Unit No: Unit I
Paper : Discrete Structures
Lesson: Relations and Function
Lesson Developer: Mona Adlakha
College/ Department: Ram Lal Anand College, University of Delhi

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2.1 Relations

2.1.1 Introduction

When talking about discrete objects we often talk about some kind of relationship among objects or sets of objects. *Objects are related if they share a common property* and are not related otherwise.

The relations consider the existence or non existence of a certain connection between pairs of objects taken in a definite order. Formally, a relation can be defined in terms of these "ordered pairs".

Example 1: Examples of relations (or related objects):

i) For a set of computer programs, two programs are related if they share some common data and are not related otherwise.

ii) Among group of people, two people are related if their age is the same.

iii) Consider a set of integers X = {1, 3, 5, 7, 9, 11, 13, 15, 17}. The integers in the set E = {3, 9, 15} are related if they are divisible by 3. Thus integers 3, 9, 15 are related and 1, 5, 7, 11, 17 are not. The relation is *"divisibility by 3"*.

iv) Considering the same set of integers X = {1, 3, 5, 7, 9, 11, 13, 15, 17} (taking another property to determine if the elements are related or not). The integers in the set E = {11, 13, 15} are related if the relation is *"elements are >10 and < 17"*. Thus integers 11, 13, 15 are related and 1, 3, 5, 7, 9, 17 are not.

2.1.2 Cartesian Product

Consider two arbitrary sets A and B. The set of all ordered pairs *(a, b)* where *a* ϵ *A* and *b* ϵ *B* is called the **product** or **Cartesian product**, of *A* and *B*. The product can be written in short designation as *A x B* which is read as "*A cross B*".

By definition,

$$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$$

One can also write A^2 instead of *A x A*.

Example 2: Let *A = {a, b}* and *B = {1, 2, 3}*. Then

A x B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}
B x A = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}
A x A = {(a, a), (a, b), (b, a), (b, b)}

Example 3: *{a, b, c}* x *{a, b} = {(a, a), (a, b), (b, a), (b, b), (c, a), (c, c)}*

Note:

- As we can see in the Example 2, *A x B ≠ B x A.* The Cartesian product deals with ordered pairs, so the order in which the sets are considered is important.

- Using *n(S)* for the number of elements in a set *S*, for finite sets *A* and *B*

$$n(A \times B) = n(A) \cdot n(B)$$

n(S) is called the **cardinality** of a set.

Value addition:
Cartesian Plane

Let **R** denotes the set of real numbers, so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers.

The geometric representations of $\mathbf{R}^2$ are points in a plane, as in Figure 2.1.1 (a). Each point P represents an ordered pair of real numbers (m, n) and vice versa. The vertical line through P meets the *x* axis at *m*, and the horizontal line through *P* meets the *y* axis at *n*. $\mathbf{R}^2$ is generally called the **Cartesian plane**.

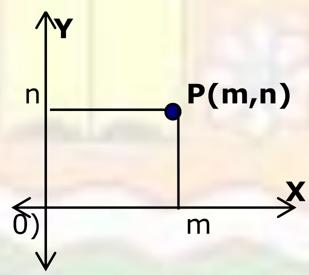


Figure 2.1.1 (a): Cartesian Plane

Extending the above concept, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the *three-dimensional space*.

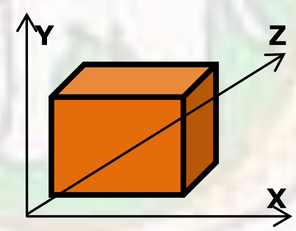


Figure 2.1.1 (b): A cube in 3-D space

Source: Book - Lipschutz & Lipson, Chap 2, definition of Cartesian plane

2.1.3 Binary Relations

Let A and B be sets. A **binary relation** or a **relation** from *A* to *B* is a subset of *A x B*.
A binary relation essentially means that some of the elements in set *A* are related to some of the elements in set *B*.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from set B. That is, for each pair $a \in A$ and $b \in B$, exactly one of the following is true:

(i) $(a, b) \in R$; we then say "*a* is *R*-related to *b*", written as *aRb*.

(ii) $(a, b) \notin R$; we then say "*a* is not *R*-related to *b*", written as $a\not{R}b$.

A relation R from A to B is written as R: A → B. R is a subset of A x B.

Example 4: Let *S = {Rohit, Seema, Akash, Sanya}* be a set of students and *C = {CS101, CS102, CS103, ENG1, MATH1}* be a set of courses.

The Cartesian product *S x C* gives all the possible pairing of students and courses.
S x C =
{(Rohit, CS101), (Rohit, CS102), (Rohit, CS103), (Rohit, ENG1), (Rohit, MATH1),
(Seema, CS101), (Seema, CS102), (Seema, CS103), (Seema, ENG1), (Seema, MATH1),
(Akash, CS101), (Akash, CS102), (Akash, CS103), (Akash, MATH1), (Akash, ENG1),
(Sanya, CS101), (Sanya, CS102), (Sanya, CS103), (Sanya, MATH1), (Sanya, ENG1)}

Let the relation *R* describe the courses the students are taking.
R =
{(Rohit, CS101), (Rohit, CS102), (Rohit, CS103), (Rohit, ENG1),
(Seema, CS101), (Seema, CS102), (Seema, CS103), (Seema, ENG1),
(Akash, CS101), (Akash, CS102), (Akash, CS103), (Akash, MATH1),
(Sanya, CS101), (Sanya, CS102), (Sanya, CS103), (Sanya, MATH1)}

Assume relation T describes the courses the students are having difficulty in.
T = {(Rohit, CS101), (Rohit, CS102), (Seema, CS102), (Akash, CS101), (Akash, MATH1)}

As we can see, relations *R* and *T* are subsets of the Cartesian product *S x C.*

Value addition:
Empty Relation, Universal Relation
Since empty set, ∅, is a subset of A x B, therefore R = ∅ is a relation called **empty relation**. Also, if R = A x B, then R is called the **universal relation**.
Source: Book - Lipschutz & Lipson, Chap 2, definition of empty &

universal relation

Example 5: Let *A = {1, 2, 3}* and *B = {w, x, y, z}*.
Then, relation *R = {(1, x), (2, y), (2, z), (3, z)}* is a binary relation from A to B.

Example 6: Given (x + y, 3x) = (3, 6), find x and y.
Solution: Two ordered pairs are equal iff the corresponding components are equal. We thus obtain the equations:

x + y = 3 and 3x = 6
Solving the equations we get, x = 2 and y = 1.

Relations are defined in terms of ordered pairs *(a, b)* of elements, where *a* is designated as the first element and *b* as the second element. In particular,

***(a, b) = (c, d)* if and only if *a = c* and *b = d*.**

Also, *(a, b) ≠ (b, a)* unless a = b. This is in contrast with sets where the order of elements is irrelevant; for example in sets, {3, 5} = {5, 3}.

Domain and Range of a Relation

Let R be a relation from A to B, R: A → B.

The **domain** of a relation *R* ⊆ A x B is the set of all first elements of the ordered pairs which are in the set *R*.

The **range** of *R* ⊆ A x B is the set of all second elements of the ordered pairs which belong to *R*.

Domain & Codomain: Set A is domain and set B is codomain.

Example 7: Considering the relation *R* of Example 4.
The domain of R is *{Rohit, Seema, Akash, Sanya}* and the range of *R* is *{CS101, CS102, CS103, ENG1, MATH1}*.

The domain of relation *T* is *{Rohit, Seema, Akash}* and the range of *T* is *{CS101, CS102, MATH1}.*

Example 8: Considering the relation *R* of Example 5.
The domain of *R* is *{1, 2, 3}* and the range of *R* is *{x, y, z}*.
Codomain of R is *{w, x, y, z}*.

Value addition:

Inverse of a relation

Let us take a relation *R* from a set *A* to a set *B*. The **inverse** of *R*, denoted by R^{-1}, is the relation from *B* to *A*.

R^{-1} ⊆ B x A. Therefore R^{-1} consists of those ordered pairs which, when reversed, belong to *R*; i.e.,

R^{-1} = {(b, a): (a, b) ∈ R}

- The **domain of R^{-1}** is the range of *R*.
- The **range of R^{-1}** is the domain of *R*.
- For a relation *R*, **$(R^{-1})^{-1}$ = *R***.
- If *R* is a relation on *A*, then R^{-1} is also a relation on *A*.

Example 9: The inverse of *R = {(1, x), (2, y), (2, z), (3, z)}* from *A = {1, 2, 3} to B = {w, x, y, z}* is:

$$R^{-1} = \{(x, 1), (y, 2), (z, 2), (z, 3)\}$$

In this example, domain of R^{-1} is *{x, y, z}*.
In this example, range of R^{-1} is *{1, 2, 3}*.

Source: Book - Rosen, definition of Inverse of a relation

Value addition:

Notations of Binary Relations

Similar to set notations of intersection, union and difference; binary relations also have similar notations. Since binary relations are set of ordered pairs, the following notations follow directly from sets:

- **Intersection** of two relations, $\mathbf{R_1 \cap R_2}$
- **Union** of two relations, $\mathbf{R_1 \cup R_2}$
- **Difference** of two relations, $\mathbf{R_1 - R_2}$
- **Symmetric difference** of two relations, $\mathbf{R_1 \oplus R_2}$ (also written as $\mathbf{R_1 \Delta R_2}$)
  Similar to sets, $R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)$

Let us take an example to make this clearer.

Example 10: Consider the relation R & T given in Example 4.
R describe the courses the students are taking =
R = {(Rohit, CS101), (Rohit, CS102), (Rohit, CS103), (Rohit, ENG1), (Seema, CS101), (Seema, CS102), (Seema, CS103), (Seema, ENG1), (Akash, CS101), (Akash, CS102), (Akash, CS103), (Akash, MATH1), (Sanya, CS101), (Sanya, CS102), (Sanya, CS103), (Sanya, MATH1)}

T describes the courses the students are having difficulty in.
T = {(Rohit, CS101), (Rohit, CS102), (Seema, CS102), (Akash, CS101), (Akash, MATH1)}

Intersection, **R ∩ T** (the courses students are taking and are having difficulty in) =
{(Rohit, CS101), (Rohit, CS102), (Seema, CS102), (Akash, CS101), (Akash, MATH1)}
Note that in this example, R ∩ T = T.

Union, R U T (the courses students are taking or are having difficulty in) =

= *{(Rohit, CS101), (Rohit, CS102), (Rohit, CS103), (Rohit, ENG1), (Seema, CS101), (Seema, CS102), (Seema, CS103), (Seema, ENG1), (Akash, CS101), (Akash, CS102), (Akash, CS103), (Akash, MATH1), (Sanya, CS101), (Sanya, CS102), (Sanya, CS103), (Sanya, MATH1)}*

Note that in this example, R U T = R.

Difference, R - T (the courses students are taking and are *not* having difficulty in)

= *{(Rohit, CS103), (Rohit, ENG1), (Seema, CS101), (Seema, CS103), (Seema, ENG1), (Akash, CS102), (Akash, CS103), (Sanya, CS101), (Sanya, CS102), (Sanya, CS103), (Sanya, MATH1)}*

Symmetric difference, R ⊕ T = (R U T) – (R ∩ T)

= *{(Rohit, CS103), (Rohit, ENG1), (Seema, CS101), (Seema, CS103), (Seema, ENG1), (Akash, CS102), (Akash, CS103), (Sanya, CS101), (Sanya, CS102), (Sanya, CS103), (Sanya, MATH1)}*

Note that in this example, the symmetric difference is the same as the difference, i.e., R ⊕ T = R - T.

Source: Book - C. L. Liu, Chap 3, definition of Union, Intersection, Difference, Symmetric difference of a relation

Value addition:

Complement of a relation

Let *R* be a relation from a set *A* to a set *B*. The **complement** of *R*, denoted by $\overline{R}$, is the set of ordered pairs such that,

$$\overline{R} = A \times B - R$$
$$= \{(a, b) \in A \times B : (a, b) \notin R\}$$

- For a relation *R*, $\overline{(\overline{R})} = \overline{(A \times B - R)} = A \times B - (A \times B - R) = R$.
- *Also,* $R \cup \overline{R} = R \cup (A \times B - R) = A \times B$
- *Also,* $R \cap \overline{R} = \varnothing$

Example 11: Find the complement of *R = {(1, x), (2, w), (2, y), (3, w), (3, x)}*. Where *R* is a relation from set *A = {1, 2, 3}* to set *B = {w, x, y}*.

Solution:

$$\overline{R} = \{(1, w), (1, y), (2, x), (3, y)\}$$

Source: Book - Rosen, definition of complement of a relation

2.1.4 Different Representations of Binary Relations

Binary relations can be represented in a number of ways:

- As a list of ordered pairs (as discussed above)
- In Tabular form
- As a Relation Matrix
- As an Arrow diagram
- In Graphical form as a Directed graph (the pictorial representation of a relation)

Let set $A = \{a_1, a_2, a_3\}$, set $B = \{b_1, b_2, b_3\}$ and R be a binary relation from A to B. Let

$$R = \{(a_1, b_2), (a_1, b_3), (a_3, b_2)\}.$$

This relation is represented as a **list of ordered pairs**. Next we will show the alternate representations of the above binary relation, *R*.

2.1.4.1 Tabular Form

A binary relation can be represented in the form of a table (as shown in Figure 2.1.2), where the rows of the table correspond to the elements in *A* and the columns of the table correspond to the elements in *B*.

	b_1	b_2	b_3
a_1		√	√
a_2			
a_3		√	

(a)

	b_1	b_2	b_3
a_1	0	1	1
a_2	0	0	0
a_3	0	1	0

(b)

Figure 2.1.2: Representation of the relation $R = \{(a_1, b_2), (a_1, b_3), (a_3, b_2)\}$ in Tabular Form

As in Figure 2.1.2 (a), a check mark in a cell means that the element in the row containing the cell is related to the element in the column containing the cell.

Alternatively, as in Figure 2.1.2 (b), a '1' in a cell means that the element in the row containing the cell is related to the element in the column containing the cell, else a '0' is written in the cell.

2.1.4.2 Relation Matrix

If the elements of *A* and *B* appear in a certain order, the relation *R* can be represented in the form of a matrix whose elements are 0's and 1's. A 1 is written in the i^{th} row and j^{th} column if the element $a_i \in A$ is related to the element $b_j \in B$, otherwise a 0 is written.

This matrix of the relation *R* is usually termed as a **relation matrix** and is denoted by $\boldsymbol{M_R}$. Mathematically M_R is defined as:

$$M_R = [m_{ij}], \text{ where } m_{ij} = \begin{cases} 1, \text{ if } (a_i, b_j) \in R \\ 0, \text{ if } (a_i, b_j) \notin R \end{cases}$$

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Figure 2.1.3: The Relation matrix of the relation *R* = {(a_1, b_2), (a_1, b_3), (a_3, b_2)}

2.1.4.3 Arrow Diagram

Write down the elements of *A* and the elements of *B* in two disjoint discs, and then draw an arrow from *a* ∈ *A* to *b* ∈ *B* whenever *a* is related to *b*. This is called the **arrow diagram** of the relation.

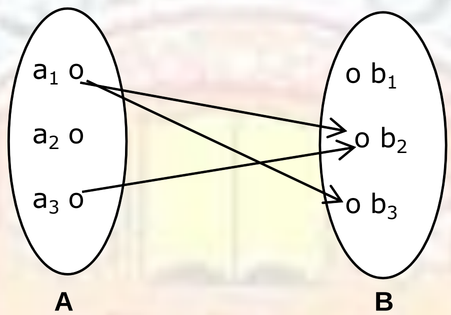


Figure 2.1.4: The Arrow diagram of the relation *R* = {(a_1, b_2), (a_1, b_3), (a_3, b_2)}

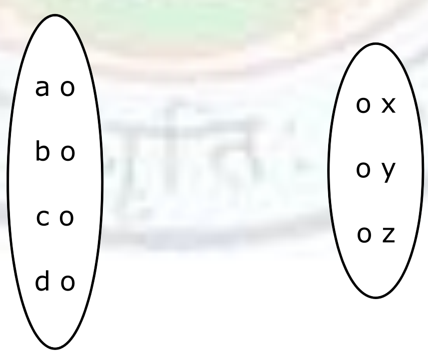
Value addition: Animation

Steps to draw an Arrow Diagram for a binary relation

Let us take a binary relation *R* = {(*b*, *x*), (*b*, *z*), (*c*, *y*), (*d*, *z*)} from set *A* = {*a*, *b*, *c*, *d*} to set *B* = {*x*, *y*, *z*}. We will now see the different steps to that need to be followed to make the arrow diagram of the relation *R*.

