dimensional space with no two of its edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint). Such a drawing is called a planar representation of the graph or a *plane drawing*.

A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

EXAMPLE 2.1: Is K_4 (shown is Figure 2.3(a) with two edges crossing) planar?

Solution: K_4 is planar because it can be drawn without crossings, as shown is Figure 2.3(b).

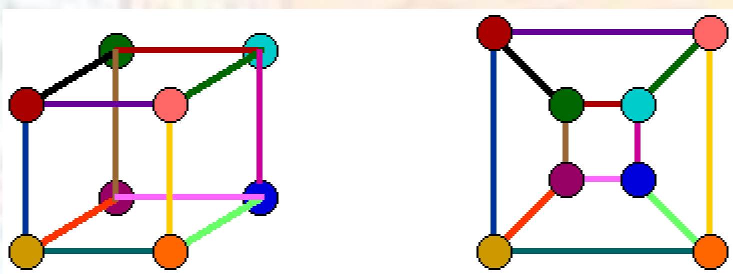


(a) (b)

Figure 2.3: The Graph K_4 is planar

EXAMPLE 2.2: Is Q_3, shown in Figure 2.4(a), planar?

Solution: Q_3 is planar, because it can be drawn without any edges crossings, as shown in Figure 2.4(b).



(a) (b)

Figure 2.4: The graph Q_3 and its planar representaion (Source: Web Link 1)

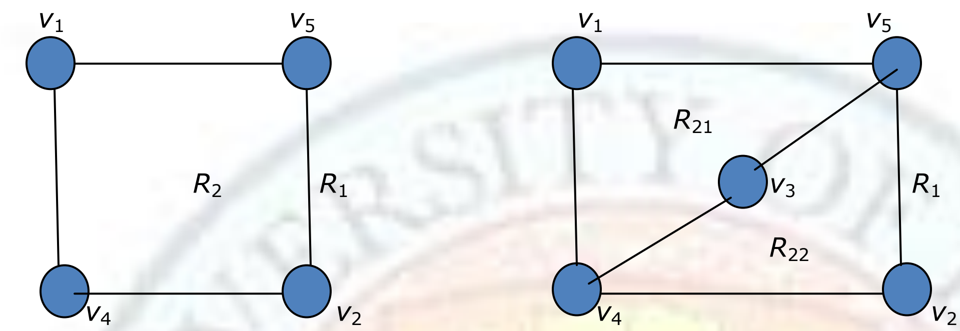
We can show that a graph is planar by displaying a planar representation. It is harder to show that a graph is nonplanar. Let's try proving for $K_{3,3}$.

EXAMPLE 2.3: Is $K_{3,3}$, shown in Figure 2.2, planar?

Solution: Any attempt to draw $K_{3.3}$ in the plane with no edges crossing is doomed. In any plane representation of $K_{3.3}$ the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions R_1 and $R_{2,}$ as shown in Figure 2.5(a). The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_1 and v_4 and between v_3 and v_5 separate R_2 into two sub regions, R_{21} and R_{22}, as shown in Figure 2.5(b).

Next, note that there is no way to place the final vertex v_6 without forcing a crossing. For if v_6 is in R_1, and then the edge between v_6 and v_3 cannot be drawn without a crossing. If v_6 is in R_{21}, then the edge between v_2 and v_6 cannot be drawn without a crossing. If v_6 is in R_{22}, then the edge between v_1 and v_6 cannot be drawn without a crossing.

A similar argument can be used when v_3 is in R_1. It follows that $K_{3.3}$ is not plane.



(a) (b)

Figure 2.5: Showing that $K_{3.3}$ is not planar

Example 2.3 solves the utilities-and-houses problem that was described at the beginning of this section. The three houses and three utilities cannot be connected in the plane without a crossing.

Planarity of graphs plays an important role in the design of electronic circuits. We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges. We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.

2.2.1 Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region. For instance, the planar representation of the graphs shown in Figure 2.6 and 2.7 split the plane into four and five regions respectively. In Figure 2.6, these are colored as red, green, and blue and white (the outer unbounded region). In the figure 2.7, the regions have been labeled R_1 to R_5 (note the unbounded region R_5).

DEFINITION: The number of edges on the boundary of a region is called **degree** of a region.

An edge is counted in the degrees of two regions, which have this edge as their boundary. In Figure 2.7, for example, edge(V_2-V_6) contributes one each to the degrees of regions R_1 and R_2. When an edge occurs twice on the boundary of a region, i.e. it can be traced twice while tracing boundary of the region, it contributes two to the degree of the same region.

Such an edge exists entirely in one region. In Figure 2.7, for example, edge(V_1-V_9) contributes two to the degree of regions R_1. The numbers in the brackets indicates degree of each region in Figure 2.7. Each of the four regions has degree 3 in Figure 2.6.

