**Paper: Discrete Structure**
**Unit: 1**
**Lesson: Set 1**
**Lesson Developer: Pooja Vashisth**
**College/ Department: SPMC, Computer Science, University of Delhi**

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## 1.1 Discrete Mathematics

We are aware of the following number systems: Natural Numbers, Integers, Rational Numbers, Real Numbers and Complex Numbers. All counting numbers 1,2,3,4,… constitute the natural numbers. Number 1 is the smallest natural number but we cannot find the largest one. Including 0 with the Natural Numbers give us the Whole Numbers 0,1,2,3,4,… , where again we have the smallest whole number as 0 but no largest number. Integers include all natural numbers, 0, and the negative of natural numbers,…,-3,-2,-1. Here, we cannot find both the smallest and the largest integer. Then we have the Rational Numbers which are of the type p/q where p and q are integers and q is non-zero. There are certain numbers that satisfy the above property of rational numbers but still cannot be expressed in the form p/q. Such numbers are known as irrational numbers. The rational and irrational numbers together form the real numbers. Lastly, we have the complex numbers which are of the form: $a + ib$ where $i^2 = -1$. So, now we have the property that between any two real numbers there are infinitely many real numbers, but if we consider between any two integers then we have only a finite number of integers between them, and if they are consecutive integers, then no integer lies between them. This means that the real number system is continuous whereas the integers are discontinuous or discrete in nature. This property of integers leads to a mathematical study of such discrete systems in Mathematics. These systems have evolved into a branch of mathematics termed as **Discrete Mathematics.** Discrete Mathematics includes the study of discrete objects like graphs.

**What is Discrete Mathematics?**

**Discrete Mathematics** is the study of mathematical structures that are fundamentally discrete rather than continuous. For example, Real numbers have the property of varying "smoothly", the objects studied in discrete mathematics – such as integers, graphs, and statements in logic – do not vary smoothly in this way, but have distinct, separated values. Discrete mathematics therefore excludes topics in "continuous mathematics" such as calculus and analysis. Discrete objects can often be enumerated by integers. More formally, discrete mathematics has been characterized as the branch of mathematics dealing with countable sets (sets that have the same cardinality as subsets of the integers, including rational numbers but not real numbers). However, there is no exact, universally agreed, definition of the term "discrete mathematics."

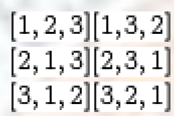
**Why Discrete Mathematics?**

Discrete mathematics helps in discussing languages used in mathematical reasoning, basic concepts, and their properties and relationships among them. Discrete mathematics is also concerned with techniques to solve counting problems. The counting problems include: How many routes exist from point A to point B in a computer network ? How much execution time is required to sort a list of integers in increasing order ? What is the probability of winning a lottery ? What is the shortest path from point A to point B in a computer network ? etc.

Research in discrete mathematics increased in the latter half of the twentieth century partly due to the development of digital computers which operate in discrete steps and store data in discrete bits. Concepts and notations from discrete mathematics are useful in studying and describing objects and problems in branches of computer science, such as computer algorithms, programming languages, cryptography, automated theorem proving, and software development. Conversely, computer implementations are significant in applying ideas from discrete mathematics to real-world problems, such as in operations research.

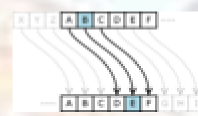
Although the main objects of study in discrete mathematics are discrete objects, analytic methods from continuous mathematics are often employed as well.

| **Value addition: Did you Know** |
|---|
| **Discrete Mathematics and Applications** |
| Discrete mathematics involves techniques that apply to objects that can only take on specific, separated values. |




Permutation of numbers (combinatorics)

Theory of computation



Cryptography


Graph theory

**Source: http://psychology.wikia.com/wiki/Mathematics**

## 1.2 Introduction to Set Theory

A major theme of the text is to study discrete objects and relationships among them. The term discrete object is a rather general one. Besides, number systems of certain types, it includes a large variety of items such as people, books, computers, transistors, and so on. In our daily lives as well as in our technical work we frequently deal with these items, while making statements such as, "The people in this room are Mathematics majors in their first year." This statement hints at the possibility of an abstraction as it refers to a collection of people who possess the two attributes of being a mathematics major and that they are in first year. This example illustrates the many occasions on which we deal with several classes of objects and wish to refer to those objects that belong to all classes. Here, in the above statement we are referring to a collection of distinct and well defined objects that possess some attributes. Such a collection of distinct, well defined objects is known as a **set. Set theory** is the branch of mathematics and computer science that studies sets, which are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics.

### 1.2.1 History

The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

The language of set theory is used in the definitions of nearly all mathematical objects, such as functions, and concepts of set theory are integrated throughout the mathematics curriculum. Elementary facts about sets and set membership can be introduced in primary school, along with Venn and Euler diagrams, to study collections of commonplace physical objects. Elementary operations such as set union and intersection can be studied in this context. More advanced concepts such as cardinality are a standard part of the undergraduate mathematics curriculum.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

Since the 5th century BC, beginning with Zeno in the West and early Indian mathematicians in the East, there had been mathematicians struggling with the concept of infinity. The Indian mathematicians in the 4th century BC proposed the concept of there being different types of infinities: infinite in length (one dimension), infinite in area (two dimensions), infinite in volume (three dimensions), and infinite perpetually (infinite dimensions). In the 10th century AD, Udayana, founder of the Navya-Nyaya school of Indian logic, developed theories on "restrictive conditions for universals" and "infinite regress" that anticipated aspects of modern set theory. Especially notable is the work of Bernard Bolzano in the first half of the 19th century. The modern understanding of infinity began in 1867-71, with Cantor's work on number theory. An 1872 meeting between Cantor and Richard Dedekind influenced Cantor's thinking and culminated in Cantor's 1874 paper.

Cantor's work initially polarized the mathematicians of his day. Cantorian set theory eventually became widespread, due to the utility of Cantorian concepts, such as one-to-one correspondence among sets, his proof that there are more real numbers than integers, and the "infinity of infinities" ("Cantor's paradise") the power set operation gives rise to.

The next wave of excitement in set theory came around 1900, when it was discovered that Cantorian set theory gave rise to several contradictions, called antinomies or paradoxes. Bertrand Russell and Ernst Zermelo independently found the simplest and best known paradox, now called Russell's paradox and involving "the set of all sets that are not members of themselves." This leads to a contradiction, since it must be a member of itself and not a member of itself. In 1899 Cantor had himself posed the question: "what is the cardinal number of the set of all sets?" and obtained a related paradox.

The momentum of set theory was such that debate on the paradoxes did not lead to its abandonment. The work of Zermelo in 1908 and Abraham Fraenkel in 1922 resulted in the canonical axiomatic set theory ZFC, which is thought to be free of paradoxes. The work of analysts such as Henri Lebesgue demonstrated the great mathematical utility of set theory. Axiomatic set theory has become woven into the very fabric of mathematics as we know it today.

| **Value addition: Did you know** |
|---|
| **Father of Set theory** |

George Cantor was born in 1845 in the western merchant colony in Saint Petersburg, Russia, and brought up in the city until he was eleven. Georg, the eldest of six children, was an outstanding violinist, having inherited his parents' considerable musical and artistic talents. Cantor's father had been a member of the Saint Petersburg stock exchange; when he became ill, the family moved to Germany in 1856, first to Wiesbaden then to Frankfurt, seeking winters milder than those of Saint Petersburg. In 1860, Cantor graduated with distinction from the Realschule in Darmstadt; his exceptional skills in mathematics, trigonometry in particular, were noted. In 1862, Cantor entered the Federal Polytechnic Institute in Zürich, today the ETH Zurich. After receiving a substantial inheritance upon his father's death in 1863, Cantor shifted his studies to the University of Berlin, attending lectures by Leopold Kronecker, Karl Weierstrass and Ernst Kummer. He spent the summer of 1866 at the University of Göttingen, then and later a very important center for mathematical research. In 1867, Berlin granted him the PhD for a thesis on number theory, *De aequationibus secundi gradus indeterminatis*

**Source: http://en.wikipedia.org/wiki/Georg_Cantor**

## 1.2.2 Basic Concepts

Set is an unordered collection of similar type of objects. A set can be finite or infinite depending on the number of elements in the set. Capital letters like X,Y,Z,A,B etc. are usually used to denote sets and lower case letters like x, y, z, p, q… denotes the elements or member of the sets.

## Related Definitions:

1. Set membership: If x is an element of set P then we say that x belongs to P and denote this by $x \in P$.
2. Subset: If all the members of set *X* are also members of set *P*, then *X* is a **subset** of *P* or P contains X, denoted $X \subseteq P$ or $P \supseteq X$. For example, {1, 2} is a subset of {1, 2, 3}, but {1, 4} is not. From this definition, it is clear that a set is a subset of itself.
3. Proper Subset: The term **proper subset** is defined to exclude the possibility of equality in the definition of a subset. It is denoted by $X \subset P$ or $P \supset X$, this means that X is a proper subset of P.
4. Singleton Set: A set P is said to be a singleton set if it contains one and only one element that is P = { a }, where 'a' is any single element contained in the set P.
5. Empty Set: A set P is said to be an empty set if it does not contain any element. It is denoted by $\phi$ or P= { }.
6. Universal set: A universal set denoted by U contains all the elements of the corresponding domain.
7. n(A): The number of elements in a set A is denoted as n(A).

- **Properties of Sets - Set is a:**

    1. Collection of objects
    2. Order of objects in the set does not matter

**Value addition: Analogy**

**Properties of SETS:**

1. For any set A, we have $\phi \subseteq A \subseteq U$.

Every set A is a subset of the universal set U since, by definition, all the members of A belongs to U. Also the empty set $\phi$ is a subset of A.

2. If $A \subseteq B$ and $B \subseteq C$ than $A \subseteq C$.

If every element of a set A belongs to set B, and every element of B belongs to C, then clearly every element of A belongs to C

3. For any set A, we have $A \subseteq A$

Every set A is a subset of itself, trivially, the elements of A belongs A.

4. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

If $A \subseteq B$ and $B \subseteq A$ then A and B have the same elements, i.e. A=B. Conversely, if A=B then $A \subseteq B$ and $B \subseteq A$ since every set is a subset of itself.

**Source: Discrete Mathematics, Schaum' Series, McGRAW-HILL, INC**

## Following special symbols will also be used further:

**N** = the set of natural numbers: 1, 2, 3…

**Q** = the set of rational numbers

**W =** the set of whole numbers: 0, 1, 2, 3…

**Z** = the set of integers: -2,-1, 0, 1, 2…..

**R** = the set of real numbers.

## There are two ways to represent sets:

**1. Set Builder form:**

One way is to state the properties which characterize the elements in the set e.g.

Example 1: A = {x: x is a color in the rainbow}

Example 2: B= { 2x + 1: $x \in$ **Z** }

**2. Roster form:**

Another way is to list its member's e.g.