Animation Frame 1:

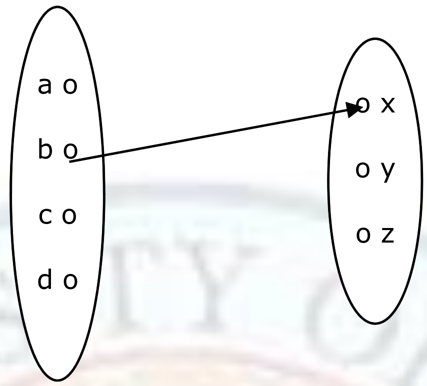
Step 1: Write down the elements of *A* and the elements of *B* in two disjoint discs



Animation Frame 2:

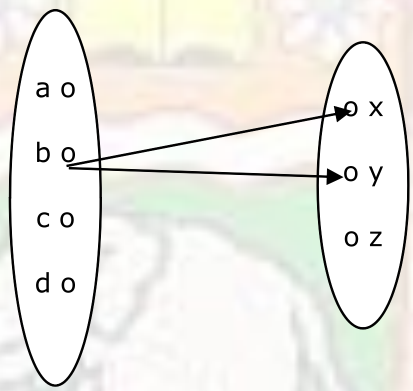
Step 2 - 5: Drawing an arrow from *p* ∈ *A* to *q* ∈ *B* whenever *p* is related to *q*.

Step 2: Drawing an arrow from $b \in A$ to $x \in B$ as they are related.



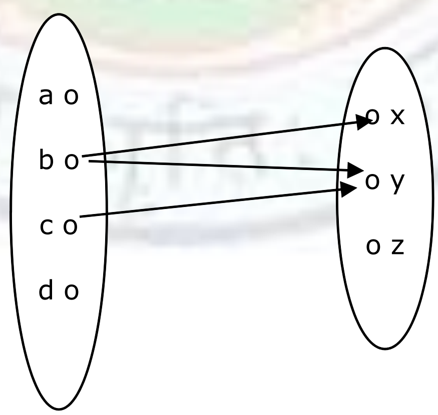
Animation Frame 3:

Step 3: Drawing an arrow from $b \in A$ to $x \in B$ as they are related.



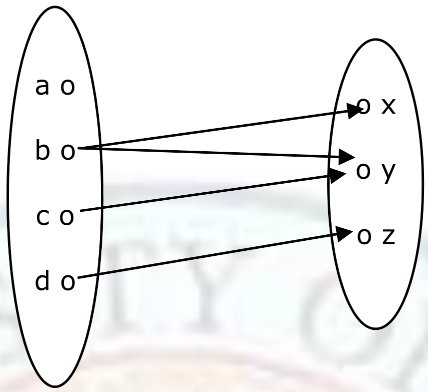
Animation Frame 4:

Step 4: Drawing an arrow from $b \in A$ to $x \in B$ as they are related.



Animation Frame 5:

Step 5: Drawing an arrow from $b \in A$ to $x \in B$ as they are related. We now get the final arrow diagram for *R* = *{(b, x), (b, z), (c, y), (d, z)}*.



It must be clear by now how an arrow diagram has to be drawn.

Source: Created by author

2.1.4.4 Directed Graph or Diagraph

A binary relation can also be represented in graphical form as explained below.

For set *A* = *{1, 2, 3}*, set *B* = *{x, y, z}* and relation *R* from *A* to *B*. Let *R* = *{(1, y), (1, z), (3, y)}*. In Figure 2.1.5, the left-hand column are elements in *A*, the right-hand column are the elements in *B*, and an arrow from a point in the left-hand column to a point in the right-hand column indicates that the corresponding element in *A* is related to the corresponding element in *B*.

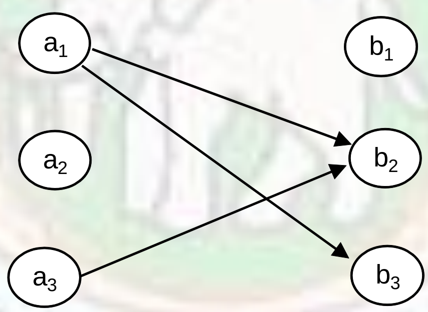


Figure 2.1.5: Directed graph of the relation $R = \{(a_1, b_2), (a_1, b_3), (a_3, b_2)\}$

Here the points are known as **vertices** or **nodes** and the arrows are known as **edges** of the graph. Since the graph contains arrows (directions) it is called a **directed graph** or **diagraph**.

Value addition:

Relation on a Set

$R \subseteq A \times A$ is called **a relation on a set** A.

Let A = {1, 2, 3}.

Then R = {(1,1), (1,2), (2,1), (3,3)} ⊆ A x A
Source: Book - Lipschutz & Lipson, Chap 2, definition of relation on a set

A relation *R,* when *R* is a relation from a finite set to itself, can also be represented as a directed graph. First write down the elements in a set, and then draw an arrow from each element *x* to each element *y* whenever x is related to y.

Example 12: Considering the relation *R* on the set *A = {1, 2, 3, 4}* where

R = {(1, 3), (2, 1), (3, 1), (3, 3), (4, 1), (4, 2)}.

Following is the directed graph of *R.*

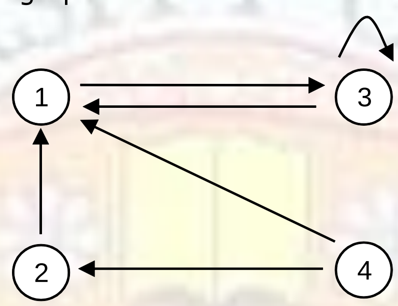


Figure 2.1.6: Directed graph of the relation
R = {(1, 3), (2, 1), (3, 1), (3, 3), (4, 1), (4, 2)}

Edge from node **3** to node **3** is called a **loop**.

2.1.5 Ternary, Quaternary and n-ary Relations

Ternary Relation

Ternary relations describe the relationship between three sets.

A **ternary relation** among three sets *A, B* and *C* is defined as a subset of the Cartesian product *(A x B) x C*. This is a set of all ordered triples of the form:

((a, b), c): (a, b) ∈ A x B and *c ∈ C*

Example 13: Let *S = {Mahima, Ragini, Sunny}* be a set of students, *C = {Maths, English, Science}* be a set of courses and *G = {A, B, C}* be a set of grades.

Assume the ternary relation *X* (among sets *S, C* and *G*), describes the grades the students obtained in the courses they took.

Say *X = {((Mahima, Maths), A), ((Mahima, English), B), ((Ragini, Maths), B), ((Sunny, Maths), A), ((Sunny, Science), A)}*

Value addition:
Property of Ternary Relation
(A x B) x C = A x (B x C) Is this True? This is left for the reader to evaluate.

Source: Created by author/reviewer

Quaternary Relation

Quaternary relations represent the relationship between quadruples of objects.

A **quaternary relation** among four sets *A, B, C* and *D* is defined as a subset of $((A \times B) \times C) \times D)$.

n-ary Relation

In general, an **n-ary relation** among the sets $A_1, A_2, A_3, \ldots, A_n$ is defined as a subset of $(A_1 \times A_2) \times A_3) \ldots \times A_n)$.

That is, an n-ary relation among the sets $A_1, A_2, A_3, \ldots, A_n$ is a set of ordered n-tuples in which the first component is an element of A_1 , the second component is an element of A_2, …., and the n^{th} component is an element of A_n.

2.2 Properties of Binary Relations

A **binary relation** *R* over a set *A* is a subset of $A \times A$. If $(a, b) \in R$ we also write *aRb*.

Example 1:

i) Let $A = \{0, 1\}$. $A \times A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
The binary relations on A are: Ø; $A \times A$; equalTo = {(0, 0), (1, 1)}, because 0=0 & 1=1 and; lessThan = {(0, 1)}, because 0<1.

ii) Let *A* be a set of positive integers. We can define a relation *R* on *A* such that (a, b) is in *R* iff $a - b \geq 10$. Thus (12, 2) is in *R* but (2, 12) and (11, 2) are not.
Similarly we can find more elements in R.

Next we discuss the types of relation defined on a non-empty set *A*.

2.2.1 Reflexive Relation

A relation *R* is said to be **reflexive** if *(a, a)* is in R for all *a* in *A*.

i.e *R* is reflexive if $aRa, \forall a \in A$

Example 2:

i) Let *R* be a relation on a set of positive integers *A* such that (a, b) is in *R* iff *a* divides *b*. Since each integer divides itself, *R* is reflexive.

ii) Let *T* be a relation on a set of integers *A* such that (a, b) is in *T* iff $a > b$. Now, $(a, a) \notin T$ as $a \not> a$. *T* is clearly not a reflexive relation.

If the relation *T* is defined as $T = \{(a, b): a \geq b\}$. Then *T* is reflexive.

iii) Let *A* be a set of students and *R* be a binary relation on A such that (*a, b*) is in *R* if *a* nominates *b* as a candidate for class representative.
R is reflexive if everyone nominates himself or herself.
On the other hand *R* is not reflexive if one or more students did not.

It is very simple to determine if a binary relation is a reflexive relation when the relation is represented in matrix/ tabular form. The relation is reflexive if and only if all the elements on the main diagonal (from top left to bottom right) contain 1s (in a matrix) or check marks (in a table).

For example, the binary relation in **Figure 2.2.1 (a)** is reflexive while that in Figure 2.2.1(b) is not.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(a) **(b)**

Figure 2.2.1: (a) Binary relation is reflexive.
(b) Binary relation is not reflexive.

Similarly in a diagraph (see Figure 2.2.2), if each vertex contains a loop, then the corresponding relation is reflexive.

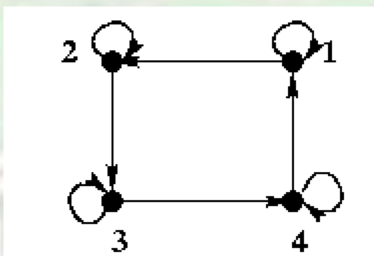


Figure 2.2.2: Relation corresponding to the Diagraph is reflexive

2.2.2 Irreflexive Relation

A relation *R* on a set *A* is **irreflexive** if *(a, a)* is not in R for all *a* in *A*.

i.e. *R* is irreflexive if $(a, a) \notin R, \forall a \in A$.

Thus *R* is non-reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.

Example 3:

i) Consider *R* on *A* such that $R = \{(a, b) \mid a \neq b\}$.
Here *R* is the inequality relation on set A. *R* is irreflexive since for every $a \in A$, $(a, a) \notin R$.
Also, *R* is not reflexive.

ii) Let there be a non-empty set, $A = \{a \mid a$ is a computer science student$\}$. Let R be a relation on A. $R = \{(a, b) \mid a, b \in A$ and a scored less than $b\}$.
Here R is not reflexive since $(a, a) \notin R$, $\forall$ a in A.
Also, R is irreflexive since for every a in A, $(a, a) \notin R$.

iii) Let $A = \{a, b, c\}$ and let $R = \{(a, a)\}$.
Here R is not reflexive since $(b, b) \notin R$ and $(c, c) \notin R$.
R is also not irreflexive since $(a, a) \in R$.

Value addition:

Reflexive vs Irreflexive Relations

If a relation is reflexive, it cannot be irreflexive and vice versa. It may also be noted that, as in example 3 iii), a relation may neither be reflexive nor irreflexive. The Venn Diagram below shows the correlation between reflexive and irreflexive relations.

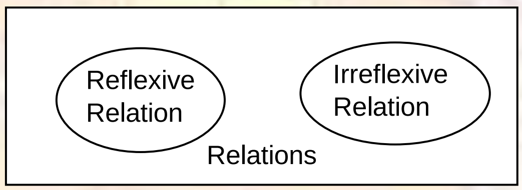


Figure 2.2.3: Venn Diagram showing the correlation between reflexive and irreflexive relations

Source: Created by the author

2.2.3 Symmetric Relation

A relation R on a set A is **symmetric** if whenever aRb implies bRa, that is, if whenever (a, b) is in R then (b,a) is also in R.

i.e. R is symmetric if $(a, b) \in R$ implies $(b, a) \in R$, $\forall\, a, b \in A$.

Thus R is not symmetric if there exists a, b $\in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

Example 4:

i) Let A be a set of students and R is a binary relation on A, such that,
$R = \{(a, b)$ iff a and b are in the same class, $\forall\, a, b \in A\}$.
Since a is b's classmate, b is also a's classmate, R is symmetric.

ii) Let A be a set of positive integers and R is a binary relation on A, such that,
$R = \{(a, b)$ iff $a \geq b$, $\forall\, a, b \in A\}$.
"greaterThanEqTo" is not symmetric since, for instance, $(6, 5) \in R$ but $(5, 6) \notin R$.

When a binary relation on a set is represented in matrix/ tabular form we can easily determine if it is symmetric. The relation is symmetric if the check marks or 1s are symmetric with respect to the main diagonal.

If M = [m_{ij}] is a matrix for a symmetric relation, then

$$m_{ij} = m_{ji} \text{ for } i \neq j$$

For example, the binary relation in Figure 2.2.4 (a) is reflexive while that in Figure 2.2.4 (b) is not.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(a) **(b)**

Figure 2.2.4: (a) Binary relation is symmetric.
(b) Binary relation is not symmetric.

2.2.4 Antisymmetric Relation

A relation *R* on a set *A* is **antisymmetric** if whenever *aRb* and *bRa,* then *a*= *b*, that is, if whenever *(a, b)* and *(b, a) are* in R then *a = b*.

i.e., *R* is *antisymmetric* means: *(a, b) ∈ R* and *(b, a) ∈ R* implies *a = b, ∀ a, b ∈ A*.

Thus *R* is not antisymmetric if there exists *a, b ∈ A* such that *(a, b)* and *(b, a) ∈ R*, but *a ≠ b*.

Example 5:

i) Let *A* be a set of tests to be performed on patients in a hospital and *R* is a binary relation on *A*, such that,

R = {(*a*, *b*) | test *a* must be performed before test *b* and *a, b ∈ A*}.

Since test *a* will always precede test *b* and the reverse cannot happen, *R* is antisymmetric.

ii) Let *A* be a set of positive integers and *R* is a binary relation on *A*, such that,

R = {(*a*, *b*) iff *a ≥ b, ∀ a, b ∈ A*}.

R (the greaterThanEqTo relation) is antisymmetric.

Value addition:
Symmetric vs Antisymmetric Relations
Let *A* = {1, 2}. A x A = {(1, 1), (1, 2), (2, 1), (2, 2)}. If a relation is symmetric, it can also be antisymmetric. For example, *R* = {(1, 1), (2, 2)}. Here *(a, b) ∈ R* implies *(b, a) ∈ R, ∀ a, b ∈ A*; hence *R* is symmetric. Also, *(a, b) ∈ R* and *(b, a) ∈ R* implies *a = b, ∀ a, b ∈ A*; hence *R* is antisymmetric.

A relation may neither be symmetric nor antisymmetric.
For example, $R = \{(1, 3), (3, 1), (2, 3)\}$.
Here $(2, 3) \in R$ but $(3, 2) \notin R$; hence R is not symmetric.
Also, $(1, 3) \in R$ and $(3, 1) \in R$ but $1 \neq 3$; hence R is antisymmetric.

The Venn Diagram below shows the correlation between symmetric and antisymmetric relations.

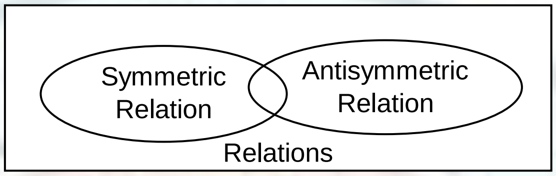


Figure 2.2.5 Venn Diagram showing the correlation between symmetric and antisymmetric relations

Source: Created by the author

2.2.5 Asymmetric Relation

A relation R on a set A is **asymmetric** if whenever aRb then $b \not R a$, that is, if (a, b) is in R then (b,a) is not in R.

i.e., R is *asymmetric* means: If $(a, b) \in R$ then $(b,a) \notin R, \forall\, a, b \in A$.

Thus R is not asymmetric if for some $a, b \in A$, both $(a, b) \in R$ and $(b, a) \in R$.