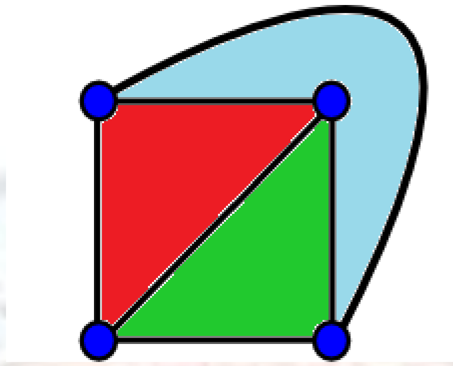


Figure 2.6: The Regions, coloured in different colurs, of a Planar Representation of a Graph

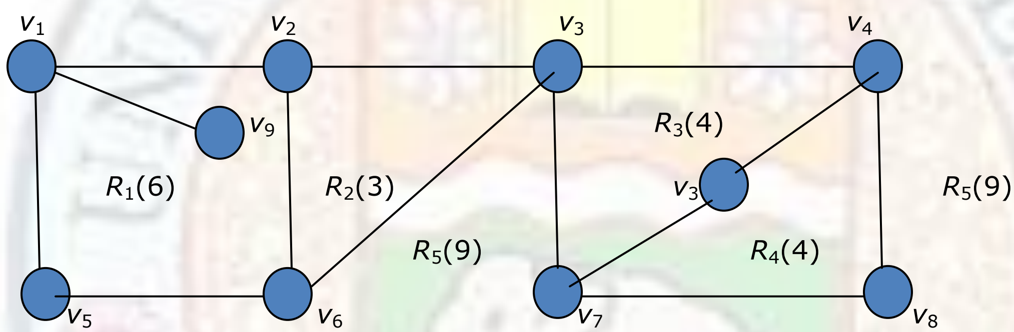


Figure 2.7: The Regions of the Planar Representation of a Graph

Euler showed that all planar representations of a graph split the plane into the same number of regions. He accomplished this by finding a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

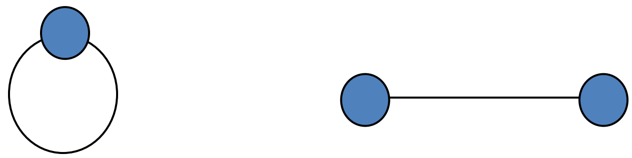
THEOREM 2.1 EULER'S FORMULA: Let G be a connected planar graph with *e* edges and *v* vertices. Let *r* be the nuber of regions in a planar representation of graph G. Then, the formula given by Euler is

$$\boxed{r = e - v + 2} \quad \textbf{Or} \quad \boxed{r - e + v = 2} \qquad (2.1)$$

Proof: (By induction on number of edges i.e., *e*.) Suppose that $e = 0$. Then for G to be a connected graph there may be only one vertex, and one region. So

$$r - e + v = 1 - 0 + 1 = 2$$

and the formula works in the $e = 0$ case. For the $e = 1$ case, there are 2 possibilities, as shown in the following figure (Figure 2.8).



$r - e + v = 2 - 1 + 1 = 2$ $\qquad$ $r - e + v = 1 - 1 + 2 = 2$

Figure 2.8: Two possible graphs (G_1) with one edge

Thus, Euler's Formula works for $e = 0$ and 1. For the inductive step, we assume that $e > 1$ and that the Euler Formula works for all graphs with n edges. Let r_n, e_n and v_n represent the number of regions, edges and vertices of the planar representation of G_n (a graph containing n edges). Then, $r_n - e_n + v_n = 2$. We will construct graph G_{n+1} containing (n+1) edges from G_n by adding an edge to G_n. Let $\{a, b\}$ be the edge that is added to G_n to obtain G_{n+1}. There are two possibilities to consider.

In the first case, both a and b are already in G_n. These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a, b\}$ to G_n without two edges crossing (we need G_{n+1} as planar). The addition of this new edge splits R into two regions. Consequently, in this case, $r_{n+1} = r_n + 1$, $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n$. Thus, $r_{n+1} - e_{n+1} + v_{n+1} = 2$. This case is illustrated in Figure 2.9.

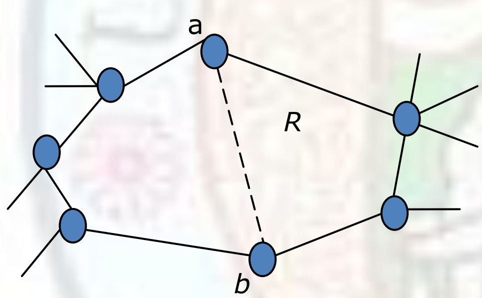


Figure 2.9: Adding an edge (connecting two already existing vertices) to G_n to get G_{n+1}

In the second case, one of the two vertices of the new edge is not already in G_n. Suppose that a is in G_n but that b is not. Adding this new edge does not produce any new regions, because b must be in a region that has a on its boundary. Consequently, $r_{n+1} = r_n$, $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n + 1$. Thus the formula relating the number of regions, edges, and vertices still remains true. In other words, $r_{n+1} - e_{n+1} + v_{n+1} = 2$. This case is illustrated in Figure 2.10.