Example 1: A = {Violet, Indigo, Blue, Green, Yellow, Orange, Red}

Example 2: B= {…-7, -5, -3, -1, 1, 3, 5, 7, …}

denotes the set A whose element are all the colors in rainbow. Note that the elements are separated by commas and are enclosed in the braces {}.

| **Value addition: Examples** |
|---|
| **Basic Concepts: SETS** |
| Example:<br><br>1. List the elements of the following sets; here N is the set of positive integers:<br>a) X = {x : x ∈ N, 3 < x < 12}<br>b) X = { x : x ∈ N, x is even, x < 15}<br>c) X = { x : x ∈ N, 4 + x = 3}<br><br>Ans:<br><br>a) X consist of positive integer between 3 and 12 so X = {4, 5, 6, 7, 8, 9, 10 , 11}<br>b) X consists of even positive integer less then15 so X = {2, 4, 6 , 8, 10, 12, 14}<br>**c)** There is no positive integer x which satisfy the condition 4 + x = 3 so X contains no elements or X = $\phi$, the empty set. |
| **Source: http://en.wikipedia.org/wiki/examples** |

## 1.3 Operations on Sets

The following operations are defined on sets which are used to study the set theory in detail:

- **Union** of the sets *A* and *B*, denoted *A* ∪ *B*, is the set of all objects that are a member of *A*, or *B*, or both. Formally, *A* ∪ *B* = {x: x ∈ A or x ∈ B, or both}. The union of {1, 2, 3} and {2, 3, 4} is the set {1, 2, 3, 4}.
- **Intersection** of the sets *A* and *B*, denoted *A* ∩ *B*, is the set of all objects that are members of both *A* and *B*. Formally, *A* ∩ *B* = { x: x ∈ A and x ∈ B}. The intersection of {1, 2, 3} and {2, 3, 4} is the set {2, 3}.
- **Complement** of set *A* relative to set *U*, denoted $A^c$ or A′, is the set of all members of *U* that are not members of *A*. This terminology is most commonly employed when *U* is a universal set, as in the study of Venn diagrams. This operation is also called the **set difference** of *U* and *A*, denoted *U* \ *A*. The complement of {1,2,3} relative to {2,3,4} is {4} , while, conversely, the complement of {2,3,4} relative to {1,2,3} is {1} . Complement of a universal set U is a null set.
- **Symmetric difference** of sets *A* and *B* is the set of all objects that are a member of exactly one of *A* and *B* (elements which are in one of the sets, but not in both). For instance, for the sets {1, 2, 3} and {2, 3, 4}, the symmetric difference set is {1, 4}. It is the set difference of the union and the intersection, denoted as A Δ B = (*A* ∪ *B*) \ (*A* ∩ *B*).

| **Value addition: Pure Sets** |
|---|
| **Basic Concepts: SETS** |
| **(i)** A set is **pure** if all of its members are sets; all members of its members are sets, and so on.<br><br>**For example**, the set containing only the empty set is a *nonempty pure set.*<br><br>**(ii) The union and intersection operation satisfy the following:**<br><br>1. Associative law<br><br>2. Commutative law<br><br>3. Idempotent law<br><br>4. Identity law<br><br>5. Distributive law<br><br>**(iii) The complement of a set satisfy:**<br><br>1. De Morgan's law<br><br>2. Complement law<br><br>3. Involution law<br><br>The above laws are explained in detail later on in the text. |
| **Source: http://en.wikipedia.org/wiki/operations** |

## Venn Diagram

Sets can be represented by means of diagrams known as Venn diagram. It's a graphic technique for visualizing set theory concepts using overlapping circles and shading to indicate intersection, union and complement. It was introduced in the late 1800s by British logician, John Venn, although it is believed that the method originated earlier.

A Venn diagram is a pictorial representation of sets in the form of closed regions in the plane. The universal set U is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle.

**For example:** A set **A** is represented as:

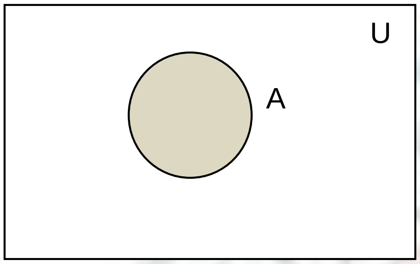



**Figure 1.3.1**

If B is another set then to represent A $\subseteq$ B the disk representing A will be entirely within the disk representing B.

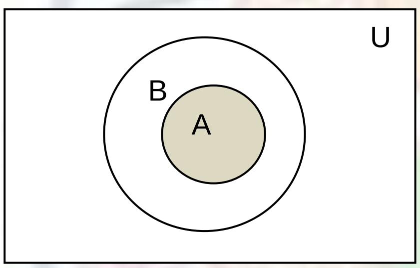



**Figure 2.3.2**

If A and B are disjoint i.e. no element in common then the disk representing A will be separated from the disk representing B.

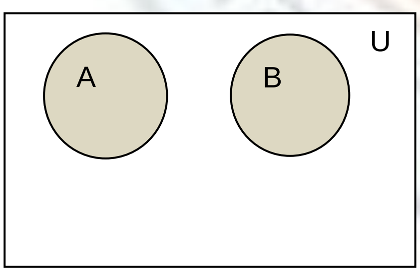



**Figure 3.3.3**

The above figure represents $A \cap B = \{ \} = \phi$

**Intersection** of sets

$A \cap B$ is the shaded region in the Venn diagram

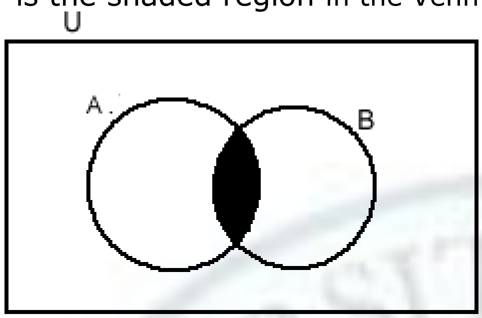



**Figure 4.3.4**

- If the intersection of two sets is $\varnothing$, the two sets are **mutually exclusive.**

**Union** of sets

- $A \cup B$ is elements in EITHER set or both represented by the shaded region

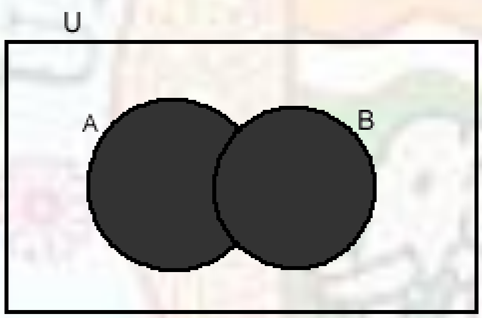



**Figure 1.3.5**

**Complement** of a set

- A' represents the elements NOT IN set A that is the shaded region

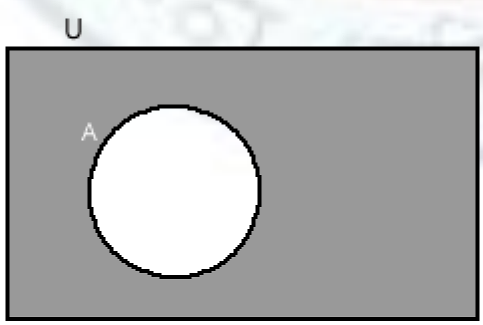



**Figure 1.3.6**

| **Value addition: Activity** |
|---|
| **Properties of Sets** |
| The Following are equivalent: A $\subseteq$ B, $A \cap B$ = A and AUB=B<br><br>**Solution:**<br><br>Suppose A $\subseteq$ B and let x $\in$ A. Then x $\in$ B, hence x $\in$ $A \cap B$ and A $\subseteq$ $A \cap B$. By definition $A \cap B$ $\subseteq$ A. Therefore $A \cap B$ = A. On the other hand, suppose $A \cap B$ = A and let x $\in$ A. then x $\in$ ($A \cap B$), hence x $\in$ A and x $\in$ B. Therefore, A $\subseteq$ B. Both result shows that A $\subseteq$ B is equivalent to $A \cap B$ =A.<br><br>Suppose again that A $\subseteq$ B. Let x $\in$ (AUB), then x $\in$ A or x $\in$ B. If x $\in$ A, then x $\in$ B because A $\subseteq$ B. In either case, x $\in$ B. Therefore AUB $\subseteq$ B, as B $\subseteq$ AUB. Therefore AUB = B. Now suppose AUB = Band let x $\in$ A. Then x $\in$ AUB by definition of union of sets. Hence x $\in$ B= AUB. Therefore A $\subseteq$ B. Both results shows that A $\subseteq$ B is equivalent to AUB = B.<br><br>Thus A $\subseteq$ B, $A \cap B$ = A and AUB = B are equivalent. |
| **Source: Discrete Mathematics, Schaum' Series, McGRAW-HILL, INC** |

**Principle of Inclusion/Exclusion**

Let A and B be two non-empty sets. The number of elements in A U B is denoted by

n(A U B) is equal to

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Consider the following example as shown in the Venn diagram below:

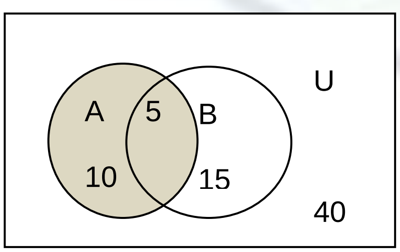



**Figure 1.3.7**

n(A) = 10 + 5 = 15

n(B) = 5 + 15 = 20

n($A \cap B$) = 5

n(A U B) = 10 + 5 + 15 = 30

n(U) = 40

We verify the principle of inclusion/exclusion because using principle as well:

n( A U B) = n (A) + n(B) - n($A \cap B$)

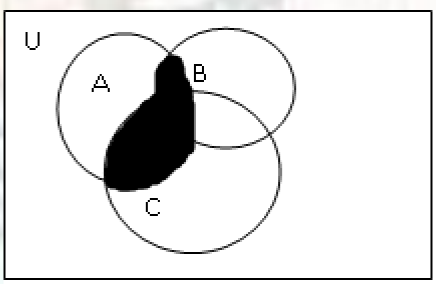
= 15 + 20 – 5

= 30

**Value addition: Using Venn diagram**

**Properties of Sets**

- **Distributive Property** of Intersection over Union
  - Remember the Distributive Property (of division over addition) from algebra? a(b + c) = ab + ac
  - For sets: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