Example 6: Let A be a set of integers and R is a binary relation on A, such that,

$R = \{(a, b) \mid a < b$ where $a, b \in A\}$.

i) Since $x < y$, $(x, y) \in R$ and $(y, x) \notin R$ (as $y \nless x$).
Hence R is not symmetric but R is asymmetric.

ii) If $x \neq y$, then either $(x, y) \in R$ or $(y, x) \in R$.
Hence R is not antisymmetric but R is asymmetric.

2.2.6 Transitive Relation

A relation R on a set A is **transitive** if whenever aRb and bRc then aRc, that is, if whenever $(a, b), (b, c)$ are in R then (a, c) is also in R.
i.e., R is *transitive* means: $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R, \forall\, a, b, c \in A$.

Thus R is not transitive if there exists $a, b, c \in A$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Example 7:

i) Let $A = \{1, 2, 3\}$.

Let $X = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$. $X \subseteq A \times A$ is a relation on A. We check transitiveness.
$(1, 1) \in X, (1, 2) \in X \Rightarrow (1, 2) \in X$ which is true.
$(1, 2) \in X, (2, 3) \in X \Rightarrow (1, 3) \in X$ which is true.
Hence X is transitive.

Let $Y = \{(1, 2), (1, 3), (2, 3)\}$ be the "less-than" relation on A. Y is also a transitive relation. This is because $(1, 2) \in Y, (2, 3) \in Y \Rightarrow (1, 3) \in Y$.

Let $Z = \{(1, 2), (2, 3)\}$. Z is not transitive as $(1, 2) \in Z, (2, 3) \in Z$ but $(1, 3) \notin Z$

ii) Let A be a set of people.
Let R be a relation on A such that $(a, b) \in R$ iff a is the ancestor of b. R is clearly a transitive relation.
Let T be a relation on A such that $(a, b) \in T$ iff a is the father of b. T is not a transitive relation

Now that we are familiar with the different properties of binary relations, let us test the properties of some relations.

Example 8: Describe the properties of the following relations on set A = {0, 1}:
i) Ø ii) $A \times A$ iii) *equalTo* = {(0, 0), (1, 1)} iv) *lessThan* = {(0, 1)}

Solution:

i) Ø (the empty set) does not contain any elements. We cannot define its symmetry, transitivity.
Ø is neither reflexive, nor irreflexive, nor symmetric, nor antisymmetric, and not transitive.

ii) The Cartesian product, $A^2 = A \times A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

$(0, 0), (1, 1) \in A^2 \Rightarrow A \times A$ is reflexive.

$(0, 1), (1, 0) \in A^2 \Rightarrow A \times A$ is symmetric.

Let us next test the transitivity of $A \times A$. We check whether all elements fulfill the transitivity property:
$(0, 0), (0, 1) \in A^2 \Rightarrow (0, 1) \in A^2$ and
$(0, 1), (1, 0) \in A^2 \Rightarrow (0, 0) \in A^2$ and
$(0, 1), (1, 1) \in A^2 \Rightarrow (0, 1) \in A^2$ and
$(1, 0), (0, 0) \in A^2 \Rightarrow (1, 0) \in A^2$ and
$(1, 0), (0, 1) \in A^2 \Rightarrow (1, 1) \in A^2$ and
$(1, 1), (1, 0) \in A^2 \Rightarrow (1, 0) \in A^2$
All of the above are true, hence the Cartesian product, $A \times A$ is transitive.

Hence ***A* x *A*** is reflexive, symmetric and transitive.

iii) *equalTo* = {(0, 0), (1, 1)}

(0, 0), (1, 1) ∈ *equalTo* ⇒ *equalTo* is reflexive.

(0, 0) ∈ *equalTo* and (1, 1) ∈ *equalTo* ⇒ *equalTo* is symmetric.

Test the transitivity of *equalTo*.
Since *(0, 0)* ∈ *equalTo* and (1, 1) ∈ *equalTo* ⇒ *equalTo* is transitive.

(0, 0) ∈ *equalTo* and 0=0;
(1, 1) ∈ *equalTo* and 1=1 ⇒ *equalTo* is antisymmetric.

Hence ***equalTo*** is reflexive, symmetric, transitive, antisymmetric.

iv) *lessThan* = {(0, 1)}

(0, 0) ∉ *lessThan* and (1, 1) ∉ *lessThan* ⇒ *lessThan* is irreflexive.

Since *(0, 1)* ∈ *lessThan* ⇒ *lessThan* is transitive.

lessThan is antisymmetric since *(0, 1)* ∈ *lessThan* and there is no other element in *lessThan*.

Thus ***lessThan*** above is irreflexive, transitive, antisymmetric.

Note: *lessThan* relation on any set will always be irreflexive, transitive, antisymmetric.

Value addition:

Composition of relations

If *R* and *S* are binary relations on a set *A*, then the ***composition*** of *R* and *S* is

$$R \circ S = \{(a, c) \mid aRb \text{ and } bSc \text{ for some } b \in A\}.$$

Example 9: Find the composition of the relation *R* and *S* where

R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and

S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}

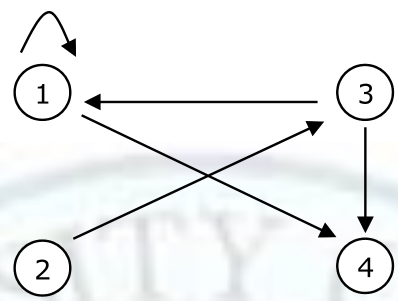
Solution: *R* ∘ *S* is constructed by using all the ordered pairs in *R* and *S,* where the second element of the ordered pair in *R* matches with the first element of the ordered pair in *S*. For example, the ordered pair (3, 1) in *R* and (1, 0) in *S* produce the ordered pair (3, 0) in *R* ∘ *S.* Thus by computing all the ordered pairs,

$$R \circ S = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}$$

Similarly we can define *R* ∘ *R* = {(1, 1), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4)}

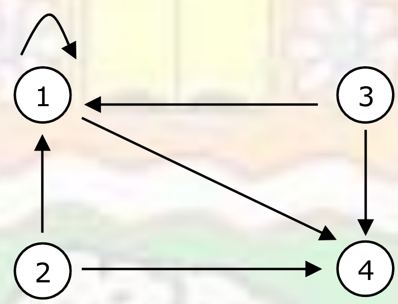
Path of a directed graph (diagraph): The path of a diagraph is a sequence of edges (e_1, e_2, …, e_i, e_j, …, e_n) such that the terminal node (vertex) of edge e_i is the initial node of edge e_j (where $1 \le i < j \le n$).

Diagraph of *R* gives a path of length 1, as shown below:



***Diagraph of R* = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)}**

Diagraph of *R* ° *R* gives a path of length 2 between various nodes of the directed graph (shown below).



***Diagraph of R* ° *R* = {(1, 1), (1, 4), (2, 1), (2, 4), (3, 1), (3, 4)}**

The powers of a relation *R* can be recursively defined from the composition of two relations.

Let *R* be a relation on set *A*. The power R^n, *n* = 1, 2, 3,… can be defined recursively by: $R^1 = R$ **and** $R^{n+1} = R^n \circ R$.

Example 10: Let *R* be the relation on the set of all people such that (*a, b*) ∈ *R*, if person *a* is the parent of person *b*.

Then (a, c) ∈ *R* ° *R* iff there is a person *b* such that (*a, b*) ∈ *R and* (*b, c*) ∈ *R,* i. e., iff there is a person *b* such that *a* is the parent of *b* and *b* is the parent of *c*. In other words (a, c) ∈ *R* ° *R* iff *a* is the grandparent of *c.*

Example 11: i) *R* ° Ø = Ø.

ii) isMotherOf ° isFatherOf = IsPaternalGrandmotherOf

iii) isSonOf ° isSiblingOf = isNephewOf

The property of *transitivity* can also be expressed in terms of composition of

relations.
Source: Book- Rosen, definition of Composition of Relations Book- C. L. Liu, chapter 4, definition of path of a diagraph

2.2.7 Equivalence Relation

Let *A* be a nonempty set, *R* be a relation on *A*. *R* is said to be an equivalence relation if *R* is:

(1) Reflexive
(2) Symmetric and
(3) Transitive

Example 12: Let I be a set of positive integers and "Congruence modulo" be relation on I defined below:

Let m be a fixed positive integer.
For a, b ∈ I; a & b are said to be congruent modulo m, denoted by $a \equiv b$ *modulo m*, if m divides a-b. For example, taking m = 5

5-0 is divisible by 5
5-1 is not divisible by 5
5-2 is not divisible by 5
5-3 is not divisible by 5
5-4 is not divisible by 5
5-5 is divisible by 5

Therefore, $5 \equiv 0$ modulo 5
$5 \equiv 5$ modulo 5
Similarly, $15 \equiv 5$ modulo 5

"Congruence modulo" is an equivalence relation over the set of integers as shown below:

(1) Reflexive: $a \equiv a$ *modulo m*, ∀ a ∈ I
(2) Symmetric: If $a \equiv b$ *modulo m* then $b \equiv a$ *modulo m*
(3) Transitive: If $a \equiv b$ *modulo m* and $b \equiv c$ *modulo m* then $a \equiv c$ *modulo m*

Value addition:
Equivalence relations
If a relation R on a set A is reflexive then (a, a) ∈ *R* ∀ a ∈ A. If R is symmetric then (a, b) ∈ *R* ⇒ (b, a) ∈ *R* ∀ a, b ∈ A. If R is transitive then (a, b) ∈ *R* & (b, c) ∈ *R* ⇒ (a, c) ∈ *R* ∀ a, b, c ∈ A. We can write transitivity for (a, b) and (b, a) which are in R: i.e. (a, b) ∈ *R* & (b, a) ∈ *R* ⇒ (a, a) ∈ *R* Hence, reflexivity is implied by symmetry and transitivity. Never the less we keep all the three properties for Equivalence relation
Source: Created by author/ reviewer.

2.3 Closure of Relations

Let us consider an example.

Example 1: Let A = {1, 2, 3} and R_1 = {(1, 1), (2, 2)} be a relation on A. R_1 is not reflexive.

But if we consider R_1^c = {(1, 1), (2, 2), (3, 3)}, it is reflexive. By including (3, 3) in R_1 we get a relation R_1^c which is reflexive. R_1^c is the closure of R_1.

In general, the closure of a relation is the smallest extension of the relation that has a certain specific property such as the reflexivity, symmetry or transitivity. They are defined in the following sections.

2.3.1 Reflexive Closure

A relation ***R'*** is the **reflexive closure** of a relation ***R*** if and only if:
(a) ***R'*** is reflexive
(b) $R \subseteq R'$
(c) for any relation ***R''***, if $R \subseteq R''$ and ***R''*** is reflexive, then $R' \subseteq R''$, that is, ***R'*** is the smallest relation that satisfies (a) and (b).

The reflexive closure of a relation ***R*** is denoted by ***r(R)***.

Example 2: Let us consider the "less-than" relation, *R*, on the set of integers **I**, i.e.,

$$R = \{(a, b) \mid a, b \in I \text{ and } a < b\}.$$

Let $E = \{(a, a) \mid a \in I\}$ be the equality or the diagonal relation on **I**.

R is not reflexive on **I**. E is reflexive on **I**.

Then the reflexive closure $r(R)$ of R is the union of R and the equality relation E on ***I***, i.e., $r(R) = R \cup E$

Hence, $r(R) = \{(a, b) \mid a, b \in I \text{ and } a \leq b\}$.

The **digraph of the reflexive closure** of a relation is obtained from the digraph of the relation by adding a self-loop at each vertex if one is already not there.

Value addition: Animation
Diagraph of Reflexive Closure

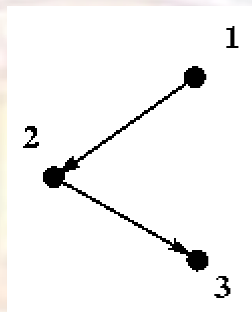
Animation frame 1

Draw the diagraph of the reflexive closure of relation:

R_1 = {(1, 2), (2, 3)} on the set {(1, 2, 3)}.

Animation frame 2

Solution: Let us first draw the diagraph of the relation R_1 = {(1, 2), (2, 3)}

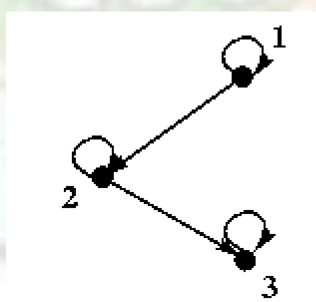


Diagraph of the relation R_1 = {(1, 2), (2, 3)}

Animation frame 3

To obtain the digraph of the reflexive closure, *r*(R_1), of R_1 – in the digraph of the relation, add a self-loop at each vertex if one is already not there.

Animation frame 4



Diagraph of the reflexive closure, *r*(R_1), of relation

R_1 = {(1, 2), (2, 3)}

Source: Created by the author

2.3.2 Symmetric Closure

A relation ***R'*** is the **symmetric closure** of a relation ***R*** if and only if:
(a) ***R'*** is symmetric
(b) $\boldsymbol{R \subseteq R'}$
(c) for any relation ***R''***, if $\boldsymbol{R \subseteq R''}$ and ***R''*** is symmetric, then $\boldsymbol{R' \subseteq R''}$, that is, ***R'*** is the smallest relation that satisfies (a) and (b).

The symmetric closure of a relation ***R*** is denoted by ***s(R)***.

Example 3: The "less-than" relation on the set of integers **I**, gives

$R = \{(a, b) \mid a, b \in \text{I and } a < b\}$.

Let $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ be the inverse relation of R.

R is not symmetric. R^{-1} is also not symmetric.

Then the symmetric closure $s(R)$ of R is the union of R and the inverse relation, R^{-1}, of R, i.e., $s(R) = R \cup R^{-1}$

$= \{(a, b) \mid a < b\} \cup \{(b, a) \mid a < b\}$

Hence, $s(R) = \{(a, b) \mid a, b \in \text{I and } a \neq b\}$

The **digraph of the symmetric closure** of a relation is obtained from the digraph of the relation by adding for each arc the arc in the reverse direction if one is already not there.

Value addition: Animation

Diagraph of Symmetric Closure

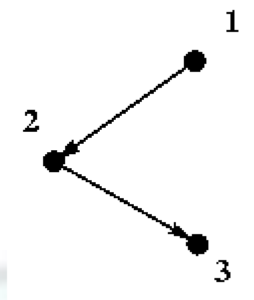
Animation frame 1

Draw the diagraph of the symmetric closure of relation

$R_1 = \{(1, 2), (2, 3)\}$ on the set $\{(1, 2, 3)\}$.

Animation frame 2

Solution: Let us first draw the diagraph of the relation $R_1 = \{(1, 2), (2, 3)\}$

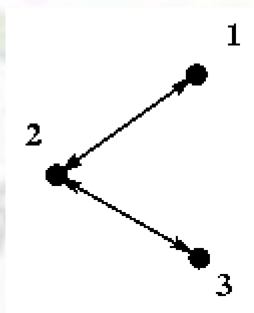


Diagraph of the relation R_1 = {(1, 2), (2, 3)}

Animation frame 3

To obtain the digraph of the symmetric closure, *$s(R_1)$,* of R_1 – in the digraph of the relation, add for each arc the arc in the reverse direction if one is already not there.

Animation frame 4



Diagraph of the symmetric closure, $s(R_1)$, of relation

R_1 = {(1, 2), (2, 3)}

Source: Created by the author

2.3.3 Transitive Closure

A relation ***R'*** is the **transitive closure** of a relation ***R*** if and only if:
(a) ***R'*** is transitive
(b) $R \subseteq R'$
(c) for any relation ***R''***, if $R \subseteq R''$ and ***R''*** is transitive, then $R' \subseteq R''$, that is, ***R'*** is the smallest relation that satisfies (a) and (b).

The transitive closure of a relation ***R*** is denoted by ***t(R)***.

Example 4: Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (4, 1)} be a relation on A. R is not transitive as (1, 3), (2, 4), (3, 1) and (4, 2) are not in R. If we include these elements in R we get its transitive closure.

Theorem: Let R be a relation on the set A with n elements. Then,

$$t(R) = R \cup R^2 \cup \ldots \cup R^n$$

where $R^2 = R \circ R$, $R^n = R^{n-1} \circ R$

Examples 5:

a) The transitive closure of a parent-child relation is the ancestor-descendant relation.

b) The "less-than" relation on the set of integers **I**, the "less-than" relation itself.