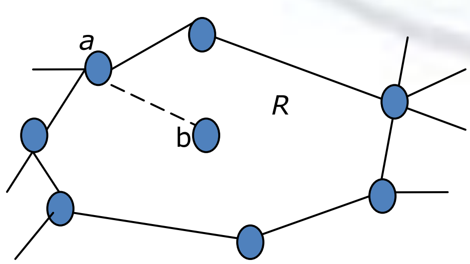


Figure 2.10: Adding an edge and a new vertex to G_n to get G_{n+1}

Hence $r_n = e_n - v_n + 2$ or $r_n - e_n + v_n = 2$ for all n. This completes the proof of the theorem.

Example 2.4: Suppose that a connected planar simple graph has 10 vertices, each of degree 4. Into how many regions does a representaion of this planar graph split the plane?

Solution: The graph has 10 vertices, each of degree 4, so $v = 10$. As sum of the degrees of the vertices is equal to twice the number of edges, we have $4v = 2e$, or $e = 20$. Consequently, from Euler's formula, the number of regions, r, is

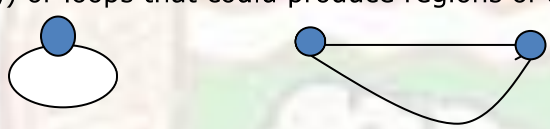
$$r = e - v + 2 = 20 - 10 + 2 = 12.$$

It is truly remarkable that, with no exception, all planar graphs satisfy Euler's formula.

We will now establish some more results using Euler's Formula.

COROLLARY 2.1.1: If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

PROOF: A simple graph has no multiple edges that could produce regions of degree two (Figure 2.11(b)) or loops that could produce regions of degree 1 (Figure 2.11(a)).



(a) (b)

Figure 2.11: Regions of degree 1 and degree 2 (not allowed in a simple graph)

Thus, a connected planar simple graph drawn in a plane divides the plane into (say) r regions each of degree atleast three. The degree of unbounded region is also atleast three as $v \geq 3$.

Also, the sum of the degrees of all regions is equal to twice the number of edges as each edge contributes two to this sum - either it is counted once each in two different regions (when it occurs on the boundary dividing these two regions, e.g. edge v_3-v_7 in Figure 2.7) or it appears twice in the same region (e.g. edge v_1-v_9 in Figure 2.7).

From this discussion, it follows

$$2e = \Sigma_{\text{all regions R}} \deg(R) \geq 3r$$
$$\Rightarrow (2/3)e \geq r$$

Using Euler's formula, $r = e - v + 2$, we obtain

$$r = e - v + 2 \leq (2/3)e$$
$$\Rightarrow e/3 \leq v - 2$$
$$\Rightarrow e \leq 3v - 6 \qquad (2.2)$$

Hence proved.

COROLLARY 2.1.2: If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

PROOF: A simple graph has no multiple edges thus *G* has one or two vertices, the result is true. If *G* has at least three vertices, by Corollary 2.1.1 above we know that $e \le 3v - 6$, so

$$2e \le 6v - 12 \qquad (2.3)$$

If the degree of every vertex were at least six, then because $2e = \sum_{v \in V} \deg(v)$, we would have $2e \ge 6v$. This contradicts inequality 2.3. Thus there must be a vertex with degree no greater than five.

COROLLARY 2.1.3: If G is a connected planar simple graph having *e* edges and *v* vertices with $v \ge 3$ and no circuits of length three, then $e \le 2v - 4$.

PROOF: G being a simple graph has no multiple edges that could produce regions of degree two or loops that could produce regions of degree 1 and it has no circuits of length three, thus the degree of a region must be at least four. Also, the sum of the degrees of all regions is equal to twice the number of edges. From this discussion, it follows

$2e = \sum_{\text{all regions } R} \deg(R) \ge 4r$
⇨ $(2/4)e \ge r$

Using Euler's formula, $r = e - v + 2$, we obtain

$r = e - v + 2 \le (2/4)e$
⇨ $e/2 \le v - 2$
⇨ $e \le 2v - 4 \qquad (2.4)$

Hence proved.

Now let's use these corollaries.

Example 2.5: Show that K_5 (Figure 2.12) is nonplanar.

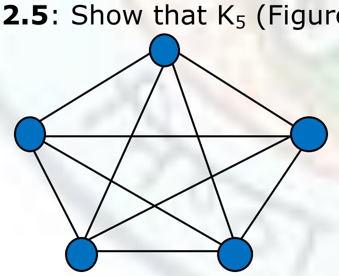


Figure 2.12: The Graph K_5

Solution: The graph K_5 has five vertices and 10 edges. Putting these values in inequalities 2.2, we get

$e \le 3v - 6$
⇨ $10 \le 3x5 - 6 = 9$

which is a contradiction. Hence, K_5 (Figure 2.12) is nonplanar.