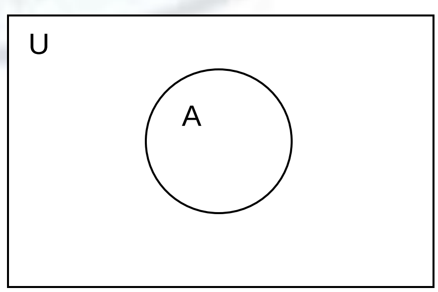



- **Idempotent laws**

If A is any set,then

i. $A \cup A = A$

ii. $A \cap A = A$



- **Identity laws**

i. $A \cup \phi = A$

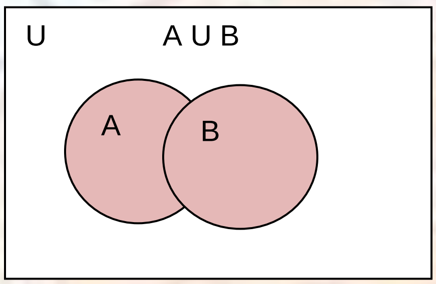
ii. $A \cap U = A$

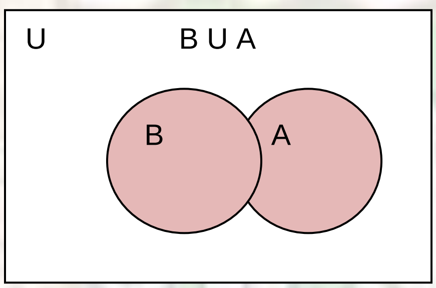
- **Commutative laws**

If A, B are two sets, then:

i. $A \cup B = B \cup A$

ii. $A \cap B = B \cap A$



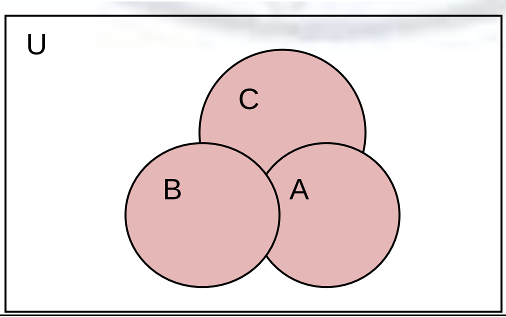




- **Associative laws**

If A, B, C are three sets, then:

i. $A \cup (B \cup C) = (A \cup B) \cup C$

ii. $A \cap (B \cap C) = (A \cap B) \cap C$



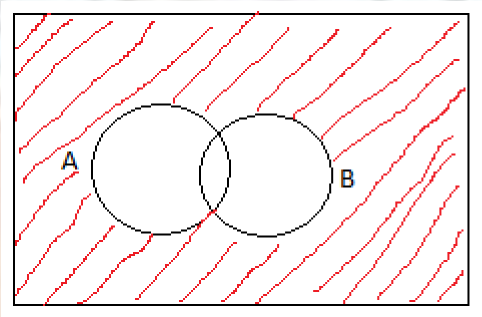
- **De Morgan's Laws**

  (A U B)′ = A′ ∩ B′

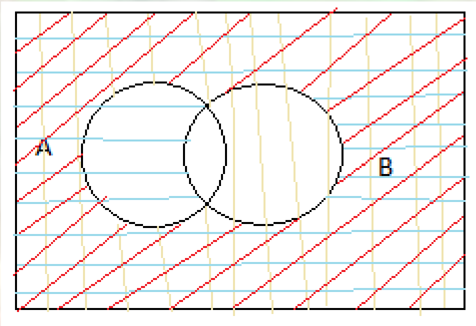
  - The complement of the union of two sets equals the intersection of their complements.

  (A ∩ B)′ = A′ U B′

  - The complement of the intersection of two sets equals the union of their complements.




(A U B)′ = A′ ∩ B′




**Proof:** (A U B)′ = A′ ∩ B′

(Taking forward and backward steps together)

Let $x \in (A \cup B)'$

$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \quad \text{or} \quad x \notin B$

$\Leftrightarrow x \in A' \text{ and } x \in B'$

$\Leftrightarrow x \in A' \cap B'$

We have shown that (A U B)′ ⊆ A′ ∩ B′. Retracting the steps forward and backward we get x ∈ A′ ∩ B′ ⇔ x ∈ (A U B)′. Therefore, A′ ∩ B′ ⊆ (A U B)′. Therefore, (A U B)′ = A′ ∩ B′.

(A ∩ B)′ = A′ U B′

**Proof:**

(Taking forward and backward steps together)

Let $x \in (A \cap B)'$

$\Leftrightarrow x \notin A \cap B$

$\Leftrightarrow x \notin A$ and $x \notin B$

$\Leftrightarrow x \in A'$ or $x \in B'$

$\Leftrightarrow x \in A' \cup B'$

$\therefore (A \cap B)' = A' \cup B'$

**Source: Discrete Mathematics, Schaum' Series, McGRAW-HILL, INC**

| **Value addition: Learning Targets** |
| --- |
| **You should be able to (for set properties and Venn Diagram) :** |

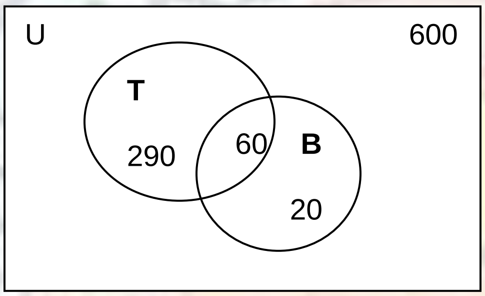
1. Draw a Venn Diagram for two or more given sets
   - **Sample:** If U = all students, T = students taller than 5'8", and B = students with blue eyes, draw a Venn Diagram showing the sets U, T, and B .
2. Indicate the intersection of two or more sets in a Venn Diagram
   - **Sample:** Shade the intersection of sets T and B in your diagram.
3. Indicate the union of two or more sets in a Venn Diagram
   - **Sample:** Shade the union of sets T and B in your diagram.
4. Indicate the complement of a set in a Venn Diagram
   - **Sample:** Shade the complement of set T in your diagram.
5. Give a verbal description of the elements in the intersection of two or more given sets
   - **Sample:** Describe the people in the intersection of sets T and B.
6. Give a verbal description of the elements in the union of two or more given sets
   - **Sample:** Describe the people in the union of sets T and B.
7. Give a verbal description of the elements in the complement of a given sets
   - **Sample:** Describe the people in the complement of set T.
8. Use a Venn Diagram, along with the intersection, union, and complement of sets, to solve counting problems.
   - **Sample A:** If n(T) = 350 and n($T \cap B$) = 60, how many tall students do

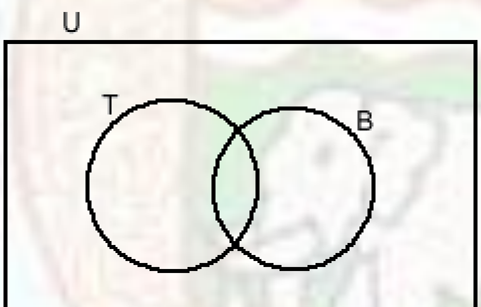
not have blue eyes?
- **Sample B:** If n(U) = 600, and n(T) = 350, how many short students are there?

9. Use the Inclusion/Exclusion Principle to solve counting problems
   - **Sample A:** If n(T) = 350, n(B) = 80, and n($T \cap B$) = 60, how many students are either tall or blue-eyed?
   - **Sample B:** If n(U) = 600, n(T) = 350, n(B) = 80, and n($T \cap B$) = 60, how many students are short and not blue-eyed?

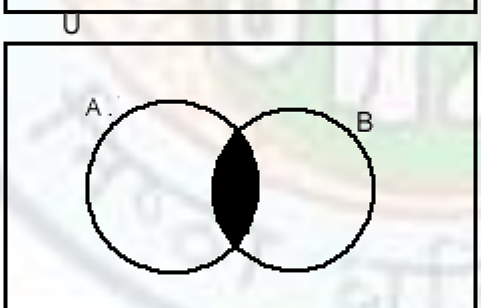
**Answers:**

**Basic Venn diagram for the problems 1 to 9.**

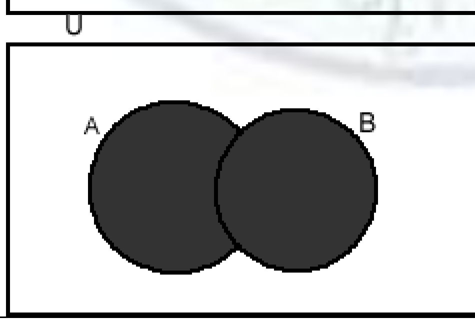


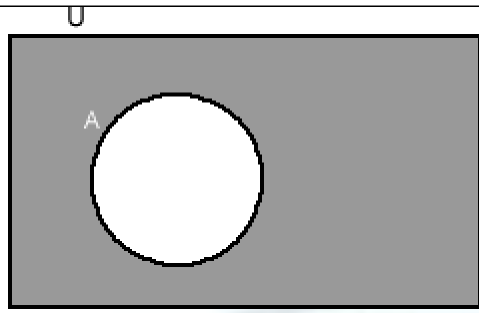

1.




2.




3.

4. 


5. The intersection of sets T and B contains tall people with blue eyes.
6. The union of sets T and B contains people who are tall or have blue eyes.
7. The complement of set T contains short people.
8. 
   A. 290 tall people do not have blue eyes.
   B. There are 250 (i.e. 600-350) short students.
9. 
   A. 370 (i.e. 290+60+20) students are either tall or have blue eyes.
   B. 230 (i.e. 600-370) students are short and do not have blue eyes.

**Source: http://www.batesville.k12.in.us/Physics/ProbStats/Combinatorics/15-1_Notes.htm#answers**

## Axiomatic set theory

We have assumed, so far, that a set having valid elements does exist. But it raises the following questions:

1. What are the valid elements of a set?
2. How can we say whether an entity is a set or not?
3. When can we say that a set exists?

Axiomatic theory was provided to answer these questions:

1. A set exists if the proposition that asserts its existence is logically true.
2. If an assumption that a set exist leads to a contradiction then the set does not exist.
3. Sets can have elements but those elements must be sets themselves otherwise they would not exist.

Axiomatic set theory works with propositions i.e. logical statements as per the rules of logic and that can take two values say, 'true' and 'false'. The various axioms of Axiomatic Set Theory are axioms of existence, equality, pair separation, union, power, infinity image, foundation and choice.

The most widely studied systems of axiomatic set theory imply that all sets form a cumulative hierarchy. Such systems come in two flavors, those whose ontology consists of:

- *Sets alone*. This includes the most common axiomatic set theory, Zermelo–Fraenkel set theory (ZFC), which includes the axiom of choice. Fragments of ZFC include:
    - Zermelo set theory, which replaces the axiom schema of replacement with that of separation;
    - General set theory, a small fragment of Zermelo set theory sufficient for the Peano axioms and finite sets;
    - Kripke-Platek set theory, which omits the axioms of infinity, powerset, and choice, and weakens the axiom schemata of separation and replacement.
- *Sets and proper classes*. This includes Von Neumann-Bernays-Gödel set theory, which has the same strength as ZFC for theorems about sets alone, and Morse-Kelley set theory, which is stronger than ZFC.

For the above systems, allowing urelements (objects that can be members of sets while having no members themselves) does not give rise to any interesting mathematics.

The systems NFU (allowing urelements) and NF (lacking them), though having their origin in type theory, are not based on a cumulative hierarchy. NF and NFU include a "set of everything," relative to which every set has a complement. Here urelements matter, because NF, but not NFU, produces sets for which the axiom of choice does not hold.

Systems of constructive set theory, such as CST, CZF, and IZF, embed their set axioms in intuitionistic logic instead of first order logic. Yet other systems accept standard first order logic but feature a nonstandard membership relation. These include rough set theory and fuzzy set theory, in which the value of an atomic formula embodying the membership relation is not simply **True** and **False**. The Boolean-valued models of ZFC are a related subject.