Examples 6: Find the transitive closure of relation R on the set $A = \{1, 2, 3\}$, where $R = \{(1, 2), (2, 3), (3, 3)\}$

Solution: Set A has $n = 3$ elements. The transitive closure, $t(R)$, of relation R is thus,

$t(R) = R \cup R^2 \cup R^3$

Now $R^2 = R \circ R = \{(1, 3), (2, 3), (3, 3)\}$ and

$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$

Thus, $t(R) = R \cup R^2 \cup R^3 = \{(1, 2), (2, 3), (3, 3), (1, 3)\}$

Value addition:

Transitive Closure using matrices

If we consider the matrix representation of a relation, R:

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

R = {(1, 2), (2, 3), (3, 4), (4, 1)}

$$M \times M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

M x M is the matrix for $R \circ R$ = $\boldsymbol{R^2}$ **= {(1, 3), (2, 4), (3, 1), (4, 2)}**

M x M gives the elements of R, which on including in *R* gives *t(R)*.

We find R^3 and R^4 similarly.

$$R^3 = R^2 \circ R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

i.e., $\boldsymbol{R^3}$ **= {(1, 4), (2, 1), (3, 2), (4, 3)}**

$$R^4 = R^3 \circ R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

i.e., $\boldsymbol{R^4}$ **= {(1, 1), (2, 2), (3, 3), (4, 4)}**

Finally we have $t(R) = R \cup R^2 \cup R^3 \cup R^4$

$$t(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

t(R) **= {(1, 1), (1, 2), {(1, 3), {(1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2) , (4, 3), (4, 4)}**

Source: Created by author/ reviewer

The **digraph of the transitive closure** of a relation is obtained from the digraph of the relation by adding for each directed path the arc that shunts the path if one is already not there.

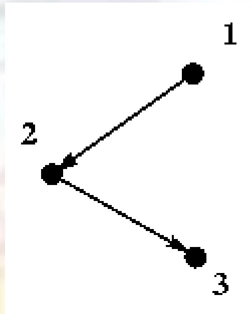
Value addition: Animation
Diagraph of Transitive Closure
Animation frame 1

Draw the diagraph of the transitive closure of relation

R_1 = {(1, 2), (2, 3)} on the set {(1, 2, 3)}.

Animation frame 2

Solution: Let us first draw the diagraph of the relation R_1 = {(1, 2), (2, 3)}

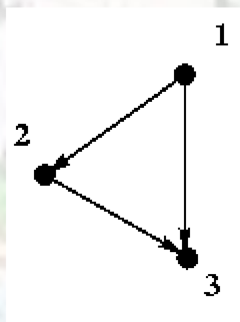


Diagraph of the relation R_1 = {(1, 2), (2, 3)}

Animation frame 3

To obtain the digraph of the transitive closure, *t(R_1),* of R_1 – in the digraph of the relation, add for each directed path, the arc that shunts the path if one is already not there.

Animation frame 4



Diagraph of the transitive closure, *t(R_1)*, of relation

R_1 = {(1, 2), (2, 3)}

Source: Created by the author

2.3.4 Properties of Closure

The properties of closure are stated here as theorems.

Theorem: Let ***E*** denote the equality relation, and $\boldsymbol{R^{-1}}$ the inverse relation of binary relation ***R***, all on a set ***A***, where $R^{-1} = \{(a, b) \mid (b, a) \in R\}$. Then,

1) $r(R) = R \cup E$
2) $s(R) = R \cup R^{-1}$
3) $t(R) = \cup^{\infty}_{i=1} R^{i} = \cup^{\infty}_{i=n} R^{i}$, if $|A| = n$
4) R is reflexive if and only if $r(R) = R$.
5) R is symmetric if and only if $s(R) = R$.
6) R is transitive if and only if $t(R) = R$.

2.4 Partial Ordering Relations

2.4.1 Partial Ordering Relations and Partially Ordered Set (Poset)

Consider the set of all English words. We order words using the relation containing pairs of words (*x, y*) where word *x* comes before *y* in the dictionary. We often use relations to order some or all elements of a set. In mathematics, a partially ordered set (or poset) formalizes the intuitive concept of an ordering, sequencing, or arrangement of the elements of a set.

In a partial ordering relation two objects are related if one of them is smaller (or larger) than or inferior (superior) to the other object according to some property or criteria, precedes (or succeeds) other objects.

Ordering implies that the objects in the set are ordered according to some property or criteria. However two objects in the set may not be related in the partial ordering relation. In this case we cannot compare these objects and identify the small or inferior one. That is the reason it is called *partial ordering*.

A familiar real-life example of a partially ordered set is a collection of people ordered by genealogical descendancy. Some pairs of people bear the ancestor-descendant relationship, but other pairs bear no such relationship.

Definitions:

A binary relation R on a set S is called a **partial ordering relation** or a **partial order** if it is reflexive, transitive and antisymmetric.

A set S along with the partial order R is called a **partially ordered set** or **poset** and is denoted by **(S, R)**.

These relations are called *partial orders* to reflect the fact that not every pair of elements of a poset need be related: for some pairs, it may be that neither element precedes the other in the poset.

<u>Example 1</u>: Let Z^+ be a set of positive integers and $R = \{(a, b) | a, b \in Z^+$ and a *divides* $b\}$, i.e. *R* is the "*divides*" relation. R is a partial ordering relation as it is reflexive transitive and antisymmetric over the set Z^+. The partially ordered ser or poset is (Z^+, R).

<u>Example 2</u>: Let Z be a set of integers and $R = \{(a, b) | a, b \in Z$ and a *divides* $b\}$, i.e. *R* is the "*divides*" relation. R is not antisymmetric.

For example, 2 *divides* -2 = -2 *divides* 2, but $2 \neq -2$.

Thus R, the "*divides*" relation, is not a partial ordering relation over the set Z.

<u>Example 3</u>: Show that the "less-than-or-equal-to" (≤) relation, in the usual sense, is the partial ordering on the set of integers, **Z**.

Solution: Since $a \leq a$, $\forall a \in \mathbf{Z}$, ≤ is reflexive.

If $a \leq b$ and $b \leq a$ then $a = b$. Hence ≤ is antisymmetric.

Also, ≤ is transitive since $a \leq b$ and $b \leq c$ imply that $a \leq c$.

Thus ≤ is a partial ordering relation on the set of integers, **Z**, and $(\mathbf{Z}, \leq)$ is a poset.

<u>Example 4</u>: We can use the symbol '≤' in some other sense:

$$a \leq b \Leftrightarrow a \text{ precedes } b$$

'≤' is a partial order.

$a \leq a$: *a* precedes *a*

$a \leq b$ and $b \leq a$ then $a = b$: *a* precedes *b, b* precedes *a* then *a* and *b* are identical

$a \leq b$ and $b \leq c$ then $a \leq c$: *a* precedes *b, b* precedes *c* then *a* precedes *c*

<u>2.4.2 Hasse Diagram (Poset Diagram)</u>

<u>2.4.2.1 Introduction</u>

A finite poset can be visualized through its Hasse diagram, which depicts the ordering relation between certain pairs of elements and allows one to reconstruct the whole partial order structure.

Definition:

A graph representing a poset but with only immediate predecessor edges, and the edges are oriented up from x to y when x < y is called a **Hasse Diagram** or **Poset Diagram**.

The Hasse Diagram has been named after the twentieth-century German mathematician Helmut Hasse

Example 5: The figure below shows the Hasse diagram of the set of all subsets of a three-element set {x, y, z}, ordered by inclusion.

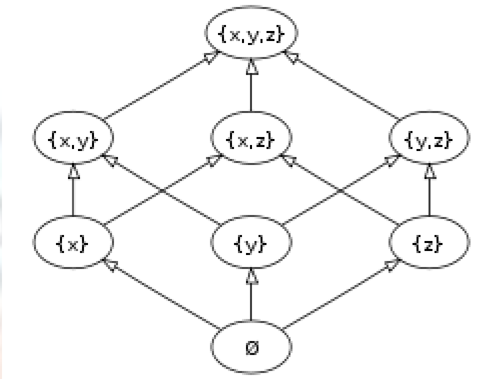


Figure 2.4.1: Hasse Diagram of the poset (power({a, b, c}), ⊂)

(Source: [2.1])

We know that binary relation can be represented as a diagraph or directed graph. When the binary relation is a partially ordered relation, its directed graph can be simplified.

2.4.2.2 Steps to draw a Hasse Diagram

Consider the relation matrix in Figure 2.4.2(a), for which a Hasse diagram is to be drawn.

Following are the steps to be followed to draw a Hasse Diagram (Poset Diagram) of a partial ordering relation:

Step 1: Draw the diagraph corresponding to the given relation or relation matrix.

Step 2: Since the relation is reflexive – we can remove all arrows pointing back to a node.

Step 3: The relation is transitive – we can remove arrows between nodes that are connected by sequence of arrows.

The resultant diagram is shown in Figure 2.4.2(b).

Step 4: In most of the resultant graphical representations, all arrows are pointing in one direction (upward, downward, left-to-right, right-to-left). We can remove all the arrowheads.

Such a graphical representation of a partial ordering relation in which all arrowheads are understood to be pointing in the upward is known as a *Hasse Diagram or Poset Diagram* of the given relation.

The required Hasse Diagram is shown in Figure 2.4.2(c).

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)

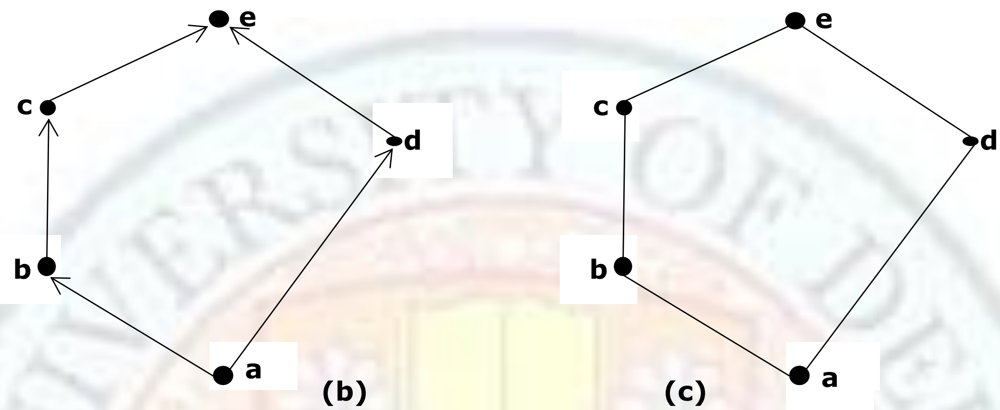


Figure 2.4.2: (a) Relation Matrix (b) Simplified representation

(c) Hasse Diagram

Value addition: Animation

Steps to draw a Hasse Diagram

<u>Slide 1</u>:

Let us take an example to understand how a Hasse diagram is to be drawn.

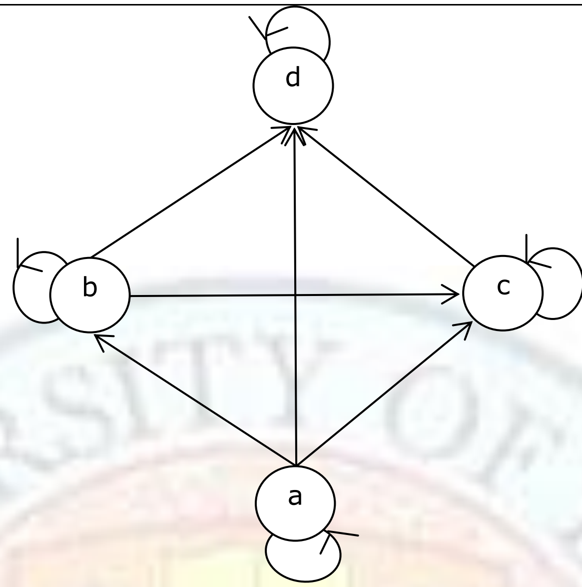
Example: Draw the Hasse Diagram for the relation R on set A = {a, b, c, d}, whose relation matrix is given below:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

<u>Slide 2</u>

Solution:

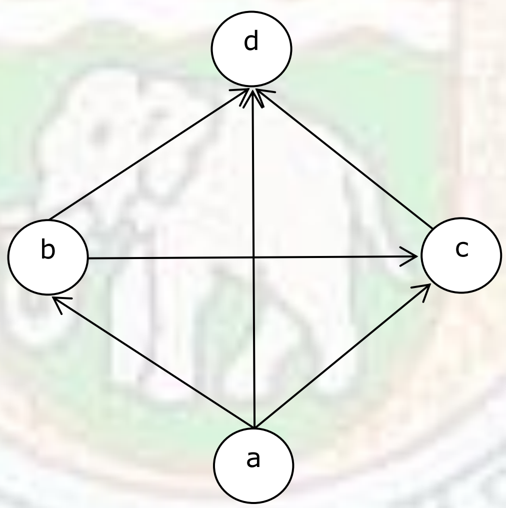
<u>Step 1</u>: Draw the diagraph corresponding to the above relation matrix.



Slide 3

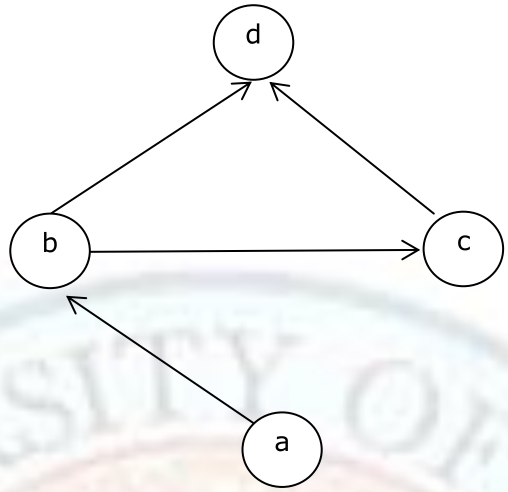
Step 2: Since the binary relation is a partially ordered relation, it can be simplified.

It is reflexive – remove all arrows pointing back to a node.



Slide 4

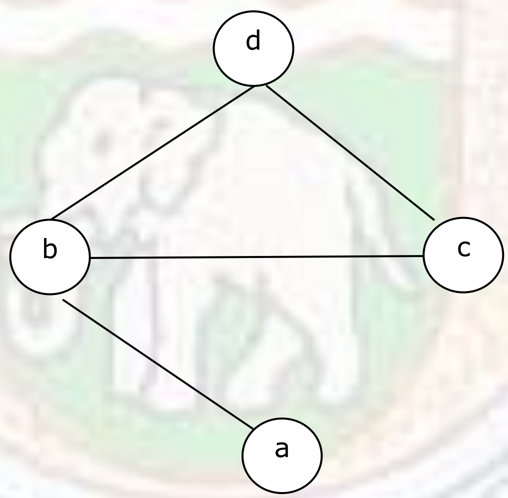
Step 3: The relation is transitive – thus remove arrows between nodes that are connected by sequence of arrows.



Slide 5

Step 4: In the above graphical representation, all arrows are pointing in one direction (here pointing upwards). We remove all the arrowheads.

We thus obtain the Hasse Diagram corresponding to the given partial ordering relation.



Hasse Diagram

Source: Created by the author

2.4.3 Total Ordering

For *a, b* distinct elements of a partially ordered set *P*, if $a \leq b$ or $b \leq a$, then *a* and *b* are ***comparable***. Otherwise they are ***incomparable***.

If every two elements of a poset are comparable, the poset is called a **totally ordered set** or **linearly ordered set** or **chain** (e.g. the natural numbers under order).

A poset in which every two elements are incomparable is called an **antichain**.

2.5 Functions

2.5.1 Introduction

In many cases we assign to each element of a set a particular element of the second set (which may be the same as the first). For example, the students in a mathematics class are assigned grades. The assignment is an example of a function.

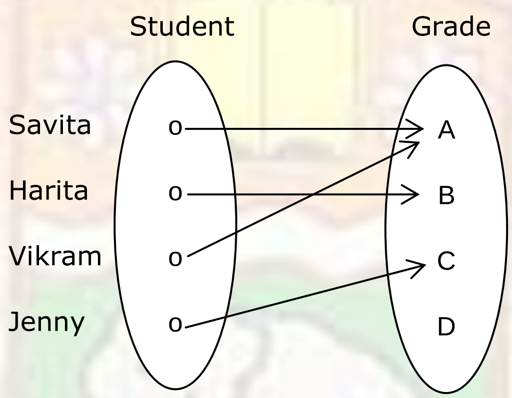


Figure 2.5.1: Assignment of grades in Mathematics

We know that a relation from a non-empty set A to a non-empty set B is a subset of A x B. A function from A to B is also a subset of A x B with some restrictions on the elements in the subset.