Example 2.6: Show that $K_{3,3}$ (Figure 2.2) is nonplanar using corollaries of euler's Theorem.

Solution: $K_{3,3}$ is bipartite, hence has no circuits of length three, Corollary 2.1.3 can be used. Here, $e = 9$ and $v = 6$. Thus we get,

$e \leq 2v - 4$

⇨ $9 \leq 2 \times 6 - 4$

which is not true. Hence, $K_{3,3}$ is nonplanar.

2.2.2 Kuratowski's theorem

We have already proved that K_5 and $K_{3,3}$ are not planar. Its clear that any graph having these graphs as subgraphs, is not planar. Also, any non-planar graph must contain a subgraph that can be obtained from either of these graphs.

DEFINITION: An **elementary subdivision** is an operation on a graph where an edge {u,v} is removed and a new vertex *w* is added together with edges {u, w} and {w, v} to obtain another graph.

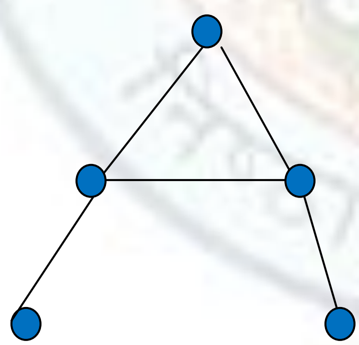
DEFINITION: Two graphs G = {V,E} and G′ = {V′, E′} are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

Value Addition: Animation

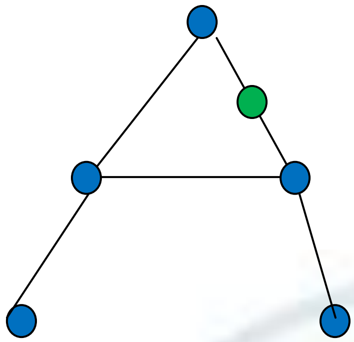
Heading text: Homeomorphic Graphs

Body text: Now we show how to use elementary subdivisions to get homeomorphic graphs.

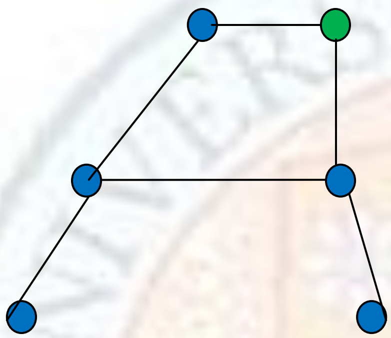
Following are the frames for which change a graph G = {V,E} to a **homeomorphic** graph G′ = {V′, E′}. Whenever a vertex/edge is added or the shape of the graph is changed, the change needs to be animated.



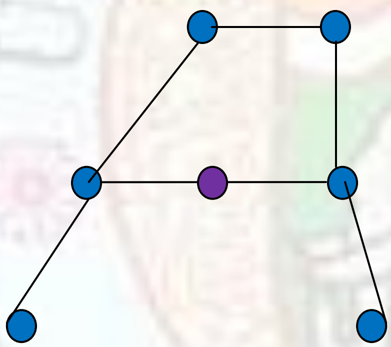
G = {V,E}



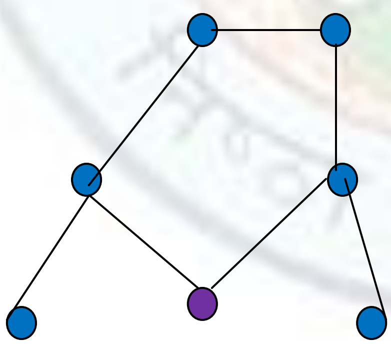
Add a vertex



G′ = {V′,E′}



Add a vertex



G′′ = {V′′,E′′}

THEOREM 2.2 Kurotawski's Theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$. We can also say that a finite graph is planar if and only if it contains no subgraph that is isomorphic to or is a subdivision of K_5 or $K_{3,3}$ where K_5 is the complete graph of order five and $K_{3,3}$ is the complete bipartite graph with three vertices in each of the halves.

Value addition: Biographic Sketch
Heading text: Kazimierz Kuratowski
Body Text: Kazimierz Kuratowski (Warsaw, February 2, 1896–June 18, 1980) was a Polish mathematician and logician. He was one of the leading representatives of the Warsaw School of Mathematics. He was a son of Marek Kuratow, a barrister, and Róża Karzewski. Kuratowski was born a subject of Tsarist Russia. He attended secondary school in Warsaw. In 1913, he enrolled in an engineering course at the University of Glasgow in Scotland, but he could complete only one year of study and could not return there after the outbreak of World War I. In 1915, Kuratowski restarted his university education at the Warsaw University, this time in mathematics. He published his first paper in 1919 and obtained his Ph.D. in 1921. He was an active member of the group known as the Warsaw School of Mathematics, working in the areas of the foundations of set theory and topology. He was appointed associate professor of mathematics in 1927, at Lviv Polytechnical University, where he stayed for 7 years, collaborating with important Polish mathematicians Banach and Ulam. In 1930, while at Lviv, he completed his work characterizing planar graphs. In 1934 he returned to Warsaw University as a full professor. Until the start of World War II, he was active in research and teaching. During the war, because of the persecution of educated Poles, Kuratowski went into hiding under an assumed name and taught at the clandestine Warsaw University. After the war, he helped revive Polish mathematics, serving as director of the Polish National Mathematics Institute. He wrote over 180 papers and three widely used textbooks.
Source: Kenneth Rosen, Discrete Mathematics and Its Applications, Sixth Edition

Example 2.7: Show that following graph (Figure 2.13) is nonplanar.