**Principle of Extension:** The sets X and Y are equal if and only if they have the same members (The principle formally states that a set is completely determined by its members).

**Principle of Abstraction:** Given any set U and any property P there is a set X such that the elements of X are exactly those members of U which have the property P (This formally states that a set can be described in terms of its properties).

| **Value addition: Interesting facts** |
| --- |
| **Extensions to sets: Rough sets** |
| In computer science, a **rough set**, first described by Zdzisław I. Pawlak, is a formal approximation of a crisp set (i.e., conventional set) in terms of a pair of sets which give the *lower* and the *upper* approximation of the original set. In the standard version of rough set |

theory (Pawlak 1991), the lower- and upper-approximation sets are crisp sets, but in other variations, the approximating sets may be fuzzy sets.

**Applications:**

Rough set methods can be applied as a component of hybrid solutions in machine learning and data mining. They have been found to be particularly useful for rule induction and feature selection (semantics-preserving dimensionality reduction). Rough set-based data analysis methods have been successfully applied in bioinformatics, economics and finance, medicine, multimedia, web and text mining, signal and image processing, software engineering, robotics, and engineering (e.g. power systems and control engineering). Recently, rough set principles have been also used in the open source database software by Infobright.

**Example:**

Rough sets are used to identify the most relevant attributes to characterize an object having a large number of describing attributes. Rough sets have been used successfully in the study of consumer behavior as it helps in identifying the most important and distinguishing attributes in the analysis and study of the consumer behavior. This technique helps in removing those attributes which provide no new information useful for analysis and hence are redundant for the system.

**Source: http://en.wikipedia.org/wiki/Rough_set**

**Value addition: Interesting facts**

**Extensions to sets: Fuzzy sets**

**Fuzzy sets** are sets whose elements have degrees of membership. Fuzzy sets were introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Classical bivalent sets are in fuzzy set theory usually called *crisp* sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics

**Example:**

A realtor wants to classify the houses he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in them. Let the available types of houses be represented by the following set.

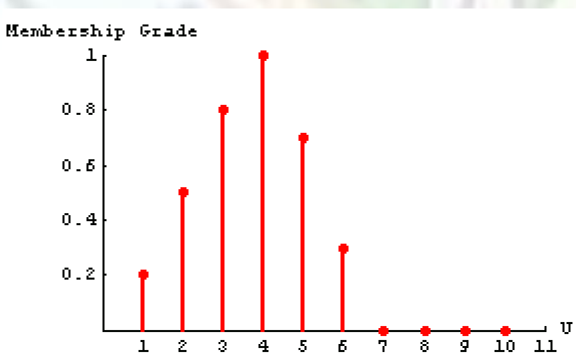
U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

The houses in this set are described by the number of bedrooms in a house. The realtor wants to describe a "comfortable house for a 4-person family," using a fuzzy set.

**Solution.** The fuzzy set "comfortable type of house for a 4-person family" may be described using a fuzzy set in the following manner.

```
HouseForFour =
 FuzzySet[{{1, .2}, {2, .5}, {3, .8}, {4, 1},
   {5, .7}, {6, .3}}, UniversalSpace → {1, 10}];
```

```
FuzzyPlot[HouseForFour, ShowDots → True];
```




**Source: http://en.wikipedia.org/wiki/Fuzzy_set**

## 1.4 Finite and Infinite Sets

**Finite set** is a set that has a finite number of members that can be counted.

More formally; *A set S is finite if it has the same cardinality as some natural number n in N (The set of all natural numbers) . We then define |S| = n and say that S has n elements.*

Example- Finite sets:
$A = \{0, 2, 4, 6, 8, \ldots, 100\}$
$C = \{x : x \text{ is an integer}, 1 < x < 10\}$

| **Value addition: Examples** |
|---|
| **Extensions to sets: Finite sets** |
| **Example:**<br>If $A$ is the set of positive integers less than 12 then $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and $n(A) = 11$.<br><br>If $C$ is the set of numbers which are also multiples of 3 then $C = \{3, 6, 9, \ldots\}$ and $C$ is not a finite set.<br><br>If $D$ is the set of integers $x$ defined by $-3 < x < 6$ then $D = \{-2, -1, 0, 1, 2, 3, 4, 5\}$ and $n(D) = 8$.<br><br>If $Q$ is the set of letters in the word *'HELLO'* then $Q = \{H, E, L, O\}$, n $(Q)$ = 4 and we see that '*L*' is not repeated. |
| **Source: Elements of Discrete Mathematics by C.L. Liu, second edition, TMH.** |

### Infinite set

An **infinite set** is a set which is not finite. It is not possible to explicitly list out all the elements of an infinite set.

Example- infinite sets:
$T = \{x : x \text{ is a triangle}\}$
*N* is the set of natural numbers
*A* is the set of fractions

### Countable Set

A set which is finite is a countable set. However, some authors (e.g., Ciesielski 1997) use the definition "equipollent to the finite ordinals," commonly used to define a denumerable set, to define a countable set. A countable set is defined as…

*A set S is countable if |S| = |N|.*

*Thus a set S is countable if there is a one-to-one mapping of N onto S, that is, if S is the range of an infinite one-to-one sequence. Otherwise, the set is uncountable.* For example, the set of all integers, {..., -1, 0, 1, 2 ...}, is a countably infinite set and the set of all alphabets {a, b, …,y, z} is a countably finite set.

## Uncountable set

In mathematics, an **uncountable set** is an infinite set that contains too many elements to be countable. The uncountability of a set is closely related to its cardinal number (number of elements in the set): a set is uncountable if its cardinal number is larger than that of the natural numbers. For example: the set or irrational numbers, real numbers.

In set theory, an infinite set is a set that is not a finite set but the infinite sets may be countable or uncountable. Some such examples are:

the set of all integers, {..., -1, 0, 1, 2 ...}, is a countably infinite set; and

the set of all real numbers is an uncountably infinite set.

## Properties

The set of natural numbers (whose existence is assured by the axiom of infinity) is infinite. It is the only set which is directly required by the axioms to be infinite. The existence of any other infinite set can be proved in Zermelo–Fraenkel set theory (ZFC) only by showing that it follows from the existence of the natural numbers.

A set is infinite if and only if for every natural number the set has a subset whose cardinality is that natural number.

If the axiom of choice holds, then a set is infinite if and only if it includes a countable infinite subset.

If a set of sets is infinite or contains an infinite element, then its union is infinite. The powerset of an infinite set is infinite. Any superset of an infinite set is infinite. If an infinite set is partitioned into finitely many subsets, then at least one of them must be infinite. Any set which can be mapped onto an infinite set is infinite. The Cartesian product of an infinite set and a nonempty set is infinite. The Cartesian product of an infinite number of sets each containing at least two elements is either empty or infinite; if the axiom of choice holds, then it is infinite.

If an infinite set is a well-ordered set, then it must have a nonempty subset which has no greatest element.

In ZF, a set is infinite if and only if the powerset of its powerset is a Dedekind-infinite set, having a proper subset equinumerous to itself. If the axiom of choice is also true, infinite sets are precisely the Dedekind-infinite sets.

If an infinite set is a well-orderable set, then it has many well-orderings which are non-isomorphic.

## Some Set Definitions:

- **Cartesian product** of *A* and *B*, denoted *A* × *B*, is the set whose members are all possible ordered pairs (*a*, *b*) where *a* is a member of *A* and *b* is a member of *B*.
- **Power set** of a set *A* is the set whose members are all possible subsets of *A*. For example, the powerset of {1, 2} is {{}, {1}, {2}, {1, 2}}.
- **Comparable and Non comparable sets:** Sets A and B are comparable if A $\subseteq$ B or B $\subseteq$ A; hence A and B are non comparable if A $\not\subset$ B or B $\not\subset$ A.
- **Disjoint Sets:** Sets A and B are disjoint if they have no elements in common, i.e., if no elements of A belongs to B and no elements of B belongs to A.

# 1.5 Uncountably Infinite Sets

## Denumerable Set

A set is denumerable iff it is equipollent to the finite ordinal numbers. (Moore 1982; Rubin 1967; Suppes 1972). However, Ciesielski (1997) calls this property "countable." The set aleph0 is most commonly called "denumerable" to "countably infinite". In set theory, the **aleph numbers** are a sequence of numbers used to represent the cardinality (or size) of infinite sets. They are named after the symbol used to denote them, the Hebrew letter aleph ($\aleph$). The cardinality of the natural numbers is $\aleph_0$(read *aleph-naught*; also *aleph-null* or *aleph-zero*), the next larger cardinality is aleph-one $\aleph_1$, then $\aleph_2$and so on. Continuing in this manner, it is possible to define a cardinal number $\aleph_\alpha$for every ordinal number α, as described below.

The concept goes back to Georg Cantor, who defined the notion of cardinality and realized that infinite sets can have different cardinalities.

## Countably Infinite

Any set which can be put in a one-to-one correspondence with the natural numbers (or integers) so that a prescription can be given for identifying its members one at a time is called a countably infinite (or denumerably infinite) set. Once one countable set S is given, any other set which can be put into a one-to-one correspondence with S, then it is also countable. Countably infinite sets have cardinal number aleph-0.

Examples of countable sets include the integers, algebraic numbers, and rational numbers. Georg Cantor showed that the number of real numbers is rigorously larger than a countably infinite set, and the postulate that this number, the so-called "continuum," is equal to aleph-1 is called the continuum hypothesis. Examples of nondenumerable sets include the real, complex, irrational, and transcendental numbers.

## Characterizations

There are many equivalent characterizations of uncountability. A set *X* is uncountable if and only if any of the following conditions holds:

- There is no injective function from *X* to the set of natural numbers.
- *X* is nonempty and any ω-sequence of elements of *X* fails to include at least one element of *X*. That is, *X* is nonempty and there is no surjective function from the natural numbers to *X*.
- The cardinality of *X* is neither finite nor equal to $\aleph_0$(aleph-null, the cardinality of the natural numbers).
- The set *X* has cardinality strictly greater than $\aleph_0$.

The first three of these characterizations can be proved equivalent in Zermelo–Fraenkel set theory without the axiom of choice, but the equivalence of the third and fourth cannot be proved without additional choice principles

## Properties

If an uncountable set *X* is a subset of set *Y*, then *Y* is uncountable.

| **Value addition: Examples** |
|---|
| **Uncountable Sets** |

### Examples

The best known example of an uncountable set is the set **R** of all real numbers; Cantor's diagonal argument shows that this set is uncountable. The diagonalization proof technique can also be used to show that several other sets are uncountable, such as the set of all infinite sequences of natural numbers (and even the set of all infinite sequences consisting only of zeros and ones) and the set of all subsets of the set of natural numbers. The cardinality of **R** is often called the cardinality of the continuum and denoted by *c*, or $2^{\aleph_0}$, or $\beth_1$ (beth-one).

The Cantor set is an uncountable subset of **R**. The Cantor set is a fractal and has Hausdorff dimension greater than zero but less than one (**R** has dimension one). This is an example of the following fact: any subset of **R** of Hausdorff dimension strictly greater than zero must be uncountable.