This rule leads to a rule of mapping elements of one set to another set.

Functions are very important in the field of discrete mathematics. They are used:

- In the definition of discrete structures like sequences and strings
- To represent how long it takes the computer to solve a problems of a particular size
- To solve recursive problems through recursive functions
- etc.

Definition:

A **function** from set *A* to set *B*, denoted as $\boldsymbol{f}$**: A→B**, is a relation from A to B, such that, each a $\in$ A belongs to a unique ordered pair (a, b) in $f \subseteq$ A x B.

We write $f(x) = y$ to mean f associates $x \in A$ with $y \in B$. Say, "f of x is y" or "f maps x to y." Here y would be a unique element in B.

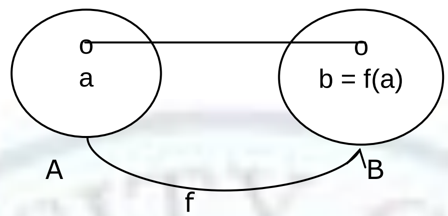


Figure 2.5.2: Diagram of Function *f* mapping elements of *A* to elements of *B*

Value addition:
Functions vs Relations
Every function is a relation but every relation is not a function. Therefore $f = \{(a, b): a \in A, b \in B \text{ and } b = f(a)\}$
Source: Created by author

Example 1: Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. And let relation

$R = \{(a, 2), (a, 4), (b, 1), (c, 2), (c, 3)\}$ be a binary relation from A to B. The relation shown in Figure 2.5.3 is not a function since element *a* and *c* of *A* do not map to a single element of *B.*

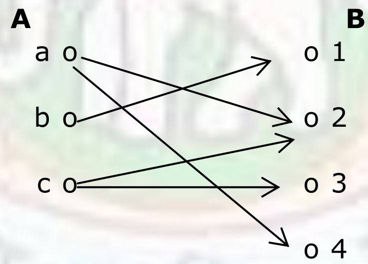


Figure 2.5.3: The above is a function but not a relation

Functions can be defined in a number of ways.

- There may be explicit assignment, as in Figure 2.5.1 students are assigned grades in mathematics.
- Give a formula, such as $f(x) = x + 5$
- Use a computer program to specify a function

Definitions:

If f is a function from A to B, A is called the **domain** of f and B is called the **codomain** of f.

If $f(a) = b$, we say that b is the **image** of a and a is a **pre-image** of b.

The **range** of f is the set of all images of elements of A.

If $D \subset B$, the **pre-image** or **inverse image** of D is the set

$$f^{-1}(D) = \{x \mid f(x) \in D\}$$

Let f be a function from A to B and let S be a subset of A.

The **image of *S*,** $f(S)$, is the subset of B that contains the images of elements of S. Thus, if $f: A \rightarrow B$ and $S \subset A$, $f(S) = \{f(s) \mid s \in S\}$.

Example 2: Consider the example in Figure 2.5.1 shows a function f: *Student* $\rightarrow$ *Grade* which assigns grades to students.

Here, domain of f is *Student* = {Savita, Harita, Vikram, Jenny}

codomain of f is *Grade* = {A, B, C, D}.

f(Savita) = f(Vikram) = A, f(Harita) = B and f(Jenny) = B

range(f) = {A, B, C}, because D has not been assigned to any *Student*

f({Savita, Vikram}) = {A}

f^{-1}({D}) = $\varnothing$

f^{-1}({A, B, C, D}) = {Savita, Harita, Vikram, Jenny}

Example 3: The figure below shows a function $f: A \rightarrow B$.

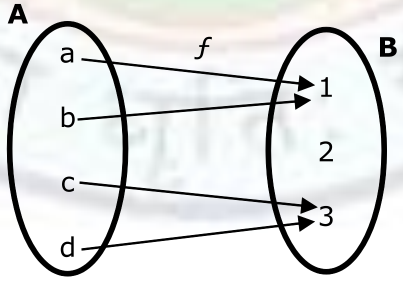


Figure 2.5.4: Function f: $A \rightarrow B$ with domain A = {a, b, c, d} and codomain B = {1, 2, 3}

Here, domain A = {a, b, c, d} and codomain B = {1, 2, 3}.

$f(a) = f(b) = 1$ and $f(c) = f(d) = 3$.

range(f) = {1, 3}, because 2 has not been assigned to any element of A

$f(\{a, b\}) = \{1\}$

$f^{-1}(\{2\}) = \varnothing$

$f^{-1}(\{1, 2, 3\}) = \{a, b, c, d\}$

Example 4: Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$. Let $f(a) = f(b) = 3$,

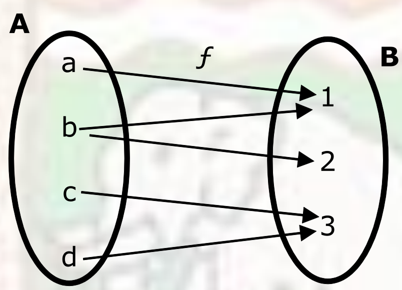
$f(c) = 2$ and $f(d) = 1$.

The image of $S = \{a, b, c, d\}$, $f(S) = \{2, 3\}$.

Value addition: Example

Function or Relation

Consider the mapping given below.



Is it a function?

No, f is not a function. Here $b \in A$ is mapped to two elements 1, 2 of B.

But, f is a relation.

Source: Created by author/reviewer

Operations on Functions

Definition:

Let f_1 and f_2 be functions from A to B (the set of real numbers). Then $f_1 \pm f_2$ and $f_1 f_2$ are also functions from A to B.

$$(f_1 \pm f_2)x = f_1(x) \pm f_2(x) \text{ and } (f_1 f_2)x = f_1(x) f_2(x)$$

Example 5: Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x$ and

$f_2(x) = x^2 - 3$. Find $f_1 + f_2$ and f_1f_2.

Solution:

$(f_1 + f_2)x = f_1(x) + f_2(x) = x + x^2 - 3 = x^2 + x + 3$

and $(f_1f_2)x = f_1(x)f_2(x) = x(x^2 - 3) = x^3 - 3x$

Composition of Functions

Definition:

Let $g: A \rightarrow B$ and $f: B \rightarrow C$. The **composition** of the functions f and g, denoted by

$f \circ g: A \rightarrow C$, is defined by $(f \circ g)(a) = f(g(a))$.

The composition $f \circ g$ is shown in Figure 2.5.5. To find $(f \circ g)(a)$, first apply function g to a to obtain $g(a)$ and then apply function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$.

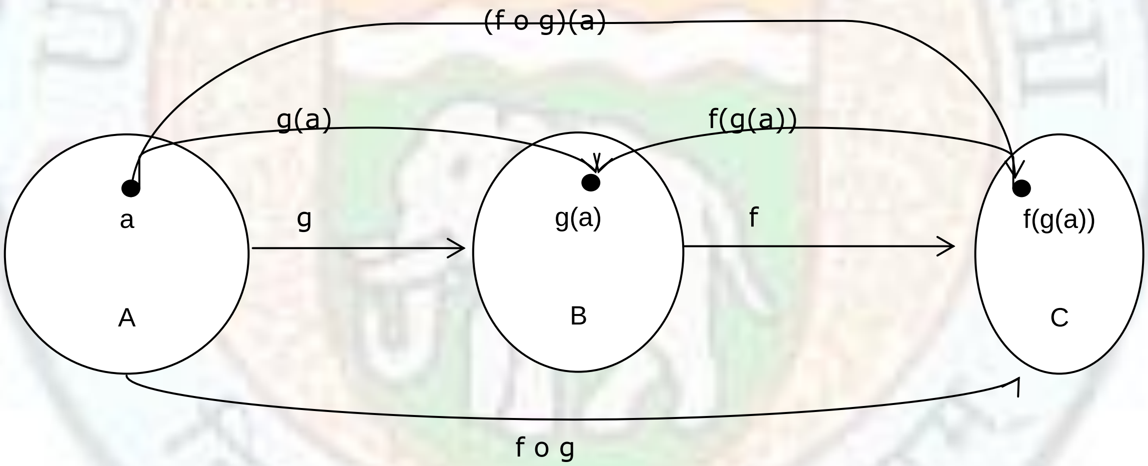


Figure 2.5.5: **The composition, $f \circ g$, of the functions f and g**

Note:

- The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.
- Even though $f \circ g$ and $g \circ f$ are defined for the functions f and g, (see Example 6), $f \circ g$ and $g \circ f$ are not equal. That is, the commutative law does not hold for the composition of functions.

Example 6: Let f and g be functions from the set of integers to itself, such that

$f(x) = 3x + 2$ and $g(x) = 7x + 1$. Find the composition of f and g, *and* the composition of g and f?

Solution:

$(f \circ g)(x) = f(g(x)) = f(7x + 1) = 3(7x + 1) + 2 = 21x + 5$

$(g \circ f)(x) = g(f(x)) = g(3x + 2) = 7(3x + 2) + 1 = 21x + 15.$

Thus we can see that, $(f \circ g)(x) \neq (g \circ f)(x)$.

2.5.2 Types of Functions

There are various types of functions.

- Constant Function
- Identity Function
- Polynomial Function
- Step Function
- Greatest Integer Function
- Strictly increasing and Strictly decreasing Functions
- Increasing Function, Decreasing Function
- Floor and Ceiling Function

Let us discuss some of these functions in detail.

Identity Function

The **identity function** on set A is the function $i_A : A \to A$, where

$i_A(x) = x$ for all $x \in A$.

A function that assigns each element to itself is called the identity function.

Strictly increasing and Strictly decreasing Functions

Definitions:

A function f whose domain and codomain are subsets of **R** is called **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f.

In other words, a function is f is called *strictly increasing* if

$$\forall x\ \forall y\ ((x < y) \Rightarrow (f(x) < f(y))),$$

where x and y are is the domain of f.

Also we can say a function is strictly increasing if $\forall x \, \forall y \, ((x > y) \Rightarrow (f(x) > f(y)))$.

Similarly, a function f is called **strictly decreasing** if $f(x) < f(y)$ whenever

$x > y$ and x and y are in the domain of f.

In other words, a function is f is called *strictly decreasing* if

$$\forall x \, \forall y \, ((x > y) \Rightarrow (f(x) < f(y)),$$

where x and y are in the domain of f.

Also, a function is strictly decreasing if $\forall x \, \forall y \, ((x < y) \Rightarrow (f(x) > f(y)))$.

Floor and Ceiling Functions

Let x be a real number. The floor function rounds x down to the closest integer less than or equal to x, and the ceiling function rounds x up to the closest integer greater than or equal to x.

These functions are often used:

- When objects are counted.
- In the analysis of the number of steps used by procedures to solve problems of a particular size.

Definition:

The **floor function** assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.

The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Note*:* The floor function is often also called the ***greatest integer function***. It is often denoted by $[x]$.

The Floor and Ceiling functions are discussed in detail, with example, in Section 2.5.5.

2.5.3 Properties of Functions

Let f: A → B be a function. There are three properties that f might possess:

- Injective (One-to-one)
- Surjective (Onto)
- Bijective or One-to-one Correspondence (One-to-one and Onto)

2.5.3.1 Injective (One-to-one) Functions

In an Injective or One-to-one function distinct elements in *A* map to distinct elements in *B*.

Definition:

A function f is said to be **Injective** or **One-to-one** (written as 1-1) if and only if $f(x) = f(y)$ implies x = y for all x, y in the domain of f.

In other words, a function is one-to-one iff $f(x) \neq f(y)$ when $x \neq y$.

Example 7:

i) Let f be a function, see Figure 2.5.6, from {a, b, c, d} to {1, 2, 3, 4, 5}, with $f(a) = 3$, $f(b) = 5$, $f(c) = 2$ and $f(d) = 1$.
This function is one-to-one since f takes different values for each of the elements in the domain.

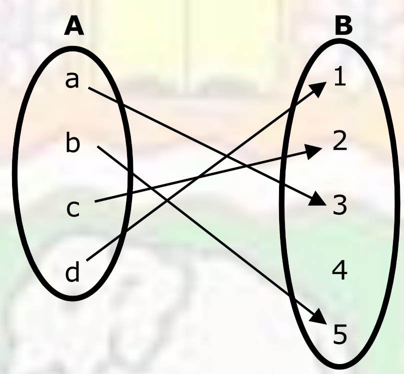


Figure 2.5.6: Function is One-to-one or Injective

ii) Consider function $f(x) = x^2$ from the set of integers to the set of integers.
Now, $f(-1) = f(1) = 1$, but $-1 \neq 1$. Thus $f(x) = x^2$ is not an injective function.

iii) Consider function $f(x) = x^2$ from the set of natural numbers to a set of natural numbers.
Now, $f(1) = 1$, $f(2) = 4$. We see that $f(1) \neq f(2) \Rightarrow 1 \neq 2$. Thus $f(x) = x^2$ is 1-1 (injective).

Note: A **strictly increasing** or a **strictly decreasing function** (discussed in Section 2.5.2) will always be one-to-one.

2.5.3.2 Surjective (Onto) Functions

In some functions, the codomain and range are the same. That is, every member of the codomain is the image of some element of the domain. Such functions are called onto functions.

Definition:

A function f from set X to set Y is said to be **surjective** or **onto** if and only if for every y $\in$ Y there is an element x $\in$ X with $f(x) = y$.

In other words, $f: X \rightarrow Y$ is **onto** or **surjective** iff $\forall y \in Y, \exists x \in X \mid f(x) = y$.

In an onto function, the universe of discourse of x is the domain of the function; and the universe of discourse of y is the codomain (in this case also the range) of the function.

Example 8:

i) Let f be a function, see Figure 2.5.7, from {a, b, c, d} to {1, 2, 3}, with $f(a) = f(b) = 3$, $f(c) = 2$ and $f(d) = 1$.
Since all three elements of the codomain are images of the elements of the domain f is onto.

Had the codomain been {1, 2, 3}, it would not have been onto.

It may be noted that this function is not one-to-one.

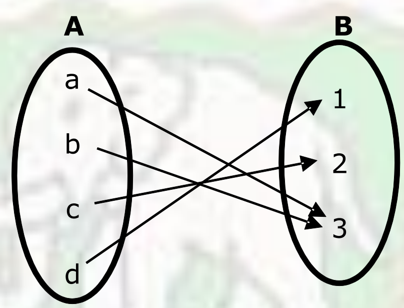


Figure 2.5.7: Function is Onto or Surjective

ii) Consider function $f(x) = x^2$ from the set of integers to the set of integers. There is no x with $x^2 = -1$, and so on. This shows that not all elements of the codomain are images of the elements of the domain f. Hence f is not onto.

iii) The function of Example 7 i), shown in Figure 2.5.6, is not onto.

2.5.3.3 Bijective Functions or One-to-one Correspondence

Definition:

A function f is **bijective** or a **one-to-one correspondence** if it is both one-to-one and onto, i.e. it is both injective and surjective.

Example 9: Let *f* be a function, see Figure 2.5.8, from {a, b, c} to {1, 2, 3}, with *f*(a) = 3, *f*(b) = 1 and *f*(c) = 2.

This function is one-to-one since *f* takes different values for each of the elements in the domain. Since all three elements of the codomain are images of the elements of the domain *f* is onto.

Hence *f* is a bijection.

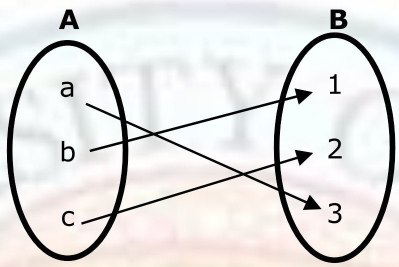


Figure 2.5.8: Function is a Bijection or One-to-one Correspondence

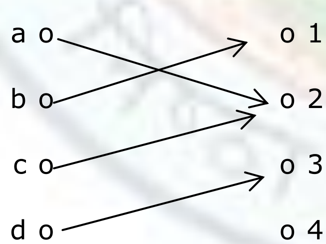
Note: The **identity function**, i_A, (discussed in Section 2.5.2) is a bijection since it is one-to-one and onto.