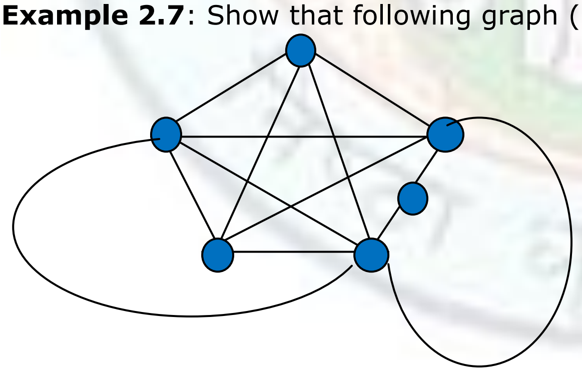


Figure 2.13: The Graph G

Solution: The subgraph shown in Figure 2.14 of the graph in Figure 2.13 is homeomorphic to K_5 (notice: green vertex), thus G is not Planar.

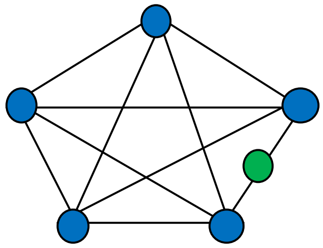
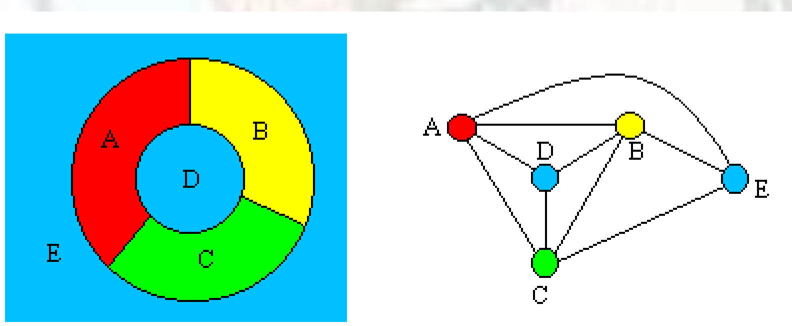


Figure 2.14: A subgraph of Graph in Figure 2.13

2.3 Graph Coloring

Consider the problem of coloring a map *M* in such a way that no adjacent regions (sharing a border) have the same color. One solution is to use a different color for each region. However, this is inefficient, considering some maps might have too many regions, e.g. map showing states of USA will require more than 50 colors. Instead a small number of colors should be used. This is where Graph Theory is used. Each map in a plane can be represented by a graph called the **dual graph** of the map. To get the dual graph, represent each region of the map by a vertex and make edges such that an edge connects two vertices if the corresponding regions share a border. Then, the problem of coloring the regions of a map is equivalent to the problem of coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color.



(a) (b)

Figure 2.15: (a) A simple map (with just five regions) and (b) its dual graph.
(Source: Weblink 2)

DEFINITION: A **coloring** of a simple graph is the assignment of a color to each vertex of the graph. The coloring is called **proper** if there are no adjacent vertices with the same color. If a graph can be properly colored with *n* colors we say that it is ***n-colorable***.

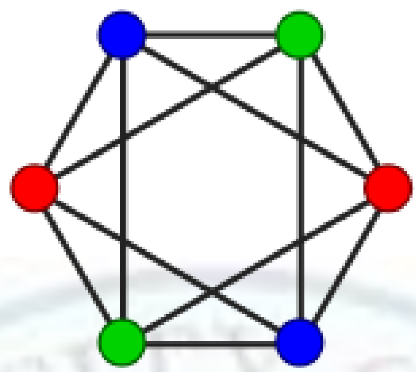


Figure 2.16: A 3-color example

DEFINITION: The **chromatic number** of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$.

The problem of finding the minimum number of colors required to color a planar map is same as finding the chromatic number of a planar graph. Following theorem provides the answer.

Value addition: Common Misconceptions
Heading text: Coloring a graph with loops
Body text: The graph must be loop-free, because if vertices are allowed to have loops, then they can be adjacent to themselves. If a vertex is adjacent to itself, then it must be colored a different color from itself, which is impossible.

THEOREM 2.3 The Four Color Theorem: A planar graph has chromatic number not greater than four.

This theorem applies to planar graphs only, however, nonplanar graphs can have arbitrarily large chromatic numbers.