Another example of an uncountable set is the set of all functions from **R** to **R**. This set is even "more uncountable" than **R** in the sense that the cardinality of this set is $\beth_2$(beth-two), which is larger than $\beth_1$.

A more abstract example of an uncountable set is the set of all countable ordinal numbers, denoted by Ω (omega) or $\omega_1$. The cardinality of Ω is denoted $\aleph_1$(aleph-one). It can be shown, using the axiom of choice, that $\aleph_1$is the *smallest* uncountable cardinal number. Thus either $\beth_1$, the cardinality of the reals, is equal to $\aleph_1$ or it is strictly larger. Georg Cantor was the first to propose the question of whether $\beth_1$is equal to $\aleph_1$. In 1900, David Hilbert posed this question as the first of his 23 problems. The statement that $\aleph_1 = \beth_1$is now called the continuum hypothesis and is known to be independent of the Zermelo–Fraenkel axioms for set theory (including the axiom of choice).

**Source: http://en.wikipedia.org/wiki/Uncountable_set**

## 1.6 Multisets

**Introduction**

A multiset is a set in which an element may appear more than once. In a regular set duplicates are not allowed but duplicates are allowed in a multiset. In other words, a **multiset** (or **bag**) is a generalization of a set. While each member of a set has only one membership, a member of a multiset can have more than one membership (meaning that there may be multiple instances of a member in a multiset, not that a single member instance may appear simultaneously in several multisets). The term "multiset" was coined by Nicolaas Govert de Bruijn in the 1970s. The use of multisets in mathematics predates the name "multiset" by nearly 90 years. Richard Dedekind used multisets in a paper published in 1888.

**Overview**

The total number of elements in a multiset, including repeated memberships, is the cardinality of the multiset. The number of times an element belongs to the multiset is the multiplicity of that member. For example, in the multiset {*a*, *a*, *b*, *b*, *b*, *c*} the multiplicities of the members *a*, *b*, and *c* are respectively 2, 3, and 1, and the cardinality of the multiset is 6. To distinguish between sets and multisets, a notation that incorporates brackets is sometimes used: the multiset {2,2,3} can be represented as [2,2,3]. In multisets, as in sets and in contrast to tuples, the order of elements is irrelevant: The multisets {*a*, *b*} and {*b*, *a*} are equal. But multisets {a, a, b} and {a, b} are different. Multiset is also known as bag and has similar operations as that of a regular set.

## Formal definition

Within set theory, a **multiset** can be formally defined as a 2-tuple (*A*, *m*) where *A* is some set and $m : A \to \mathbf{N}_{\geq 1}$ is a function from *A* to the set $\mathbf{N}_{\geq 1} = \{1, 2, 3, ...\}$ of positive natural numbers. The set *A* is called the *underlying set of elements*. For each *a* in *A* the *multiplicity* (that is, number of occurrences) of *a* is the number *m*(*a*). If a universe *U* in which the elements of *A* must live is specified, the definition can be simplified to just a multiplicity function $m_U$: $U \to \mathbf{N}$ from *U* to the set $\mathbf{N} = \{0, 1, 2, 3 ...\}$ of natural numbers, obtained by extending *m* to *U* with values 0 outside *A*. This extended multiplicity function is the multiplicity function called $1_A$ below. Like any function, the function *m* may be defined as its graph: the set of ordered pairs {(*a*, *m* (*a*)): *a* in *A*}. With these definitions the multiset written as { *a*, *a*, *b* } is defined as ({ *a*, *b* },{ (*a*, 2), (*b*, 1) }), and the multiset { *a*, *b* } is defined as ({ *a*, *b* },{ (*a*, 1), (*b*, 1) }).

The concept of a multiset is a generalization of the concept of a set. A multiset corresponds to an ordinary set if the multiplicity of every element is one (as opposed to some larger natural number). However to replace set theory by "multiset theory" so as to have multisets directly into the foundations is not easy: a privileged role would still have to be given to (ordinary) sets when defining maps, as there is no clear notion of maps (functions) between multisets. It can be done, with the result that classical theorems such as the Cantor–Bernstein–Schroeder theorem or Cantor's theorem, when generalized to multisets, are false; they remain true only in the case of finite multisets. In addition, the notion of a set as a "class of items satisfying a certain property" – i.e. the extension of a predicate – is used throughout mathematics, and this notion lacks a sensible generalization to multisets with multiple memberships.

One advantage of treating multisets in their own right (as primitive, rather than defined in terms of something else) is that one can talk naturally about multiplicity 0 without having to admit that multisets are infinite – in the classical sense – since multisets have automatically their own notion of size.

An indexed family, ( $a_i$ ), where *i* is in some index-set, may define a multiset, sometimes written { $a_i$ }, in which the multiplicity of any element *x* is the number of indices *i* such that $a_i = x$. The condition for this to be possible is that no element occurs infinitely many times in the family: even in an infinite multiset, the multiplicities must be finite numbers.

## 1.7 Mathematical Induction

**Definition**

The principle of mathematical induction is used to prove that a given proposition (formula, equality or inequality) is true for all positive integer numbers greater than or equal to some integer n.

Let us denote the proposition in question by P (n), where n is a positive integer. The proof involves two steps:

Step 1: We first establish that the proposition P (n) is true for the lowest possible value of the positive integer n.

Step 2: We assume that P (k) is true and establish that P (k+1) is also true

**Mathematical induction** is considered to be of the following two types:

1. **Simple Induction**

Mathematical induction is a method that allows one to prove that a statement is true for all natural numbers.

Using induction means that by proving the first statement (in an infinite sequence of statements) is true, and then proving that the "next" statement after any given point is true, then all must be true.

**Example :**

Use the Principle of Mathematical Induction to prove:

$\sum^{n}_{k=1} (2k + 3) = n(n + 4)$ (for all $n \geq 1$).

Solution. Assume

$P(n) : 5 + 7 + 9 + \cdots + (2n + 3) = n(n + 4)$

**Base Step:** P(1) states that $2 \cdot 1 + 3 = 1(1 + 4)$, which is true since both sides equal 5.

**Inductive Hypothesis:** Suppose P(k) is true; that is

$5 + 7 + 9 + \cdots + (2k + 3) = k(k + 4)$ (*)

**Inductive Step:** $P(k) \rightarrow P(k+1)$: We want to show $P(k+1) : 5+7+9+ \cdots +(2k+3)+(2k+5) = (k+1)(k+5)$.

To do so, we add 2k + 3 to both sides of (*) and simplify:

$LHS = 5 + 7 + 9 + \cdots + (2k + 3) + (2k + 5) = k(k + 4) + (2k + 5)$

$= k2 + 6k + 5 = (k + 1)(k + 5) = RHS$

which proves P(k + 1). Thus, P(k) ! P(k + 1) is true.

Therefore, by the Principle of Mathematical Induction, $5+7+9+\cdots+(2n+3) = n(n+4)$ is true for all $n \geq 1$.

2. **Strong Induction**

Simple induction involved taking a single "base" case, and proving the next case in the sequence on that single foundation. By contrast, strong induction involves proving the next case in the sequence based on *every* previous case, not just a particular one. More formally:

Whenever P(m), P(m+1), p(m+2),...,P(k) are all true, then P(k+1) is also true.

Therefore P(n) is true for all integers n ≥ k.

It's important to remember this difference, as otherwise the methodology used is the same as for "weak" induction.

**Examples**

**Problem 1:**

Use mathematical induction to prove that

**1 + 2 + 3 + ... + n = n (n + 1) / 2**

for all positive integers n.

**Solution to Problem 1:**

- Let the statement P (n) be

  1 + 2 + 3 + ... + n = n (n + 1) / 2

- STEP 1: We first show that p (1) is true.

  Left Side = 1

  Right Side = 1 (1 + 1) / 2 = 1

- Both sides of the statement are equal hence p (1) is true.
- STEP 2: We now assume that p (k) is true

  1 + 2 + 3 + ... + k = k (k + 1) / 2

- and show that p (k + 1) is true by adding k + 1 to both sides of the above statement

  1 + 2 + 3 + ... + k + (k + 1) = k (k + 1) / 2 + (k + 1)

  = (k + 1)(k / 2 + 1)

  = (k + 1)(k + 3) / 2

- The last statement may be written as

  1 + 2 + 3 + ... + k + (k + 1) = (k + 1)(k + 3) / 2

- Which is the statement p(k + 1).

**Problem 2:**

Prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = n (n + 1) (2n + 1)/ 6$$

For all positive integers n.

**Solution to Problem 2:**

- Statement P (n) is defined by

  $1^2 + 2^2 + 3^2 + \ldots + n^2 = n (n + 1) (2n + 1)/ 2$

- STEP 1: We first show that p (1) is true.

  Left Side = $1^2 = 1$

  Right Side = $1 (1 + 1) (2*1 + 1)/ 6 = 1$

- Both sides of the statement are equal hence p (1) is true.
- STEP 2: We now assume that p (k) is true

  $1^2 + 2^2 + 3^2 + \ldots + k^2 = k (k + 1) (2k + 1)/ 6$

- and show that p (k + 1) is true by adding $(k + 1)^2$ to both sides of the above statement

  $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2 = k (k + 1) (2k + 1)/ 6 + (k + 1)^2$

- Set common denominator and factor k + 1 on the right side

  $= (k + 1) [ k (2k + 1)+ 6 (k + 1) ] /6$

- Expand k (2k + 1)+ 6 (k + 1)

  $= (k + 1) [ 2k^2 + 7k + 6 ] /6$

- Now factor $2k^2 + 7k + 6$.

  $= (k + 1) [ (k + 2) (2k + 3) ] /6$

- We have started from the statement P(k) and have shown that

  $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2 = (k + 1) [ (k + 2) (2k + 3) ] /6$

- Which is the statement P(k + 1).

**Problem 3:**

Use mathematical induction to prove that

$$1^3 + 2^3 + 3^3 + ... + n^3 = n^2 (n + 1)^2 / 4$$

for all positive integers n.

**Solution to Problem 3:**

- Statement P (n) is defined by

  $1^3 + 2^3 + 3^3 + ... + n^3 = n^2 (n + 1)^2 / 4$

- STEP 1: We first show that p (1) is true.

  Left Side = $1^3 = 1$

  Right Side = $1^2 (1 + 1)^2 / 4 = 1$

- hence p (1) is true.
- STEP 2: We now assume that p (k) is true

  $1^3 + 2^3 + 3^3 + ... + k^3 = k^2 (k + 1)^2 / 4$

- add $(k + 1)^3$ to both sides

  $1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = k^2 (k + 1)^2 / 4 + (k + 1)^3$

- factor $(k + 1)^2$ on the right side

  $= (k + 1)^2 [ k^2 / 4 + (k + 1) ]$

- set to common denominator and group

  $= (k + 1)^2 [ k^2 + 4 k + 4 ] / 4$

  $= (k + 1)^2 [ (k + 2)^2 ] / 4$

- We have started from the statement P(k) and have shown that

  $1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = (k + 1)^2 [ (k + 2)^2 ] / 4$

- Which is the statement P(k + 1).