Value addition: Animation

Quiz on Properties of Functions

Now that you are aware of the different properties of functions, let us test your understanding. Different figures would be displayed; you need to guess the type of correspondence, if any. Click on the **"CHECK ANSWER"** button to see if you guessed the correct answer.

Question 1:

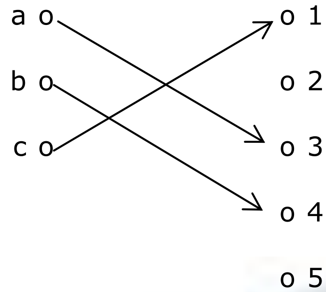


CHECK ANSWER

Answer 1:
Neither one-to-one, nor onto

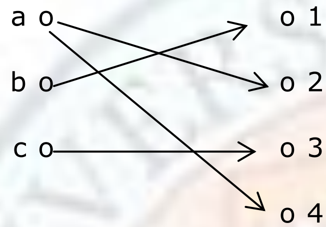
Question 2:

Answer 2:
One-to-one, not onto



CHECK ANSWER

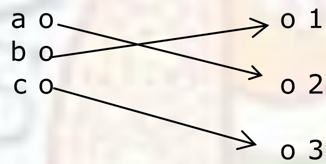
Question 3:



CHECK ANSWER

Answer 3:
Not a function.
(There are two arrows coming from *a*, i.e., *a* is associated with two different range elements. This *is a relation*, not a function.)

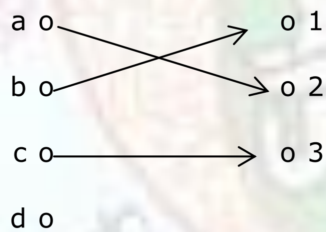
Question 4:



CHECK ANSWER

Answer 4:
Bijection – both one-to-one, and onto

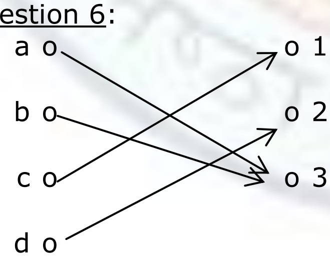
Question 5:



CHECK ANSWER

Answer 5:
Not a function
(Element *a, b, c* of the domain have a pair in the range. Element *d* is in the domain, but it has no range element that corresponds to it. This is not a function. Also *not a relation*.)

Question 6:



CHECK ANSWER

Answer 6:
Onto, not One-to-one

Source: Created by the author

2.5.3.4 Inverse Functions

Next we will use the one-to-one, onto correspondence, *ƒ,* from the set *A* to the set *B* to define another function. Since *ƒ* is an onto function, every element of *B* is the image of some element in *A*. Also, since *f* is a one-to-one function, every element of *B* is the image of a unique element of *A*. We can thus define a new function from *B* to *A* that reverses the correspondence given by *ƒ*.

Definition:

Let *ƒ* be a one-to-one onto function from the set *A* to the set *B*. The **inverse function** of *ƒ* is the function that assigns to an element *b* belonging to *B* the unique element *a* in *A* such that $f(a) = b$. The inverse function of *ƒ* is denoted by f^{-1}. Hence, $\boldsymbol{f^{-1}(b) = a}$ when $f(a) = b$.

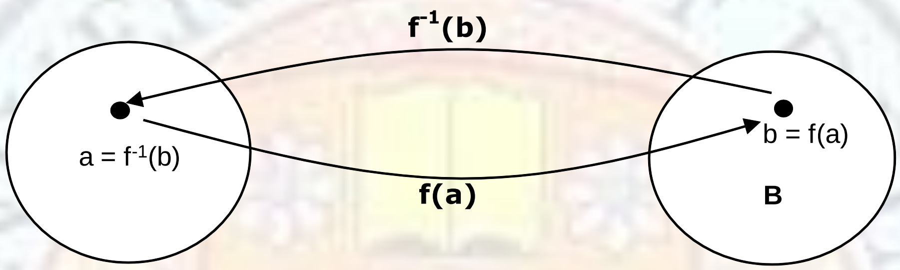


Figure 2.5.9: The inverse function, f^{-1} of *ƒ*

Note: If a function *ƒ* is not a one-to-one or onto, we cannot define an inverse function of *ƒ*.

A bijection is called **invertible** because its inverse can be defined. A function is **not invertible** if it is not a bijection, because the inverse of such a function does not exist.

Example 10: Let *ƒ* be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ such that

$f(a) = 4$, $f(b) = 2$, $f(c) = 3$, and $f(d) = 1$. If *ƒ* is invertible, find its inverse.

Solution: Since *ƒ* is a one-to-one correspondence it is invertible.

The inverse function f^{-1} is: so $f^{-1}(1) = d$, $f^{-1}(2) = b$,

$f^{-1}(3) = c$, and $f^{-1}(4) = a$.

Example 11: Let $f: \mathbf{R} \to \mathbf{R}$ with $f(x) = x^4$. Is *ƒ* invertible?

Solution: Since $f(-3) = f(3) = 81$, *ƒ* is not one-to-one. Hence *ƒ* is not invertible.

Example 12: Let $f: \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x+5$. If *ƒ* is invertible, find its inverse.

Solution: The function f is a one-to-one and onto, hence it has an inverse.

Suppose y is the image of x, i.e., $y = x + 5$. Then $x = y - 5$.

This means that $y - 5$ is the unique element of **Z** that is mapped to y by f.

Hence, $f^{-1}(y) = y - 5$.

Value addition:

Composition of a function and its Inverse gives Identity Function

When the composition of a function and its inverse is formed, in either order, an identity function is obtained.

Let f: A → B be 1-1, onto. Thus f^{-1}: B → A exists. $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$

We will now prove the above statement.

If $f(a) = b$ then $f^{-1}(b) = a$. Also, if $f^{-1}(b) = a$ then $f(a) = b$.

Hence, $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$

and $(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$.

Consequently $\boldsymbol{f^{-1} \circ f = I_A}$ and $\boldsymbol{f \circ f^{-1} = I_B}$, where I_A and I_B are the identity functions on the sets A and B, respectively.

Also $\boldsymbol{f \circ I_A = f;\quad I_B \circ f = f}$ ***and*** $\boldsymbol{(f^{-1})^{-1} = f}$

Source: Book - Rosen, Inverse and Composition of Functions

2.5.4 Graphs of Functions

A set of ordered pairs in A X B can be associated to each function from A to B. This set of pairs is called the graph of the function and can be displayed pictorially to help understand the behavior of a function.

Definition:

Let f be a function from the set A to the set B. The **graph** of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

i.e. A function $f: A \rightarrow B$ is a relation from A to B (i.e. a subset of A X B) such that each $a \in A$ belong to a *unique* ordered pair (a, b) in f.

Example 13:

i) Let us display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

The graph of f is the set of ordered pairs of the form $(n, 2n + 1)$, where n is an integer. This graph is displayed in the figure below:

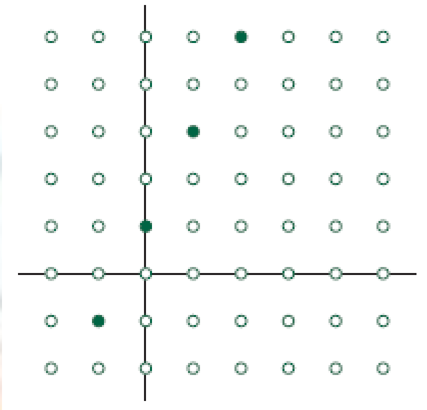


Figure 2.5.10: Graph of function $f(n) = 2n + 1$

Source: Chapter 2, Discrete Mathematics and Its Applications, 6th Ed.,

by Kenneth Rosen

ii) Let us display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.
The graph of f is the set of ordered pairs of the form (x, x^2), where x is an integer. This graph is displayed in the figure below:

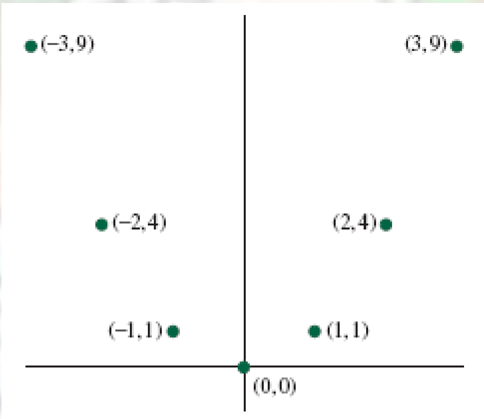


Figure 2.5.11: Graph of function $f(x) = x^2$

Source: Chapter 2, Discrete Mathematics and Its Applications, 6th Ed.,

by Kenneth Rosen

Geometric Characteristics of Functions

The property of a function (one-to-one, onto, one-to-one correspondence/ invertible) can be determined from the graph of the function according to the following:

- Function $f: A \to B$ is *one-to-one* if there are no two distinct ordered pairs (a_1, b) and (a_2, b) in the graph of f; hence each horizontal line can *intersect* the graph of f in *at most one point*.

- If $f: A \to B$ is *onto* means that for every b $\in B$ there must be at least one a $\in A$ such that (a, b) belongs to the graph of f; hence each horizontal line must *intersect* the graph of f in *at least one point*.

- If f is both *one-to-one* and *onto, i.e.,* one-to-one correspondence/ invertible, then each horizontal line will *intersect* the graph of f in *exactly one point*.

Thus from Figure 2.5.10, we can infer that $f(n) = 2n + 1$ is one-to-one correspondence (both one-to-one and onto).

Similarly from Figure 2.5.11, we can infer that $f(x) = x^2$ is neither one-to-one nor onto.

2.5.5 Floor and Ceiling Functions

The floor and ceiling functions were defined in Section 2.5.2. Let us discuss them in detail with examples.

Example 14: Some values of the floor and ceiling functions are given below:

$\lfloor 1/8 \rfloor = 0$; $\lceil 1/8 \rceil = 1$; $\lfloor -1/8 \rfloor = -1$; $\lceil -1/8 \rceil = 0$;

$\lfloor 7.3 \rfloor = 7$; $\lceil 7.3 \rceil = 8$; $\lfloor 10 \rfloor = 10$; $\lceil 10 \rceil = 10$

Let us analyze the graphs of the floor and ceiling functions.

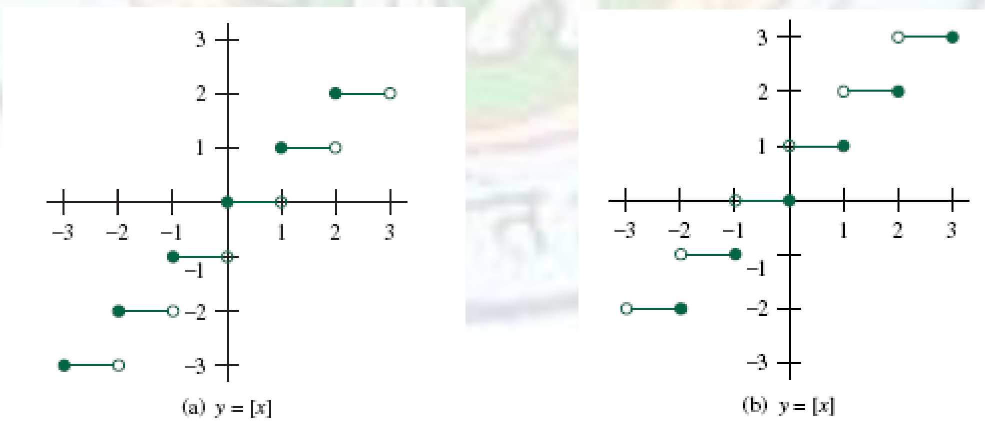


Figure 2.5.12: Graph of (a) Floor and (b) Ceiling Functions

Source: Chapter 2, Discrete Mathematics and Its Applications, 6th Ed., *by Kenneth Rosen*

In graph in Figure 2.5.12(a), of the floor function $\lfloor x \rfloor$, we see that the function has the same value throughout the interval $[n, n + 1)$, namely n, and then it jumps up to $n + 1$ when

$x = n + 1$.

In graph in Figure 2.5.12(b), of the ceiling function $\lceil x \rceil$, we see that the function has the same value throughout the interval $(n, n + 1]$, namely $n + 1$, and then it jumps up to $n + 2$ when x is a little larger than $n + 1$.

The floor and ceiling functions are useful in a wide variety of applications involving data storage and data transmission. Lets see an example.

Example 15: The data transmitted over a data network or stored on a computer disk are usually represented as a string of bytes. How many bytes are required to encode 100 bits of data?

Solution: Since each byte has 8 bits, we need to find the number of bytes required for 100 bits of information. We will use the ceiling function.

Now, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$. Thus 13 bytes are required.

Properties of Floor and Ceiling Functions

Useful Properties of the Floor and Ceiling Functions(n is an integer)
(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$ (1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \le n$ (1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$ (1d) $\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2) $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$ (3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ (4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Source: Chapter 2, Discrete Mathematics and Its Applications, 6th Ed.,

by Kenneth Rosen

A useful approach for considering statements about the floor function is to let

$x = n + \epsilon$, where $n = \lfloor x \rfloor$ is an integer, and ϵ, the fractional part of x, satisfies the inequality $0 \leq \epsilon < 1$.

Similarly, when considering statements about the ceiling function, it is useful to write

$x = n - \epsilon$, where $n = \lceil x \rceil$ is an integer and $0 \leq \epsilon < 1$.

2.6 Pigeonhole Principle

2.6.1 Introduction

Suppose a flock of pigeons fly into pigeonholes to roost. The pigeonhole principle states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 2.6.1).

Here n (= 10) pigeons in m (= 9) holes, so that some hole has more than one pigeon.

Figure 2.6.1 There are more pigeons than Pigeonholes

(Source: http://en.wikipedia.org/wiki/File:TooManyPigeons.jpg)

Theorem – The Pigeonhole Principle

If there are n pigeons (items) are put into m pigeonholes (boxes) with $n > m$ (i.e. $n \geq m+1$), then at least one pigeonhole (box) must contain more than one pigeon (item).

Proof: Suppose that none of the m pigeonholes contains more than one object. Then the total number of objects would be at most m. This is a contradiction, since there are at least $m+1$ objects.

The first formalization of the idea was made by the mathematician, Johann P. G. Lejeune Dirichlet in 1834 under the name Schubfachprinzip ("drawer principle" or "shelf principle"). This principle is also called the **Dirichlet's box principle** or **Dirichlet's drawer principle**.

Value addition: Biological Sketch
Johann Peter Gustav Lejeune Dirichlet

Johann Peter Gustav Lejeune Dirichlet (1805 - 1859) was born in a French family Cologne, Germany (at that time in the French Empire). At the age of 12, he had developed a passion for mathematics and spent his pocket-money on buying mathematics books. He studied at the University of Paris and held positions at University of Breslau and University of Berlin. Dirichlet is said to be the fist person to master Gauss's *Disquisitiones arithmeticae*, a work he treasured and kept constantly with him as others might do with the Bible.

Dirichlet's first paper he proved the *n=5* case of Fermat's Last Theorem, that there was no nontrivial solution in integers to $x^5 + y^5 = z^5$. He has made numerous contributions in the field of number theory and analysis. Analytic number theory may be said to begin with the work of Dirichlet, and in particular with his work on the existence of primes in a given arithmetic progression.

His work on units in algebraic number theory contains important work on ideals. He also proposed in modern definition of a function – "*If a variable y is so related to a variable x that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x.* "

Source: [2.3] http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Dirichlet.html

Examples 1:

i) In a group of 367 people, there must be at least two with the same birthday.

Explanation: There are 366 days in a leap year and thus there can be 366 birthdays in a year. Hence applying the pigeonhole principle, if n = 367 people have m = 366 birthdays (n > m), there must be at least one birthday that is shared by two people.

ii) A trivial application of pigeonhole principle is that - "there must be at least two left gloves or two right gloves in a group of three gloves".

Explanation: In a group of n = 3 gloves, m = 2 types of gloves are distinct (left glove & right glove). Hence according to the pigeonhole principle, since n > m, at least two gloves would be of the same kind.

iii) In a group of 27 English words, at least two will begin with the same letter.

Explanation: The English alphabet has m = 26 letters. In a group of 27 words, there will be n = 27 first letters. Applying the pigeonhole principle, since n > m, there must be at least two first letters that are the same. Hence at least two out of 27 words will begin with the same letter.

<u>Example 2</u>: Find the minimum number of elements that one needs to take from the set

A = {1, 2, 3, …, 9} to be sure that two of the numbers add upto 10.