Value addition: Historical Context
Heading text: The Four Color Theorem
Body Text: Over the last 150 years, some big shots in the world of mathematics have been involved with this problem. Around 1850, Francis Guthrie (1831-1899) showed how to color a map of all the counties in England using only four colors. He became interested in the general problem, and talked about it with his brother, Frederick. Frederick talked about it with his math teacher, Augustus DeMorgan (you might have heard of DeMorgan's laws in logic and proof), who sent the problem to William Hamilton (for whom Hamiltonian mechanics is named). Hamilton was evidently too interested in other things to work on the four color problem, and it lay dormant for about 25 years. In 1878, Arthur Cayley made the scientific community aware of the problem again, and shortly thereafter, British mathematician Sir Alfred Kempe devised a 'proof' that was unquestioned for over ten years. However, in 1890, another British mathematician, Percy John Heawood, found a mistake in Kempe's work. The problem remained unsolved

until 1976, when Kenneth Appel and Wolfgang Haken produced a proof involving an intricate computer analysis of 1936 different configurations.

Some mathematicians have expressed dissatisfaction with Appel's and Haken's proof. There is still no 'cute' proof of the four color problem.

Source: Weblink (1)

To show that the chromatic number of a graph is *k*, we are required to show following two things:

1. The graph can be colored with *k* colors.
2. The graph can not be colored using fewer than *k* colors.

Example 2.8: What is the chromatic number of the graph in Figure 2.7?

Solution: The chromatic number of this graph is 3, as it can be properly colored using three colors as shown in Figure 2.15 below. The number three is evident by the cycle v_2-v_3-v_6, as these three vertices should be colored using a different color.

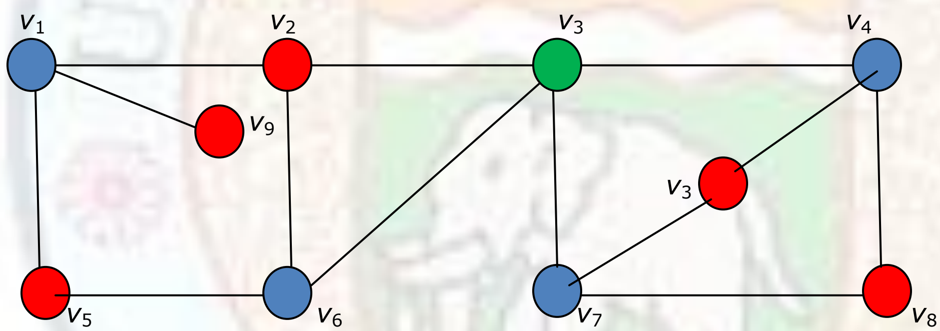


Figure 2.17: Coloring of graph in Figure 2.7

Example 2.9: What is the chromatic number of the trianlge graph and the cube graph?

Solution: The triangle graph has chromatic number 3 as shown below in Figure 2.16. Note that it is a cycle with three vertices.

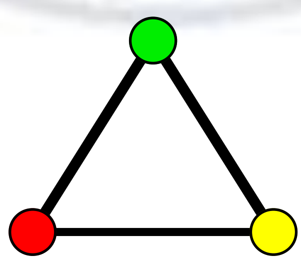


Figure 2.18: Coloring of the triangle graph.

The cube graph has chromatic number 2 as shown below in Figure 2.17. Notice that this graph is a bipartite graph.

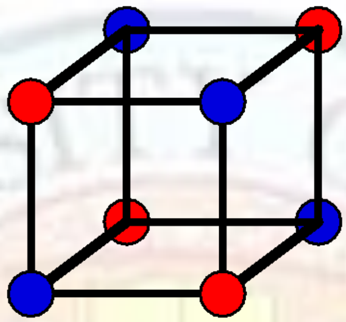


Figure 2.19: Coloring of the cube graph.

Example 2.10: Is the following planar graph 3-colorable?

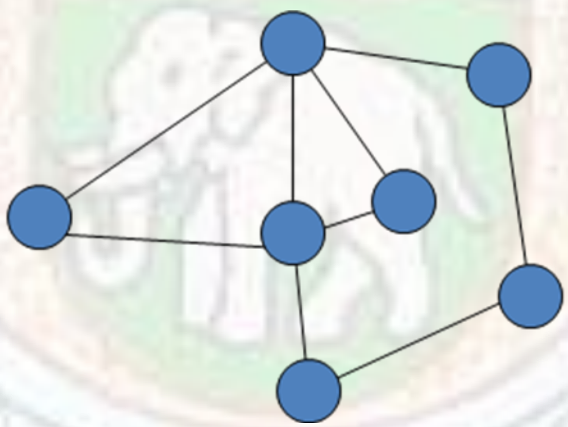


Figure 2.19: Graph G

Solution: The graph G can be colored using three colors, as shown below, hence is 3-colorable.