**Problem 4:**

Prove that for any positive integer number n , $\mathbf{n^3 + 2n}$ is divisible by 3

**Solution to Problem 4:**

- Statement P (n) is defined by

  $n^3 + 2n$ is divisible by 3

- STEP 1: We first show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$

  $1^3 + 2(1) = 3$

  3 is divisible by 3

- hence p (1) is true.
- STEP 2: We now assume that p (k) is true

  $k^3 + 2k$ is divisible by 3

  is equivalent to

  $k^3 + 2k = 3M$ , where M is a positive integer.

- We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms

  $(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

  $= [k^3 + 2k] + [3k^2 + 3k + 3]$

  $= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

- Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

| **Value addition: Learn More** |
|---|
| **More Advanced Problems** |
| **Problem 1:**<br><br>Prove that $\mathbf{3^n > n^2}$ for n = 1, n = 2 and use the mathematical induction to prove that $3^n > n^2$ for n a positive integer greater than 2. |

**Solution to Problem 1:**

- Statement P (n) is defined by

  $3^n > n^2$

- STEP 1: We first show that p (1) is true. Let n = 1 and calculate $3^1$ and $1^2$ and compare them

  $3^1 = 3$

  $1^2 = 1$

- 3 is greater than 1 and hence p (1) is true.
- Let us also show that P(2) is true.

  $3^2 = 9$

  $2^2 = 4$

- Hence P(2) is also true.
- STEP 2: We now assume that p (k) is true

  $3^k > k^2$

- Multiply both sides of the above inequality by 3

  $3 * 3^k > 3 * k^2$

- The left side is equal to $3^{k+1}$. For k >, 2, we can write

  $k^2 > 2k$ and $k^2 > 1$

- We now combine the above inequalities by adding the left hand sides and the right hand sides of the two inequalities

  $2k^2 > 2k + 1$

- We now add $k^2$ to both sides of the above inequality to obtain the inequality

  $3k^2 > k^2 + 2k + 1$

- Factor the right side we can write

$3 * k^2 > (k + 1)^2$

- If $3 * 3^k > 3 * k^2$ and $3 * k^2 > (k + 1)^{2}$ then

  $3 * 3^k > (k + 1)^2$

- Rewrite the left side as $3^{k+1}$

  $3^{k+1} > (k + 1)^2$

- Which proves tha P(k + 1) is true

---

**Problem 2:**

Prove that **$n! > 2^n$** for n a positive integer greater than or equal to 4. (Note: n! is n factorial and is given by 1 * 2 * ...* (n-1)*n.)

**Solution to Problem 2:**

- Statement P (n) is defined by

  $n! > 2^n$

- STEP 1: We first show that p (4) is true. Let n = 4 and calculate 4 ! and $2^n$ and compare them

  $4! = 24$

  $2^4 = 16$

- 24 is greater than 16 and hence p (4) is true.

- STEP 2: We now assume that p (k) is true

  $k! > 2^k$

- Multiply both sides of the above inequality by k + 1

  $k! (k + 1) > 2^k (k + 1)$

- The left side is equal to (k + 1)!. For k >, 4, we can write

$k + 1 > 2$

- Multiply both sides of the above inequality by $2^k$ to obtain

  $2^k (k + 1) > 2 * 2^k$

- The above inequality may be written

  $2^k (k + 1) > 2^{k+1}$

- We have proved that $(k + 1)! > 2^k (k + 1)$ and $2^k (k + 1) > 2^{k+1}$ we can now write

  $(k + 1)! > 2^{k+1}$

- We have assumed that statement P(k) is true and proved that statment P(k+1) is also true.

---

**Problem 3:**

Use mathematical induction to prove De Moivre's theorem

$$[ R (\cos t + i \sin t) ]^n = R^n(\cos nt + i \sin nt)$$

for n a positive integer.

**Solution to Problem 3:**

- STEP 1: For n = 1

  $[ R (\cos t + i \sin t) ]^1 = R^1(\cos 1*t + i \sin 1*t)$

- It can easily be seen that the two sides are equal.

- STEP 2: We now assume that the theorem is true for n = k, hence

  $[ R (\cos t + i \sin t) ]^k = R^k(\cos kt + i \sin kt)$

- Multiply both sides of the above equation by R (cos t + i sin t)

  $[ R (\cos t + i \sin t) ]^k R (\cos t + i \sin t) = R^k(\cos kt + i \sin kt) R$

(cos t + i sin t)

- Rewrite the above as follows

  $[ R (\cos t + i \sin t) ]^{k+1} = R^{k+1} [ (\cos kt \cos t - \sin kt \sin t) + i (\sin kt \cos t + \cos kt \sin t) ]$

- Trigonometric identities can be used to write the trigonometric expressions (cos kt cos t - sin kt sin t) and (sin kt cos t + cos kt sin t) as follows

  $(\cos kt \cos t - \sin kt \sin t) = \cos(kt + t) = \cos(k + 1)t$

  $(\sin kt \cos t + \cos kt \sin t) = \sin(kt + t) = \sin(k + 1)t$

- Substitute the above into the last equation to obtain

  $[ R (\cos t + i \sin t) ]^{k+1} = R^{k+1} [ \cos (k + 1)t + \sin(k + 1)t ]$

It has been established that the theorem is true for n = 1 and that if it assumed true for n = k it is true for n = k + 1.

**Source: http://en.wikipedia.org/wiki/ Mathematical Induction - Problems With Solutions.htm**

**Value addition: Activity**

**Exercises**

Use the Principle of Mathematical Induction to prove:

1. $\sum^{n}_{k=1} k = n(n + 1)/2$ (for all $n \geq 1$)
2. $\sum^{n}_{k=1} k = n(n + 1)/2$ (for all $n \geq 1$)

**Source: Discrete Mathematics, Schaum' Series, McGRAW-HILL, INC**

# 1.8 Principle of Inclusion and Exclusion

**Introduction**

In combinatorial mathematics, the **inclusion–exclusion principle is** also known as the **Sieve principle**. The rule of addition says how many elements are in a union of sets if the sets are mutually disjoint. But there exist cases where one has to determine the number of elements in a union of sets when some of the sets overlap.

Let us consider the union of two sets A and B. The number of elements in the union of the two sets varies according to the number of elements in each set that are common. If A and B have no common elements at all then the number of elements in their union will be the algebraic sum of the number of elements of A and the number of elements of B, which is better represented by ***n(A U B) = n(A) + n(B)*** or |A U B| = |A| + |B|

If A and B are exactly the same sets, i.e if they coincide, then ***n(AUB) = n(A)*** .Thereby any formula for the number of elements in a union of two sets must contain a reference to the number of elements that are common, ***n(A∩ B)*** and also references to the number of individual elements of the sets, ***n(A)*** and ***n(B)***.

To derive a formula for the number of elements in the union of the two sets, the number of elements in set ***A*** is ***n(A)***, which counts the elements in ***A***, the elements not in ***B*** and the elements that are both in ***A*** and ***B,*** along with ***n(B), which*** counts the elements in ***A***, the number of elements not in ***A*** and the elements that are in both ***A*** and ***B*** are considered. By adding ***n(A)*** and ***n(B)***, the elements in both ***A*** and ***B*** are counted twice. To eliminate this redundancy and get the value for ***n(AUB)***, we need to subtract the number of elements which are both in A and B, which is represented by their intersection ***A∩B,***
Therefore ***n(AUB)=n(A) + n(B) - n(A∩ B)*** or |A U B| = |A| + |B| - |A ∩ B|

## Formal Definition:

**The Inclusion Exclusion Principle for Two or Three Sets**

Let ***P, Q*** be finite sets, then

***n (PUQ) =n(P) + n(Q) - n(P∩ Q)***

and for three sets ***P, Q*** and ***R***

***n(P U Q U R) = n(P) + n(Q) + n(R) - n(P∩ Q) - n(P∩ R) - n(Q∩ R) +n(P ∩ Q ∩ R)***

In general if $A_1, \ldots, A_n$ are finite sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j\,:\,1 \le i < j \le n} |A_i \cap A_j| + \sum_{i,j,k\,:\,1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

where $|A|$ denotes the cardinality of the set $A$. For example, taking $n = 2$, we get a special case of double counting; in words: we can count the size of the union of sets $A$ and $B$ by adding $|A|$ and $|B|$ and then subtracting the size of their intersect compensating *exclusion*. When $n > 2$ the exclusion of the pairwise intersections is (possibly) too severe, and the correct formula is as shown with alternating signs.

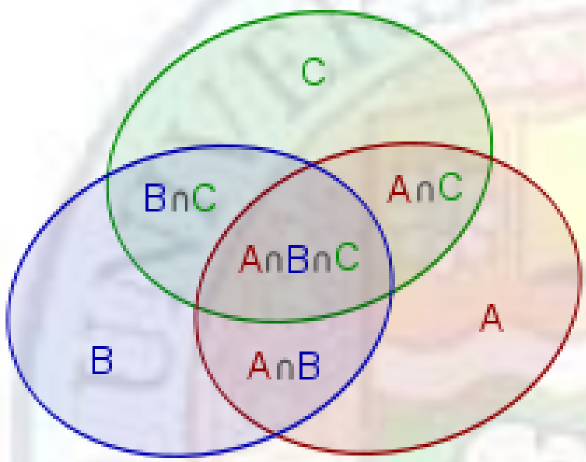
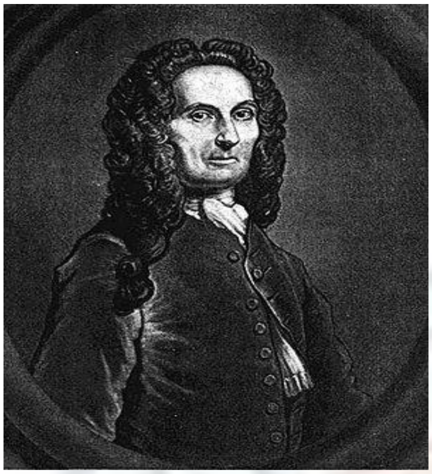



**Figure 1.8.1**

| **Value addition: Did you Know** |
|---|
| **Abraham de Moivre** |
| This formula is attributed to Abraham de Moivre; it is sometimes also named for Daniel da Silva, Joseph Sylvester or Henri Poincaré. The name comes from the idea that the principle is based on over-generous *inclusion*, followed by compensating *exclusion*.<br><br>*"Moivre" redirects here; for the French commune see Moivre, Marne.* |



Abraham de Moivre

| | |
|---|---|
| **Born** | 26 May 1667<br>Vitry-le-François, Champagne, France |
| **Died** | 27 November 1754 (aged 87)<br>London, England |
| **Residence** | England |
| **Nationality** | French |
| **Fields** | Mathematician |
| **Alma mater** | Academy of Saumur<br>Collège de Harcourt |
| **Doctoral advisor** | Jacques Ozanam |
| **Known for** | De Moivre's formula<br>Theorem of de Moivre–Laplace |
| **Influences** | Isaac Newton |

**Abraham de Moivre** (26 May 1667 in Vitry-le-François, Champagne, France – 27 November 1754 in London, England; French pronunciation: [abʁam də mwavʁ]) was a French mathematician famous for de Moivre's formula, which links complex numbers and trigonometry, and for his work on the normal distribution and probability theory. He was elected a Fellow of the Royal Society in 1697, and was a friend of Isaac Newton, Edmund Halley, and James Stirling. Among his fellow Huguenot exiles in England, he was a colleague of the editor and translator Pierre des Maizeaux.