Solution: We can construct five sets, {1, 9}, {2, 8}, {3, 7}, {4, 6}, {5}; each containing two numbers that add to 10. Thus we can consider these five sets as m = 5 boxes. Applying the pigeonhole principle, we would need to select minimum n = 6 (objects) to ensure that two of them add upto 10.

Value addition:

Generalized Pigeonhole Principle

The pigeonhole principle states that there must be at least two objects in one box when there are more objects than boxes. Even more can be said when the number of objects exceeds a multiple of the number of boxes.

If the number of objects is much larger than the number of boxes such as more than double the number of boxes, then the **Generalized (or Extended) Pigeonhole Principle** is applied. This principle states that:

If n discrete objects are to be allocated to m containers, then at least one container must hold at least $\lceil n/m \rceil$ objects.

Similarly, at least one container must hold at most $\lfloor n/m \rfloor$ objects.

Also, if m containers (pigeonhole) are occupied by km+1 or more objects (pigeons), where k is a positive integer, then atleast one container (pigeonhole) is occupied by k + 1 or more objects (pigeons).

<u>Example 3</u>: Consider a set of 21 decimal digits. Here there are n = 21 decimal digits and m = 10 distinct digits. Now, $\lceil n/m \rceil = \lceil 21/10 \rceil = 3$. Thus there must be more than 2 digits that are the same.

<u>Example 4</u>: Find the minimum number of students in a class to be sure that four of them have birthdays in the same month.

Solution: There are m = 12 months in a year. If three of them have birthdays in the same month, k + 1 = 4, or k = 3. Thus there should be km+1 = 37 students in the class so that three have birthdays in the same month.

Source: Books – Rosen; Lipschutz & Lipson, definition of Generalized Pigeonhole Principle

2.6.2 Examples Pigeonhole Principle

Following are some common examples of the Pigeonhole Principle:

Softball Team

Let us assume that 5 people want to play softball (n = 5 items), but there are only 4 softball teams (m = 4 holes) to choose from. Further, each of the five players refuses to play on a team with any of the other four players.

It is impossible to divide five people among four teams without putting two of the people on the same team. Since players refuse to play on the same team, we can use the pigeonhole principle to conclude that at most four of the five players will be able to play.

Sock Picking

Assume a box contains 10 black socks and 12 blue socks. We want to find out the maximum number of socks needed to be drawn from the box before a pair of the same colour can be made.

Let us use the pigeonhole principle. We want to have at least one pair of the same colour (m = 2 holes, one per colour). By using one pigeonhole per colour, we need only three socks (n = 3 items). Here, if the first and second sock drawn are not of the same colour, the very next sock drawn would complete one same-colour pair.

Hand Shaking

Suppose there are n number of people who can shake hands with one another (where $n > 1$). We can use the pigeonhole principle to show that there is always a pair of people who will shake hands with the same number of people.

Let m (the number of holes) correspond to the number of hands shaken. Each person can shake hands with 0 to $n - 1$ other people, thus there are $m = n - 1$ possible holes. This leaves n people to be placed in at most $n - 1$ non-empty holes, ensuring duplication.

2.6.3 Application and Uses of Pigeonhole Principle

The pigeonhole principle can be applied to a number of problems in computer science and mathematical analysis.

Collisions are expected in a hash table because the number of possible keys exceeds the number of indices in the array. The pigeonhole principle proves that no hashing algorithm, can avoid these collisions.

This principle also proves that any general-purpose lossless compression algorithm makes at least one input file smaller and another input file larger. (Else, two files would be compressed to the same smaller file and restoring them would be ambiguous.)

Summary

Relations

- Objects are related if they share a common property and are not related otherwise.
- The relations consider the existence or non existence of a certain connection between pairs of objects taken in a definite order.
- A **relation** can be defined in terms of these "ordered pairs" *(a, b)* of elements. In particular,

  ***(a, b) = (c, d)* if and only if *a = c* and *b = d*.**

- The set of all ordered pairs *(a, b)* where $a \in A$ and $b \in B$ is called the **product** or **Cartesian product**, of *A* and *B*. The product can be written as $A \times B$. By definition,

  $A \times B = \{(a, b): a \in A \text{ and } b \in B\}$

- $A \times B \neq B \times A$
- For finite sets *A* and *B*, $n(A \times B) = n(A) \cdot n(B)$
- A **binary relation** or a **relation** from *A* to *B* is a subset of $A \times B$.
- A binary relation essentially means that some of the elements in set *A* are related to some of the elements in set *B*.
- Empty set, $\varnothing \subseteq A \times B$, therefore $R = \varnothing$ is a relation called **empty relation**.
- If $R = A \times B$, then R is called the **universal relation**.
- The **domain** of a relation *R* is the set of all first elements of the ordered pairs which belong to *R*.
- The **range** of *R* is the set of all second elements of the ordered pairs which belong to *R*.
- The **inverse** of *R*, denoted by R^{-1}, is the relation from *B* to *A*. R^{-1} consists of those ordered pairs which, when reversed, belong to *R*; i.e.,

  $R^{-1} = \{(b, a): (a, b) \in R\}$

- The **complement** of *R* from set A to set B, denoted by $\overline{R}$, is the set of ordered pairs such that, $\overline{R} = \{(a, b) \in A \times B: (a, b) \notin R\}$.
- Binary relations can be represented in a number of ways:
  - As a list of ordered pairs
  - In Tabular form
  - As a Relation Matrix
  - As an Arrow diagram
  - In Graphical form as a Directed graph with vertices and edges
- $R \subseteq A \times A$ is called **a relation on a set** A.
- A **ternary relation** among three sets *A, B* and *C* is defined as a subset of the cartesian product $(A \times B) \times C$.
- **Quaternary relations** represent the relationship between quadruples of objects.
- An **n-ary relation** among the sets $A_1, A_2, A_3, \ldots, A_n$ is a set of ordered n-tuples in which the first component is an element of A_1 , the second component is an element of A_2, …., and the n^{th} component is an element of A_n.

Properties of Binary Relations

- The different types of binary relation over a set *A*: reflexive, irreflexive, symmetric, antisymmetric, asymmetric and transitive
- *R* is ***reflexive*** means: $aRa \ \forall a \in A$.
- *R* is ***irreflexive*** means: $(a, a) \notin R, \forall a \in A$. Thus *R* is non-reflexive if there exists an $a \in A$ such that $(a, a) \notin R$.
- *R* is ***symmetric*** means: *aRb* implies *bRa*, $\forall a, b \in A$.

  i.e. *R* is symmetric if $(a, b) \in R$ implies $(b, a) \in R, \forall a, b \in A$.

- R is *not symmetric* if there exists a, b $\in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.
- R is ***antisymmetric*** means: aRb and bRa implies $a = b, \forall a, b \in A$.
  i.e., R is *antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b, \forall a, b \in A$.
- R is *not antisymmetric* if there exists a, b $\in A$ such that (a, b) and $(b, a) \in R$, but $a \neq b$.
- R is ***asymmetric*** means: If $(a, b) \in R$ then $(b,a) \notin R, \forall a, b \in A$.
- R is *not asymmetric* if for some $a, b \in A$, both $(a, b) \in R$ and $(b, a) \in R$.
- R is ***transitive*** means: $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R, \forall a, b, c \in A$.
- R is *not transitive* if there exists $a, b, c \in A$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.
- If R and S are binary relations, then the ***composition*** of R and S is $R \circ S = \{(a, c) \mid aRb \text{ and } bSc \text{ for some } b \in R\}$.
- Let R be a relation on set A. The power R^n, $n = 1, 2, 3, \ldots$ can be defined recursively by: $R^1 = R$. Also $R^{n+1} = R^n \circ R$.
- The property of transitivity can also be expressed in terms of composition of relations.
- Let R be a relation on a nonempty set A. R is said to be an **equivalence relation** if R is reflexive, symmetric and transitive

Closure of Relations

- The closure of a relation is the smallest extension of the relation that has a certain specific property such as the reflexivity, symmetry or transitivity.
- For a relation R on the set A with n elements. The transitive closure, $t(R)$, is:
  $$t(R) = R \cup R^2 \cup \ldots \cup R^n$$
- **Theorem:** Let ***E*** denote the equality relation, and R^{-1} the inverse relation of binary relation ***R***, all on a set ***A***, where $R^{-1} = \{(a, b) \mid (b, a) \in R\}$. Then,
  1. $r(R) = R \cup E$
  2. $s(R) = R \cup R^{-1}$
  3. $t(R) = \cup^{\infty}_{i=1} R^i = \cup^{\infty}_{i=n} R^i$, if $|A| = n$
  4. R is reflexive if and only if $r(R) = R$.
  5. R is symmetric if and only if $s(R) = R$.
  6. R is transitive if and only if $t(R) = R$.

Partial Ordering Relations

- In a partial ordering relation two objects are related if one of them is smaller (or larger) than or inferior (superior) to the other object according to some property or criteria.
- A binary relation R on a set S is called a **partial ordering relation** or a partial order if it is reflexive, transitive and antisymmetric.
- A set S along with the partial order R is called a **partially ordered set** or **poset** and is denoted by (S, R).
- These relations are called *partial orders* to reflect the fact that not every pair of elements of a poset need be related: for some pairs, it may be that neither element precedes the other in the poset.
- A graph representing a poset but with only immediate predecessor edges, and the edges are oriented up from x to y when $x < y$ is called a **Hasse Diagram** or **Poset Diagram**.

Relations and Function

- A binary relation can be represented as a diagraph or directed graph. When the binary relation is a partially ordered relation, its directed graph can be simplified.
- Following are the steps to draw a Hasse Diagram (Poset Diagram) of a partial ordering relation:
  - **Step 1:** Draw the diagraph corresponding to the given relation or relation matrix.
  - **Step 2:** Since the relation is reflexive – we can remove all arrows pointing back to a node.
  - **Step 3:** The relation is transitive – we can remove arrows between nodes that are connected by sequence of arrows.
  - **Step 4:** In most of the resultant graphical representations, all arrows are pointing in one direction (upward, downward, left-to-right, right-to-left). We can remove all the arrowheads.
  - Such a graphical representation of a partial ordering relation in which all arrowheads are understood to be pointing in the upward is known as a Hasse Diagram or Poset Diagram of the given relation.
- For *a, b* distinct elements of a partially ordered set *P*, if $a \le b$ or $b \le a$, then *a* and *b* are ***comparable***. Otherwise they are ***incomparable***.
- If every two elements of a poset are comparable, the poset is called a **totally ordered set** or **linearly ordered set** or **chain** (e.g. the natural numbers under order).
- A poset in which every two elements are incomparable is called an **antichain**.

Functions

- We assign to each element of a set a particular element of the second set (which may be the same as the first).
- Functions are very important in the field of discrete mathematics. They are used in the definition of discrete structures like sequences and strings, to represent how long it takes the computer to solve many problems of a particular size, etc.
- A function from set A to set B is also a subset of A x B with some restrictions on the elements in the subset.
- A **function** from set *A* to set *B*, denoted as $f: A \to B$, is a relation from A to B, such that, each $a \in A$ belongs to a unique ordered pair (a, b) in $f \subseteq A \times B$.
- We write $f(x) = y$ to mean f associates $x \in A$ with $y \in B$. Say, "f of *x* is *y*" or "f maps *x* to *y*." Here *y* would be a unique element in *B*.
- Every function is a relation but every relation is not a function. Therefore,

$$f = \{(a, b): a \in A, b \in B \text{ and } b = f(a)\}$$

- All functions are relations, since they pair information, not all relations are functions.
- Functions can be defined in a number of ways - as explicit assignment, as a formula or as a computer program.
- If f function is a function from *A* to *B*, *A* is called the **domain** of f and *B* is called the **codomain** of f.
- If $f(a) = b$, we say that *b* is the image of *a* and *a* is a pre-image of *b*.
- If $f: A \to B$ and $S \subset A$ the **image** of *S*, $f(S) = \{f(s) \mid s \in S\}$.
- If $D \subset B$, the **pre-image** or **inverse image** of *D* is the set $f^{-1}(D) = \{x \mid f(x) \in D\}$.
- The **range** of f is the set of all images of elements of *A*.
- Let f_1 and f_2 be functions from *A* to B (the set of real numbers). Then $f_1 \pm f_2$ and $f_1 f_2$ are also functions from *A* to B.

$$(f_1 \pm f_2)x = f_1(x) \pm f_2(x) \quad \text{and} \quad (f_1 f_2)x = f_1(x) f_2(x)$$

- Let $g: A \to B$ and $f: B \to C$. The **composition** of the functions f and g, denoted by

$\boldsymbol{f} \circ \boldsymbol{g}: A \rightarrow C$, is defined by $(f \circ g)(a) = f(g(a))$.

- Composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.
- The commutative law does not hold for the composition of functions,
  i.e., $f \circ g \neq g \circ f$.
- There are various types of functions: Constant Function, Identity Function, Polynomial Function, Step Function, Greatest Integer Function, Strictly increasing and Strictly decreasing Functions, Increasing Function, Decreasing Function and Floor & Ceiling Functions.
- The **identity function** on set A is the function $i_A: A \rightarrow A$, where $i_A(x) = x$ for all $x \in A$.
- A function is f is called **strictly increasing** if $\forall x \, \forall y \, ((x < y) \Rightarrow (f(x) < f(y))$, where x and y are in the domain of f.
- A function is f is called **strictly decreasing** if $\forall x \, \forall y \, ((x > y) \Rightarrow (f(x) < f(y))$, where x and y are in the domain of f.
- The **floor function**, $\lfloor x \rfloor$, assigns to the real number x the largest integer that is less than or equal to x.
- The **ceiling function**, $\lceil x \rceil$, assigns to the real number x the smallest integer that is greater than or equal to x.
- Let f: A $\rightarrow$ B. There are three properties that f might possess: Injective (One-to-one), Surjective (Onto) and Bijective (one-to-one and onto).
- A function f is said to be **Injective** or **One-to-one** (written as 1-1), iff $\forall x \, \forall y$ in domain of f, $f(x) = f(y)$.
- A strictly increasing or a strictly decreasing function will always be one-to-one.
- In some functions, called onto functions, the codomain and range are the same.
- A function f from set X to set Y is said to be **surjective** or **onto** if and only if for every $y \in Y$ there is an element $x \in X$ with $f(x) = y$.
- A function, $f: X \rightarrow Y$, is **onto** or **surjective** iff $\forall y \in Y, \exists x \in X \mid f(x) = y$.
- A function f is **bijective** or a **one-to-one correspondence** if it is both one-to-one and onto, i.e. it is both injective and surjective.
- The identity function, i_A, is a bijection since it is one-to-one and onto.
- Let $f: A \rightarrow B$. The **inverse function**, $f^{-1}(b) = a$ when $f(a) = b$.
- If a function f is not a one-to-one or onto, we cannot define an inverse function of f.
- A one-to-one correspondence is called **invertible** because its inverse can be defined.
- A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.
- When the composition of a function and its inverse is formed, in either order, an identity function is obtained, i.e.,
  $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$,
  where I_A and I_B are the identity functions on the sets A and B, respectively.
- Also $\boldsymbol{f \circ I_A = f; \quad I_B \circ f = f}$ ***and*** $\boldsymbol{(f^{-1})^{-1} = f}$
- Let function $f: A \rightarrow B$. The **graph** of the function f is the set of ordered pairs
  $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.
- Function $f: A \rightarrow B$ is a relation from A to B (i.e. a subset of A X B) such that each $a \in A$ belong to a *unique* ordered pair (a, b) in f.

Pigeonhole Principle

- The **Pigeonhole Principle** states that:
  - "If there are n pigeons (items) are put into m pigeonholes (boxes) with $n > m$ (i.e. $n \geq m+1$), then at least one pigeonhole (box) must contain more than one pigeon (item)."