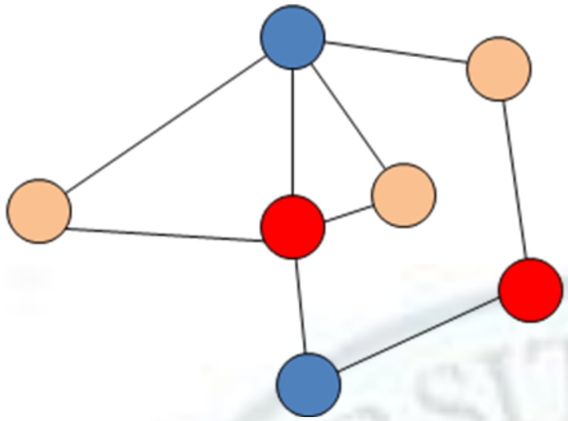


Figure 2.20: 3-color graph, G

Value addition: Activity
Heading text: Recognizing Chromatic numbers of K_n, $K_{m,n}$ and C_n
Body text: We have defined chromatic number. Now in this activity we'll try to generalize for the chromatic numbers of K_n(Complete graph with n vertices), $K_{m,n}$(Complete Bipartite graph) and C_n(Cycle with n vertices).
Chromatic Number of complete graph K_n
A coloring of K_n can be constructed using n colors by assigning a different color to each vertex. No two vertices can be assigned the same color because each pair of vertices is adjacent. Hence, the chromatic number of $K_n = n$. $\chi(K_n) = n$
Chromatic Number of complete graph $K_{m,n}$
Surprisingly the chromatic number does not depend on the value of m or n. The graph $K_{m,n}$ being complete bipartite graph, the edges connect only a vertex from the set of m vertices and a vertex from the set of n vertices. If we color all the vertices from the set of m vertices with one color and all the vertices from the set of n vertices with a second color, no two adjacent vertices will have the same color. Hence, the chromatic number of $K_{m,n} = 2$. $\chi(K_{m,n}) = 2$
Chromatic Number of Cycle C_n, where $n \geq 3$
When n is even: Two colors are needed to color C_n. To construct such a coloring, simply pick one vertex color it red(say). Proceed around the graph in clockwise direction (using a planar representation of the graph) coloring the second vertex blue, the third vertex red, and so on. The nth vertex can be colored blue, as the two vertices adjacent to it, namely the $(n-1)^{st}$ and the first vertices, are both colored red. When n is odd: Three colors are needed to color C_n. To construct such a coloring, pick an initial vertex. To use only two colors, it is necessary to alternate colors as the graph is traversed in a clockwise direction. However, the nth vertex reached has the two vertices of

different colors adjacent to it, namely the (*n-1*)st and the first vertices. Thus, a third color is needed.

Hence, for all $n \geq 3$,

$\chi(C_n) = 2$ if *n* is even integer,r and

$\chi(C_n) = 3$ if *n* is odd integer.

Value addition: Interesting Trivia
Heading text: Properties of Chromatic number, χ (G)
Body text: ■ χ (G) = 1 if and only if G is totally disconnected ■ χ (G) ≥ 3 if and only if G has an odd cycle (equivalently, if G is not bipartite) ■ χ (G) ≥ ω(G) (clique number) ■ χ (G) ≤ Δ(G)+1 (where Δ(G) is the maximum degree) ■ χ (G) ≤ Δ(G) for connected G, unless G is a complete graph or an odd cycle (Brooks' theorem). ■ χ (G) ≤ 4, for any planar graph (Four color Theorem).

2.3.1 Applications of Graph Coloring

Some applications of graph coloring are:

1. **Mobile radio frequency assignment**: The problem is to assign frequencies to mobile radios and other users of the electromagnetic spectrum. In the simplest case, two customers that are sufficiently close (say, up to n miles) must be assigned different frequencies, while those that are distant can share frequencies. The problem of minimizing the number of frequencies is then a graph coloring problem, if we construct a graph by assigning a vertex to each customer and connecting two vertices by an edge if they are located within n miles of each other.

2. **Time Table and Scheduling**: Many scheduling problems involve allowing for a number of pair wise restrictions on which jobs can be done simultaneously. For instance, in attempting to schedule classes at a university, two courses taught by the same faculty member cannot be scheduled for the same time slot. Similarly, two course that are required by the same group of students also should not conflict. The problem of determining the minimum number of time slots needed subject to these restrictions is a graph coloring problem.

3. **Register Allocation**: One very active application for graph coloring is register allocation. The register allocation problem is to assign variables to a limited number of hardware registers during program execution. Variables in registers can be accessed much quicker than those not in registers. Typically, however, there are far more variables than registers so it is necessary to assign multiple variables to registers. Variables conflict with each other if one is used both before and after the other within a short period of time (for instance, within a subroutine). The goal is to assign variables that do not conflict so as to minimize the use of non-register memory. A simple approach to this is to create a graph where the nodes represent variables and an edge represents conflict between its nodes. A coloring is then a conflict-free assignment. If the number of colors used is less than the number of registers then a conflict-free register assignment is possible.