The social status of the family de Moivre is unclear, but his father, a surgeon, was able to send him to the Protestant academy at Sedan (1678–82). De Moivre studied logic at the Academy of Saumur (1682–84), attended the Collège d'Harcourt in Paris (1684), and studied privately with Jacques Ozanam (1684–85). It appears that de Moivre never received a college degree.

A Calvinist, de Moivre left France after the revocation of the Edict of Nantes (1685) and spent the remainder of his life in England.

Throughout his life de Moivre remained poor. It is reported that he was a regular customer of Slaughter's Coffee House, St. Martin's Lane at Cranbourn Street, where he earned a little money from playing chess.

Abraham de Moivre died in London and was buried at St Martin-in-the-Fields, although his body was later moved.

De Moivre wrote a book on probability theory, *The Doctrine of Chances*, said to have been prized by gamblers. It is reported in all seriousness that de Moivre correctly predicted the day of his own death. Noting that he was sleeping 15 minutes longer each day, De Moivre surmised that he would die on the day he would sleep for 24 hours. A simple mathematical calculation quickly yielded the date, 27 November 1754. He did indeed die on that day.

De Moivre first discovered Binet's formula, the closed-form expression for Fibonacci numbers linking the $n$th power of $\varphi$ to the $n$th Fibonacci number.

**Source: http://en.wikipedia.org/wiki/Abraham_de_Moivre#Life**

**Example 1:**
Counting the number of elements in a Union.

How many integers from 1 to 1000 are either multiples of 3 or multiples of 5

**Solution**
Let **A** be the set of all integers from 1-1000 that are multiples of **3**

**A = {x: x = 3 x, x $\in$ Z}**

Let **B** be the set of all integers from 1-1000 that are multiples of **5**

**A = {x: x = 5 m, m $\in$ Z}**
From this we have **A U B** is the set of all integers from 1 to 1000 that are multiples of either 3 or 5 and we also have ***(A$\cap$ B)*** is the set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.

To use the inclusion/exclusion principle to obtain ***n(A U B)*** , we need ***n(A)***,***n(B)*** and ***n(A$\cap$ B).*** From 1 to 1000, every third integer is a multiple of 3,each of this multiple can be represented as **3p, for** any integer **p** from 1 through 333.
From the above we have that ***n(A)*** = 333 for integers 1-1000

Similarly for multiples of 5, each multiple of 5 is of the form **5q** for some integer **q** from 1 through 200.
From this we have ***n(B)*** = 200

For ***n(A$\cap$ B)*** , we need to determine the number of multiples of 15 from 1 through 1000. Each multiple of 15 is of the form **15r** for some integer **r** from 1 through 66. Now we have the values for ***n(A$\cap$ B), which*** is 66.

From all the above we can determine **n(AUB)**, using the Inclusion/Exclusion principle.

***n(AUB) = n(A) + n(B) - n(A$\cap$ B)***

***= 333 + 200 - 66***

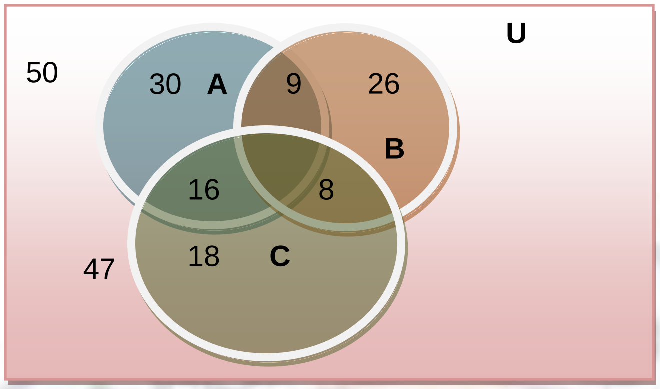
***= 467***

## Example2:

Counting the number of elements in an Intersection:

In a class of students undergoing a computer course the following were observed.
Out of a total of 50 students:
30 know Pascal,
18 know Fortran,
26 know COBOL,
9 know both Pascal and Fortran,
16 know both Pascal and COBOL,
8 know both Fortran and COBOL,
47 know at least one of the three languages.
From this we have to determine
a. How many students know none of these languages ?
b. How many students know all three languages ?

Consider the following **Venn diagram** for the given problem:




The shaded colored portion represents the |A U B U C| = 47

a. We know that 47 students know at least one of the three languages in the class of 50. The number of students who do not know any of three languages is given by the difference between the number of students in class and the number of students who know at least one language. Hence 50 - 47 = 3

b. Let us assume that A = All the students who know Pascal in class.
B = All the students who know COBOL in the class.
C = All the students who know FORTRAN in class.

By applying the Inclusion exclusion principle,

***n(AUBUC) = n(A) + n(B) + n(C) -n(A⋂B) -n(A⋂C) -n(B⋂C)+n(A⋂B⋂C)***

By Substituting the values.

**47 = 30 + 26 + 18 -9 -16 -8 + *n(A⋂B⋂C)***

Solving for ***n (A⋂B⋂C),*** we get.

***n (A⋂B⋂C)* = 6**

Hence there are 6 students in the class who know all the three languages.

| **Value addition: PRACTICE EXERCISES** |
|---|
| **More Advanced Problems** |
| **1.** How many integers from 1 through 999 do not have any repeated digits?<br><br>**2.** In a certain state, license plates consist of from zero to three letters followed by from zero to four digits, with the provision, however, that a blank plate is not allowed.<br>a. How many different license plates can the state produce? |

b. Suppose 85 letter combinations are not allowed because of their potential for giving offense. How many different license plates can the state produce?

**ANSWERS**

1. Number of integers from 1 through 999 with no repeated edges = Number of integers from 1 with no repeated digits + number of digits from 10 through 99 with no repeated digits + Number of integers from 100 through 999 with no repeated digits.

= 9 + 9. 9 + 9. 9. 8 = 9 + 81 + 648 = 738

2.

a. There are $1 + 26 + 26^2 + 26^3$ arrangements of 0 through 3 letters of the alphabet. Any of those may be paired with an arrangement of from 0 through 4 digits, and there are $(1 + 10 + 10^2 + 10^3 + 10^4)$ arrangements of from 0 through 4 digits. So by the multiplication rule and the difference rule, there are :

$(1 + 26 + 26^2 + 26^3) . (1 + 10 + 10^2 + 10^3 + 10^4) - 1 =$ 203,097,968

b. $(1 + 26 + 26^2 + 26^3 - 85) . (1 + 10 + 10^2 + 10^3 + 10^4) - 1 =$ 202,153,533

**Source: http://cpsc.ualr.edu/srini/DM/chapters/examples/ans3.6.1.html**

**Example**

Suppose there is a deck of *n* cards, each card is numbered from 1 to *n*. Suppose a card numbered *m* is in the correct position if it is the $m^{th}$ card in the deck. How many ways, *W*, can the cards be shuffled with at least 1 card being in the correct position?

Begin by defining set $A_m$, which is all of the orderings of cards with the $m^{th}$ card correct. Then the number of orders, *W*, with *at least* one card being in the correct position, *m*, is

$$W = \left| \bigcup_{m=1}^{n} A_m \right|.$$

Apply the principle of inclusion-exclusion,

$$
\begin{aligned}
W = & \sum_{m_1=1}^{n} |A_{m_1}| \\
& - \sum_{m_1,m_2\,:\,1\le m_1<m_2\le n} |A_{m_1} \cap A_{m_2}| \\
& + \sum_{m_1,m_2,m_3\,:\,1\le m_1<m_2<m_3\le n} |A_{m_1} \cap A_{m_2} \cap A_{m_3}| \\
& - \cdots \\
& + (-1)^{p-1} \sum_{m_1,\ldots,m_p\,:\,1\le m_1<\cdots<m_p\le n} \left|A_{m_1} \cap \cdots \cap A_{m_p}\right| \\
& \cdots .
\end{aligned}
$$

Each value $A_{m_1} \cap \cdots \cap A_{m_p}$ represents the set of shuffles having *p* values $m_1$, ..., $m_p$ in the correct position. Note that the number of shuffles with *p* values correct only depends on *p*, not on the particular values of *m*. For example, the number of shuffles having the 1st, 3rd, and 17th cards in the correct position is the same as the number of shuffles having the 2nd, 5th, and 13th cards in the correct positions. It only matters that of the n cards, 3 were chosen to be in the correct position. Thus there are $\binom{n}{p}$ terms in each summation.

W is the number of orderings having *p* elements in the correct position, which is equal to the number of ways of ordering the remaining $n - p$ elements, or $(n - p)!$. Thus we finally get:

$$W = \binom{n}{1}(n-1)!$$
$$- \binom{n}{2}(n-2)!$$
$$+ \binom{n}{3}(n-3)!$$
$$- \cdots$$
$$+ (-1)^{p-1}\binom{n}{p}(n-p)!$$
$$\cdots$$
$$W = \sum_{p=1}^{n}(-1)^{p-1}\binom{n}{p}(n-p)!.$$

Noting that $\binom{n}{p} = \frac{n!}{p!(n-p)!}$, this reduces to

$$W = \sum_{p=1}^{n}(-1)^{p-1}\frac{n!}{p!}.$$

A permutation where *no* card is in the correct position is called a derangement. Taking *n*! to be the total number of permutations, the probability *Q* that a random shuffle produces a derangement is given by

$$Q = 1 - \frac{W}{n!} = \sum_{p=0}^{n}\frac{(-1)^p}{p!},$$

the Taylor expansion of $e^{-1}$. Thus the probability of guessing an order for a shuffled deck of cards and being incorrect about every card is approximately 1/*e* or 36%.

| **Value addition: Did you Know** |
| --- |
| **Applications of Inclusion and Exclusion Principle** |
| In many cases where the principle could give an exact formula (in particular, counting prime numbers using the sieve of Eratosthenes), |

the formula arising doesn't offer useful content because the number of terms in it is excessive. If each term individually can be estimated accurately, the accumulation of errors may imply that the inclusion–exclusion formula isn't directly applicable. In number theory, this difficulty was addressed by Viggo Brun. After a slow start, his ideas were taken up by others, and a large variety of sieve methods developed. These for example may try to find upper bounds for the "sieved" sets, rather than an exact formula.

## Counting derangements

A well-known application of the inclusion–exclusion principle is to the combinatorial problem of counting all derangements of a finite set. A *derangement* of a set *A* is a bijection from *A* into itself that has no fixed points. Via the inclusion–exclusion principle one can show that if the cardinality of *A* is *n*, then the number of derangements is [*n*! / *e*] where [*x*] denotes the nearest integer to *x*.

This is also known as the subfactorial of *n*, written !*n*. It follows that if all bijections are assigned the same probability then the probability that a random bijection is a derangement quickly approaches 1/*e* as *n* grows.