- This principle is also called the **Dirichlet's box principle** or **Dirichlet's drawer principle**.
- If the number of objects is much larger than the number of boxes such as more than double the number of boxes, then the **Generalized (or Extended) Pigeonhole Principle** is applied. This principle states that:
    - "If *n* discrete objects are to be allocated to *m* containers, then at least one container must hold at least $\lceil n/m \rceil$ objects, where $\lceil x \rceil$ is the ceiling function."
- Pigeonhole Principle can be applied to solve many real-life counting problems like:
    - Softball team
    - Sock picking
    - Hand shaking
- This Principle can be applied to solve many problems in computer science and mathematical analysis:
    - Collisions in hash tables
    - Lossless compression algorithm

Exercises

Relations

1. Given: *A* = {a, b, c} and *B* = {β, λ}. Find (a) A x B; (b) B x A; (a) B x B.
2. Given: *A* = {1, 2, 3}, *B* = {a, b, c} and *C* = {x, y}. Find A x B x C.
3. Prove or disapprove the following:
    a) (A x B) ∩ (A x C) = A x (B ∩ C)
    b) (A U B) x (C U D) = (A x C) U (B x D)
    c) (A - B) x (C - D) = (A x C) - (B x D)
    d) (A ⊕ B) x (C ⊕ D) = (A ⊕ C) U (B ⊕ D)
4. Given that A ⊆ C and B ⊆ D, show that A x B ⊆ C x D.
5. Given that A x B = ø, what can we say about sets A and B?
6. Is it possible that A ⊆ A x A for some set A?
7. Let Let ***S*** = *{a, b, c, d}* be a set of students and ***C*** = *{CS101, CS102, CS103, ENG1, MATH1}* be a set of courses. Assume R_1 be a relation from S to C denoting the courses the students are taking. Assume R_2 be a relation from S to C describing the courses the students are having difficulty in.

    R₁ *= {(a, CS101), (a, CS102), (a, CS103), (a, ENG1), (b, CS101), (b, CS102), (b, CS103), (b, ENG1), (c, CS101), (c, CS102), (c, CS103), (c, MATH1), (d, CS101), (d, CS102), (d, CS103), (d, MATH1)}*

    R₂ *= {(a, CS101), (a, CS102), (b, CS102), (d, CS101),(d, MATH1)}*

    What is the interpretation of the relations R_1 U R_2, R_1 ∩ R_2, R_1 - R_2, R_1 ⊕ R_2?
8. Specify the domain and ranges of R_1 and R_2 given in question 7 above.

9. Show the Tabular representation of R_1 and R_2 given in question 7 above.
10. Construct the directed graph of R_1 and R_2 given in question 7 above.
11. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$ where $(a, b) \in R$ iff
    a) $a = b$.
    b) $a + b = 4$.
    c) $\gcd(a, b) = 1$.
12. Given $A = \{1, 2, 3, 4, 5, 6\}$. Let R be the relation on A defined by "x divides y", written as $x|y$.
    a) Write R as a set of ordered pairs.
    b) Draw its directed graph.
    c) Give the tabular representation of R.
    d) Represent R as a relation matrix.
    e) Find the inverse relation R^{-1} of R. Can R^{-1} be described in words?
    f) Find the complement relation $\overline{R}$ of R.

Properties of Binary Relations

1. Let A be a set of people. Let R be a relation on A such that $(a, b) \in R$ iff a is the sister of b (disregard half- sisters and cousins). Is R reflexive, symmetric, antisymmetric, transitive?
2. Let R and S be a binary relations on a set of all positive integers such that
    a) $R = \{(a, b) \mid a - b \text{ is an odd positive integer}\}$
    b) $S = \{(a, b) \mid a = b^2\}$

    Are R, S reflexive? Symmetric? Antisymmetric? Transitive?
3. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$ and $C = \{x, y, z\}$. Consider the relation R from A to B and relation S from B to C.

    $R = \{(1, b), (2, a), (3, c)\}$ and $S = \{(a, y), (b, x), (c, y), (c, z)\}$

    a) Draw the arrow diagram corresponding to the relations R and S.
    b) Find the composition relation $R \circ S$.
    c) Find the matrices M_R, M_S, and $M_{R \circ S}$.
4. Given $A = \{1, 2, 3, 4\}$. Consider the following relation on A:

    $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)\}$

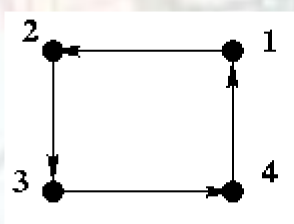
    g) Draw its directed graph.
    h) Is R reflexive? Symmetric? Antisymmetric? Transitive?
    i) Find $R^2 = R \circ R$ and R^3.

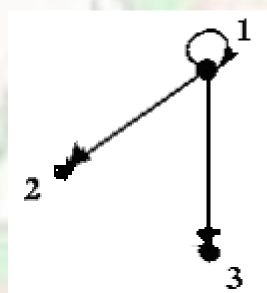
5. Determine whether the relation R on the set of all people is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive, where $(a, b) \in R$ iff
    a) *a* is taller than *b*.
    b) *a* and *b* were born on the same day.
    c) *a* has the same last name as *b*.
    d) *a* and *b* have common grandparents.
6. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answer.
7. Give an example of a relation on a set that is:
    a) Symmetric and antisymmetric.
    b) Neither symmetric nor antisymmetric.

Closure of Relations

1. Show that the transitive cover of a symmetric relation is symmetric.
2. Find the reflexive, symmetric and transitive closures of relation *R* on the set
3. $A = \{a, b, c\}$, where $R = \{(a, a), (a, b), (b, c), (c, c)\}$.
4. Draw the directed graphs of the reflexive, symmetric and transitive closures of the relations with the directed graph given below:

    i. b)





    ii.

5. Compute the reflexive, symmetric and transitive closures of relation,
    a. $R = \{(0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\}$ on the set $\{0, 1, 2, 3\}$.
6. What are the reflexive, symmetric and transitive closures of relation, *R*, on the set of integers? Here $R = \{(a, b) \mid a \neq b\}$.

Partial Ordering Relations

1. Which of these are posets? a) $(\mathbf{Z}, =)$ b) $(\mathbf{Z}, \neq)$ c) $(\mathbf{Z}, \geq)$
2. Determine whether the relations depicted by the relation matrix are partial orders.
    a) b)

Relations and Function

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

3. Draw the Hasse Diagram for the "greater than equal to" relation on set {0, 1, 2, 3, 4, 5}
4. Draw the Hasse Diagram representing the partial ordering {(a, b) | a divides b} on set {1, 2, 3, 4, 6, 8, 12}.
5. Draw the Hasse Diagram representing the partial ordering {(A, B) | A $\subseteq$ B} on the power ser P(S) where S = {a, b, c}.

Functions

1. Let X = {1, 2, 3, 4}. Determine whether or not each relation below is a function from X to X.
   a) f = {(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)}
   b) g = {(2, 1), (3, 4), (1, 4), (2, 1), (4, 4)}
2. Why is f not a function from **R** to **R** if:
   a) $f(x) = 1/x$ b) $f(x) = \sqrt{x}$
3. Determine whether f is a function from **Z** to **R** if:
   a) $f(x) = 1/(n^2 - 4)$ b) $f(x) = \sqrt{(n^2 + 1)}$
4. Find the domain and range of these functions:
   a) The function that assigns to each pair of positive integers the first integer of the pair
   b) The function that assigns to each pair of positive integers the maximum of these two numbers
5. Give an example of a function from **N** to **N** that is:
   a) one-to-one but not onto
   b) onto but not one-to-one
   c) both one-to-one and onto
   d) neither one-to-one nor onto
6. Determine whether each of these functions from {a, b, c, d} to itself is one-to-one and/ or onto:
   a) f(a) = b, f(b) = a, f(c) = c, f(d) = d
   b) f(a) = d, f(b) = b, f(c) = d, f(d) = c
7. For $f(x) = x^2 + 3$ and g(x) = x+4 functions from **R** to **R**, determine the following:
   a) $f \circ g$ b) $g \circ f$ c) $f + g$ d) fg
8. Find the inverse of the following functions:
   a) $x^3 + 7$ b) ln(2 + 4x)   c) (x + 1)/ x

9. Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.
10. Find these values:

    a) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$ b) $\lceil -3.98 \rceil$ c) $\lfloor 3.38 \rfloor$ d) $\lfloor 3/2 \cdot \lfloor 5/4 \rfloor \rfloor$

Pigeonhole Principle

1. Assume there are n distinct pairs of shoes in a shoe rack. Show that if you choose $n+1$ single shoe at random, you are certain to have a pair.
2. Assuming that no classes are held on weekends, show that in a set of six classes there must be two that meet on the same day.
3. Show that if seven numbers from 1 to 12 are chosen, then two of them will add up to 13.
4. A student must take five classes in three subjects. Many classes are offered in each discipline, but the student cannot take more than two classes of a subject. Show that the student will take at least two classes in one subject.
5. A drawer contains ten black socks and ten brown socks, all unmatched. A man takes socks out at random in the dark.
   a) How many socks must he take out to be sure that he has at least two socks of the same colour?
   b) How many socks must he take out to be sure that he has at least two brown socks?
6. Assume that there are three boys and five girls at a party. Show that if these children are seated in a row, at least two girls will sit next to each other.
7. A box contains ten red balls and ten green balls. A woman is blindfolded and randomly picks balls.
   a) How many balls must she pick to be sure of having at least three balls of the same colour?
   b) How many balls must she pick to be sure of having at least three red balls?
8. Find the minimum number n of integers to be selected from $A = \{1, 2,3, \ldots, 9\}$ so that:
   a) The sum of two of the n integers is even.
   b) The difference of two of the n integers is five.
9. Ten people come forward to volunteer for a three person committee. Every possible committee of three (that can be formed from these ten people), is written on a separate slip of paper. These slips are put in ten hats. Show that at least one hat contains 12 or more slips of papers.

10. Show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum

Glossary

Antisymmetric Relation: A relation *R* on a set *A* is **antisymmetric** if whenever *aRb* and *bRa,* then *a*= *b*, that is, if whenever *(a, b)* and *(b, a) are* in R then *a* = *b*.

Asymmetric Relation: A relation *R* on a set *A* is **asymmetric** if whenever *aRb* then $b \not R a$, that is, if *(a, b)* is in *R* then *(b, a)* is not in *R*.

Bijective Function or **one-to-one correspondence:** A function *ƒ* is **bijective** if it is both one-to-one and onto, i.e. it is both injective and surjective.

Binary relation: A **binary relation** is defined as a set of ordered pairs of elements, *(a, b)*, such that, *(a, b) = (c, d)* if and only if *a = c* and *b = d*.

Cartesian product or product: The **Cartesian product** of set *A* and *B, A x B,* is a set of all ordered pairs of elements, *(a, b)* such that *a* is in set *A* and *b* is in set *B.*

Ceiling function, $\lceil x \rceil$: The **ceiling function** gives the smallest integer larger than or equal to *x*.

Closure: The **closure** of a relation is the smallest extension of the relation that has a certain specific property such as the reflexivity, symmetry or transitivity.

Codomain: If a function *ƒ* is a function from *A* to *B*, *B* is called the **codomain** of *ƒ*.

Complement of a Relation, $\overline{R}$: The **complement** of relation *R* from a set *A* to a set *B* is the set of ordered pairs such that, $\overline{R} = \{(a, b) \in A \times B: (a, b) \notin R\}$.

Composition of functions, $f \circ g$: The **composition of the functions** f (where *ƒ*: *B* → *C)* and *g (*where *g*: *A* → *B)*, is defined by $(f \circ g)(a) = f(g(a))$.

Composition of relations, $R \circ S$: The **composition of relations** *R* and *S* (both binary on a set *A*) is the set of ordered pairs (*a, c*) such that *aRc* and *bSc* for some *b*.

Directed graph or Diagraph: A **directed graph** has a set *V* of vertices (or nodes) and a set *E* of arcs or directed edges. The set *E* is of ordered pairs *(u, v)* of vertices.

Domain of a function: If a function *ƒ* is a function from *A* to *B*, *A* is called the **domain of the function** *ƒ*.

Domain of a relation: The **domain of a relation** *R* is the set of all first elements of the ordered pairs which belong to *R*.

Empty relation: A relation R is called an **empty relation** if it is equal to an empty set, that is, R = ∅.

Equivalence relation: The relation *R* on a set *A* is said to be an **equivalence relation** if *R* is reflexive, symmetric and transitive.

Relations and Function

Floor function, $\lfloor x \rfloor$: The **floor function** gives the largest integer smaller than or equal to *x*.

Function: A **function** from set *A* to set *B*, denoted as f**: A→B**, is a relation from A to B, such that, each a ∈ A belongs to a unique ordered pair (a, b) in $f \subseteq A \times B$.

Graph: The **graph** of the function $f: A \to B$ is the set of ordered pairs $\{(a, b) \mid a \in A$ and $f(a) = b\}$.

Hasse Diagram (Poset Diagram): A graph representing a poset but with only immediate predecessor edges, and the edges are oriented up from x to y when x < y is called a **Hasse Diagram (Poset Diagram)**.

Identity function, i_A: A function that assigns each element to itself is called the **identity function**.

Image of set *S*, $f(S)$: If $f: A \to B$ and S is a subset of A, the **image of *S***, $f(S)$, is the subset of B that contains the images of elements of *S*. If $f(a) = b$, we say that *b* is the image of *a*.

Image: If $f: A \to B$ and $f(a) = b$, we say that *b* is the **image** of *a*.

Injective or One-to-one Function: A function f is said to be Injective or **One-to-one** if and only if $f(x) = f(y)$ implies x = y for all x, y in the domain of f.

Inverse function, f^{-1}: The **inverse function** of the bijective function f (where $f: A \to B$), is $f^{-1}(b) = a$ when $f(a) = b$.

Inverse of a relation, R^{-1}: The **inverse of relation** *R* (defined from set *A* to set *B*), is the relation from *B* to *A* that consists of ordered pairs which, when reversed, belong to *R.* That is, $R^{-1} = \{(b, a): (a, b) \in R\}$.

Invertible function: A bijection is called **invertible** because its inverse can be defined.

Irreflexive Relation: A relation *R* on a set *A* is **irreflexive** if *(a, a)* is not in R for all *a* in *A*.

Partial ordering relation: A binary relation R on a set S is called a partial ordering relation or a partial order if it is reflexive, transitive and antisymmetric.

Partially ordered set or poset: A set S along with the partial order R is called a partially ordered set or **poset** and is denoted by (S, R).

Path of a directed graph (diagraph): The **path** of a diagraph is a sequence of edges (e_1, e_2, …, e_i, e_j, …, e_n) such that the terminal node (vertex) of edge e_i is the initial node of edge e_j (where $1 \le i < j \le n$).

Pre-Image or inverse image: If $f: A \to B$ and $f(a) = b$, we say that *a* is the **pre-image** of *b***.** If $D \subset B$, the pre-image or inverse image of *D* is the set $f^{-1}(D) = \{x \mid f(x) \in D\}$.

Range of a function: If a function f is a function from *A* to *B*, the **range of function** f is the set of all images of elements of *A*.

Range of a relation: The **range of relation** *R* is the set of all second elements of the ordered pairs which belong to *R*.

Relations and Function

Reflexive Relation: A relation *R* is said to be **reflexive** if *(a, a)* is in R for all *a* in *A*.
Set: A set is a collection of distinct objects. The objects in the set are called *elements* or *members* of the set.

Strictly decreasing: A function f whose domain and codomain are subsets of **R** is called **strictly decreasing** if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Strictly increasing: A function f whose domain and codomain are subsets of **R** is called **strictly increasing** if $f(x) < f(y)$ whenever $x < y$ and x and y are in the domain of f.

Surjective or Onto Function: A function f from set *X* to set *Y* is said to be **surjective** or **onto** if and only if for every $y \in Y$ there is an element $x \in X$ with $f(x) = y$.

Symmetric Relation: A relation *R* on a set *A* is **symmetric** if whenever *aRb* implies *bRa,* that is, if whenever *(a, b)* is in R then *(b, a)* is also in R.

Transitive Relation: A relation *R* on a set *A* is **transitive** if whenever *aRb* and *bRc* then *aRc,* that is, if whenever *(a, b), (b, c)* are in *R* then *(a, c)* is also in *R*.

Universal relation: A relation R, from set A to set B, is called a **universal relation** if it is equal to A x B. That is, R = A x B.

Universe of Discourse: The **universe of discourse** is a collection of objects being discussed in an example. It indicates the relevant set of entities that are being dealt with.

References

1. Works Cited

2. Suggested Readings

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Semyour Lipschutz & Marc Lipson, Schaum's Outlines Discrete Mathematics, Second Edition, Tata McGraw Hill

3. Web Links

2.1 http://en.wikipedia.orghjg
2.2 http://brooks-pdx.pbworks.com/f
2.3 http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Dirichlet.html
2.4 http://www.purplemath.com
2.5 http://www.cs.odu.edu