4. **Printed Circuit Board Testing**: Consider the problem of testing printed circuit boards for unintended short circuits (caused by stray lines of solder). This gives rise to a graph coloring problem in which the vertices correspond to the nets on board and there is an edge between two vertices if there is a potential for a short circuit between the corresponding nets. Coloring the graph corresponds to partitioning the nets into *supernets*, where the nets in each supernet can be simultaneously tested for shorts against all other nets, thereby speeding up the testing process.

Summary

- A graph is ***planar*** if it can be drawn in two-dimensional space with no two of its edges crossing.
- Let G(*e*, *V*) be a connected planar graph and *r* be the number of regions in a planar representation of graph G. Then, the Euler's formula is given by $\boldsymbol{r = e - v + 2}$.
- A finite graph *G* is planar if and only if it contains no subgraph that is edge-contractible to K_5 or $K_{3,\ 3}$.
- The vertices must be colored differently if they are joined by an edge.
- The *chromatic number* of a fintite, loop-free graph *G* is the smallest positive number *k* such that *G* is *k*-colorable. The chromatic number of a graph *G* is usually denoted by $\chi(G)$.

Exercises

1.1 Define Planar Grpahs.

1.2 Define Chromatic Number of a graph.

1.3 Suppose you run a Gym in an office building and there are eight members *A,B,C,D,E,F,G, H*. You need to assign a locker where each member can put his/her belongings. The members come and leave so they are not all there at the same time. You have 1 hour time slots starting 7:00 a.m. to 2:00 p.m. A star in the table means a member can be present at that time. What is the minimum number of lockers necessary? Show how you would assign the lockers.

.	A	B	C	D	E	F	G	H
7:00	*	-	-	*	*	-	-	-
8:00	*	*	*	-	-	-	-	-
9:00	*	-	*	*	-	*	-	-
10:00	*	-	*	-	-	*	*	-
11:00	*	-	-	-	-	*	*	-
12:00	*	-	-	-	*	-	-	*
1:00	*	*	-	-	*	-	-	*
2:00	*	-	-	*	-	*	-	-

1.4 Which graphs have a chromatic number of 1?

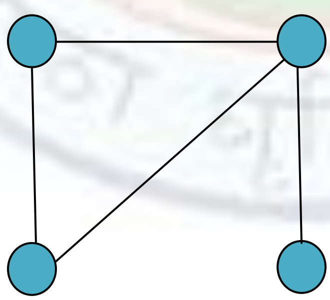
1.5 Can a simple graph having a circuit with an odd number of vertices in it be colored using two colors? Justify.

1.6 The discrete class has six groups each meeting once a month. A student can be part of at the most three groups. How can different meeting be scheduled to ensure that no member is scheduled to attend two or more meetings at the same time?

1.7 Suppose that a connected planar graph has 30 edges. If a planar representation of it divides the plane into 20 regions, how many vertices does this graph has?

1.8 Find a planar graph *G* such that $\chi(G) = 4$.

1.9 What is the chromatic number of the following graph?



Glossary

1. **Bipartite**: A graph is bipartite if the vertices can be partitioned into two sets, X and Y, so that the only edges of the graph are between the vertices in X and the vertices in Y.
2. **Multiple edges**: distinct edges connecting the same vertices
3. **Loop**: An edge from a vertex to itself is called a *loop*. Loops are not allowed in simple graph.
4. **Simple graph**: an undirected graph with no multiple edge or loops
5. **Multigraph**: an undirected graph that may contain multiple edges but no loops
6. **Adjacent**: Two vertices are adjacent if they are connected by an edge. We often call these two vertices *neighbors*.
7. **K_n Complete graph on n vertices**: a graph in which all pairs of vertices are adjacent.
8. **$K_{m,n}$, Complete bipartite graph**: the graph with the vertices partitioned into two sets, X and Y of m and n vertices respectively, so that two vertices are connected by an edge if and only if on is in X and the other is in Y.
9. **Cycle**: In a graph, a cycle is a simple closed path. A cycle containing n vertices is denoted by C_n
10. **Subgraph**: A subgraph of a graph is some smaller portion of that graph
11. **Path**: A path in a graph is a sequence of vertices from one vertex to another using the edges. The *length* of a path is the number of edges used, or the number of vertices used minus one. A *simple* path cannot visit the same vertex twice. A *closed* path has the same first and last vertex.
12. **Connected graph**: A connected graph is one in which every pair of vertices has a path between them

References

Works Cited

1. Kenneth H Rosen, Discrete Mathematics and Its Applications, Sixth Edition

Suggested Readings

1. S. Santha, Discrete Mathematics with combinatorics and graph theory, Cengage Learning
2. D.S.Malik and M.K.Sen, Discrete Mathematics, Cengage Learning

Web Links

1. http://www.math.lsa.umich.edu/mmss/coursesONLINE/graph
2. http://www.mathpages.com/home/kmath266/kmath266.htm