## Counting intersections

The principle of inclusion–exclusion, combined with de Morgan's theorem, can be used to count the intersection of sets as well. Let $\overline{A}_k$ represent the complement of $A_k$ with respect to some universal set *A* such that for each *k*. Then we have

$$\bigcap_{i=1}^{n} A_i = \overline{\bigcup_{i=1}^{n} \overline{A}_i}$$

thereby turning the problem of finding an intersection into the problem of finding a union.

**Source: http://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle**

## Summary

- A well-defined collection of definite objects is called a set.
- In roster method of representing a set, all the elements are listed in the set.
- In property method of representing a set, all the elements are represented by stating all the properties which are satisfied by the elements of the set and not by any other element.
- A set is said to be a

- finite set if it contains only finite number of elements.
- infinite set if it contains infinitely many elements.
- null set if it does not contain any element.
- singleton set if it contains only one element.

- Two sets are said to be

- Equivalent sets if the elements of one set can be put in one-one correspondence with the elements of the other set.
- Equal sets if every element of one set is in the other set and vice-versa

- A set A is said to be a subset of set B if every element of A is an element of B. If A is subset of B, then it is expressed as $A \subseteq B$.

$\phi \subseteq A$, for any set A.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

$A \subseteq A$.

If there are n elements in the set A, then it has $2^n$ subsets.

The set of all subsets of a set is called its power set.

- A set A is said to be a proper subset of set B if A is a subset of B and A is not equal to B. If A is a proper subset of B, then we write $A \subset B$.

In order to show that $A \subset B$ it is sufficient to show that each element of A is in B and there is at least one element in B, which is not in A.

- **Set Operations**

- The union of two sets A and B is defined as the set of all those elements which are in either A or B or both.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

If $x \notin A$ and $x \notin B$, then $x \notin A \cup B$.

- The intersection of two sets A and B is defined as the set of all those elements which are in both A and B.

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

If $x \notin A$ or $x \notin B$, then $x \notin A \cap B$.

If $A \cap B = \phi$, then A and B are called disjoint sets.

- The difference of two sets A and B, in this order, is the set of all those elements of A which are not in B.

Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$

If $x \notin A$ or $x \in B$, then $x \notin A - B$.

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$

- The symmetric difference of two sets A and B is defined as the union of the sets A - B and B - A.

Symbolically, $A \Delta B = (A - B) \cup (B - A)$

If $x \in A$ and $x \in B$, then $x \notin A \Delta B$.

- **Important results**

$A \cup A = A, \; A \cup \phi = A$

$A \cap A = A, \; A \cap \phi = \phi$

$A - A = \phi, \quad A - \phi = A, \; \phi - A = \phi$

$A \Delta A = \phi, \quad A \Delta \phi = A, \; \phi \Delta A = A$

$A \cup B = B \cup A$

$A \cap B = B \cap A$

$A - B \neq B - A$

$A \Delta B = B \Delta A$

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

$(A - B) - C \neq A - (B - C)$

$(A \Delta B) \Delta C = A \Delta (B \Delta C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A - (B \cup C) = (A - B) \cap (A - C)$

$A - (B \cap C) = (A - B) \cup (A - C)$

- If A is a subset of universal set X, then complement of A is defined as the set of all those elements of X which are not in A and it denoted by A' or by $A^c$. We have A' = X - A.

Symbolically,

$$A' = \{x : x \in X \text{ and } x \notin A\}$$

If $x \in A$, then $x \notin A'$

$\phi' = X,\ X' = \phi$

$A \cup A' = X,\ A \cap A' = \phi$

- **De Morgan's laws**

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

- Let A, B be finite sets. Then

$A \cap B = \phi \Rightarrow n(A \cup B) = n(A) + n(B)$

$$A \cap B \neq \phi \Rightarrow \begin{cases} n(A \cup B) = n(A) + n(B) - n(A \cap B) \\ n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A) \\ n(A) = n(A - B) + n(A \cap B) \\ n(B) = n(B - A) + n(A \cap B) \end{cases}$$

- Let A,B,C be finite sets.

We have $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$- n(C \cap A) + n(A \cap B \cap C)$

- **Mathematical Induction**

The principle of mathematical induction is used to prove that a given proposition (formula, equality, inequality…) is true for all positive integer numbers greater than or equal to some integer N.

Let us denote the proposition in question by P (n), where n is a positive integer. The proof involves two steps:

Step 1: We first establish that the proposition P (n) is true for the lowest possible value of the positive integer n.

Step 2: We assume that P (k) is true and establish that P (k+1) is also true

- **The Inclusion Exclusion Principle for Two or Three Sets**

Let ***P, Q*** be finite sets, then

***n (PUQ) =n(P) + n(Q) - n(P⋂ Q) or |P U Q| = |P| + |Q| - |P⋂Q|***

and for three sets ***P, Q*** and ***R***

***n(P U Q U R) = n(P) + n(Q) + n(R) - n(P⋂ Q) - n(P⋂ R) - n(Q⋂ R) +n(P ⋂ Q ⋂ R)***

***or***

***|P U Q U R| = |P| + |Q| + |R| - |P⋂ Q| - |P⋂ R| - |Q⋂ R| +|P ⋂ Q ⋂ R|***

In general if $A_1$, ..., $A_n$ are finite sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i,j\,:\,1\le i<j\le n} |A_i \cap A_j| + \sum_{i,j,k\,:\,1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \ \cdots \ + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

where $|A|$ denotes the cardinality of the set $A$.

## Exercise

1. Show that A ∪ (B∩C) = (A∪B) ∩ (A∪C) for the following sets.

   A={3,6,9,15,17} B={12,13,14,19,21} C={3,7,8,10,12}.

2. Proof the A∪(B∩C)=(A∪B) ∩(A∪C) condition for the sets A={3,6,7,9,10} B={1,4,5,7,8} C={3,6,8,9,10}.
3. Definition of a countable set.
4. Prove (A∩B)'=A'∪B'.
5. Let (U is the Universe) U={1,2,…,8} and let A={2,4,6}, B={1,3,5,7} and C={4,5,6}. Find the following.

   (a) A Δ B =

   (b) (B – A)' =

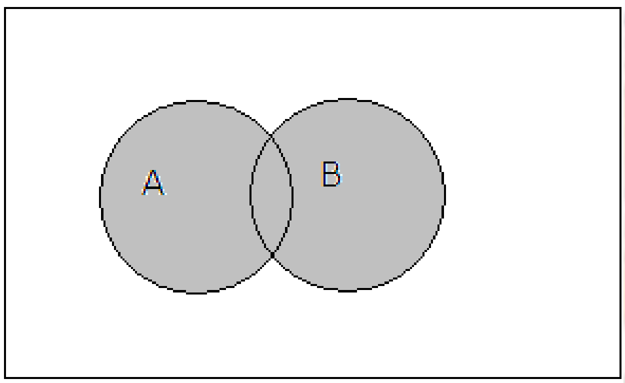
   (c) (AUC)-B=

   (d) (A-C) ∩ (B-A) =

6. State the distributive law of union over intersection. State the associative law of intersection.
7. How many subsets are there for a set with four elements and a set with three elements?
8. Define Cardinality of a Set A.

9. How many elements are there in Null Set or Empty Set or Void Set ?

10. Define Power sets
11. For all sets A,B; A $\subseteq$ B exactly if A___B = B.
12. Sets A,B are disjoint exactly if A__ B = {}
13.




In the above Venn diagram what is represented by the shaded portion

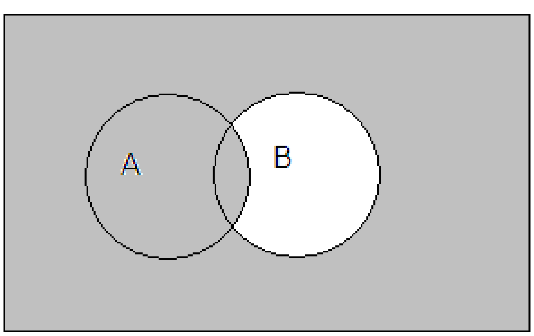
14. Out of 120 students who secured first class marks in Mathematics or English, 70 obtained first class in Mathematics and 31 in English and Mathematics. How many students secured first class marks in English only?

15. If sets A = {1, 2, 3} and B= {2, 3, 4}, then what will be symmetric difference set?

16. If sets A = {1, 2, 3} and B= {2, 3, 4}, then what will be symmetric difference set?

17. If r is a real number not equal to 1, then for every n$\geq$0, Prove by Mathematical Induction $r_0+r^1+\ldots+r^n = (1 - r^{n+1}) / (1 - r)$

18.




In the above Venn diagram what is represented by the shaded portion

19. Prove 1 + 2 + 3 + ... + n = n (n + 1) / 2 for all positive integers n

20. The inequality $2^n > n^2$ is true, whenever n is an integer greater than.....?

21. Use Mathematical Induction to prove $1+1/4+1/9+1/16+......1/n^2 <.2-1/n^2$; whenever n is a positive integer greater than 1.

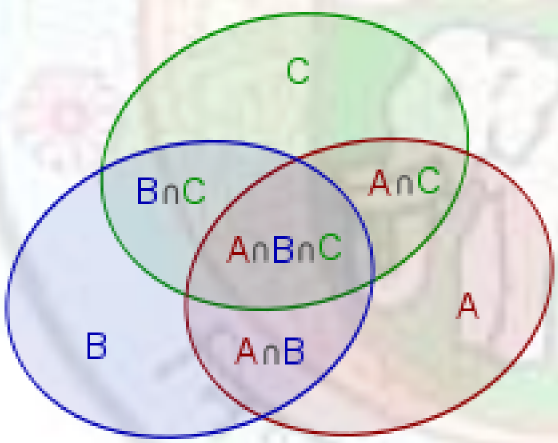
22. What is implied by the Well-Ordering Principle?

23. For every positive integer n there exist an n-digit number divisible by $X^n$ all of whose digits are odd. What is this X?

24. In combinatorial mathematics, the inclusion–exclusion principle is also known as __________ ?

25. The students in a dormitory were asked whether they had a dictionary (D) or a thesaurus (T) in their rooms. The result showed that 650 students had a dictionary, 150 did not have a dictionary, 175 had a thesaurus, and 50 had neither a dictionary nor a thesaurus. Find the number k of students who live in the dormitory?

26. Which Principal is represented in the given Venn diagram




27. Suppose that 100 of the 120 mathematics students at a college takes at least one of the languages French, German, and Russian. Also suppose

| | |
|---|---|
| 65 study French | 20 study French and German |
| 45 study German | 25 study French and Russian |
| 42 study Russian | 15 study German and Russian |

Find the number of student who study all the 3 languages?

28. In a class of students undergoing a computer course the following were observed.
Out of a total of 50 students:
30 know Pascal,
18 know Fortran,

26 know COBOL,
9 know both Pascal and Fortran,
16 know both Pascal and COBOL,
8 know both Fortran and COBOL,
47 know at least one of the three languages.

From this we have to determine: How many students know none of these languages?

29. In a certain state, license plates consist of zero to three letters followed by zero to four digits, with the provision, however, that a blank plate is not allowed. Suppose 85 letter combinations are not allowed because of their potential for giving offense. How many different license plates can the state produce?

30. How many integers from 1 through 999 do not have any repeated digits?

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