Lesson: Algebra of Continuous Functions

Paper: Analysis - II

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Algebra of Continuous Functions

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1. Learning Outcomes

After studying this chapter you should be able to

- Have a deep understanding of continuous functions.
- Understand Algebra of continuous functions.
- Gain an understanding of Properties of algebra of continuous functions.
- Will be able to solve questions related to them.

"Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them....."

Henri Poincare

2. Introduction

The idea of a function has an irregular history. Until the time of L. Euler, functions were thought of largely in terms of formulas and thus were restricted to expressions that can be generated by elementary operations. The modern view of a function as a general assignment was first seen in Euler's later work and particularly in the subsequent work of Cauchy and P. Dirichlet. A function f from A to B is a rule that assigns to each element in A one and only one element in B. We write $f : A \to B$ where A and B are nonempty sets.

A graph is a function if it passes the vertical line test. i.e., if a vertical line is drawn anywhere on the co-ordinate plane, it crosses the graph only once.

If a vertical line goes through the graph more than once, that means there is more than one y-value for the x-value. So if a graph fails the vertical line test, it is not a function.

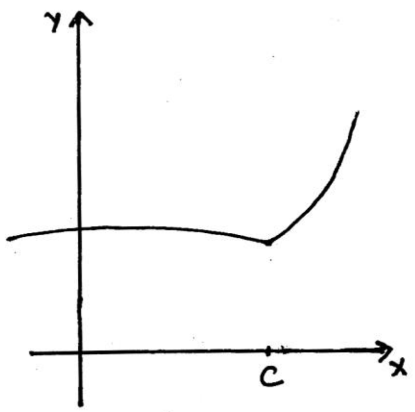
A continuous function is a function in which for "small" changes in the input result in "small" changes in the output. Continuity was first defined by Bernard Bolzano in 1817. Augustin-Louis Cauchy defined continuity for $y = f(x)$ by saying for an infinitely small increment α of the independent variable x always produces an infinitely small change $f(x + \alpha) - f(x)$ of the dependent variable y. Cauchy called "infinitely small quantities" as variable quantities.

He introduced the concept of continuous functions by requiring that indefinite small changes of x should produce indefinite small changes in y. But Bolzano (1817) and Weierstrass (1874) were more precise by saying the difference $f(x) - f(x_0)$ must be arbitrarily small for the difference $x - x_0$ sufficiently small. Thus the concept of continuity is of great important in Analysis.

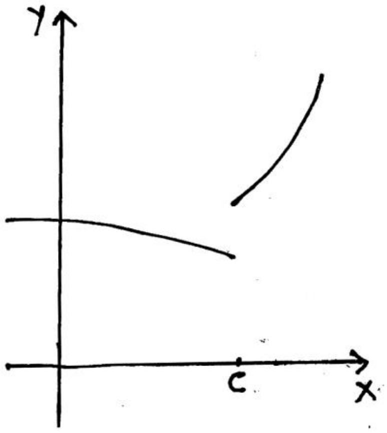
Algebra of Continuous Functions

Consider the graphs of the functions f and g. At the point $x = c$, there is a significant difference. The graph of g has a break at c, whereas the graph of f can be drawn without the pencil leaving the paper. This distinction, stated geometrically, can be made precise if we focus on the functions involved and not their graphs.

The values of the function f are all near $f(c)$ for x near c, i.e., $\lim_{x \to c} f(x)$ exists and equals $f(c)$. However, this is not true for the function g. For some x near c (either for any $x < c$ or $x > c$) the values of g are not near $g(c)$. We do not have $\lim_{x \to c} g(x) = g(c)$. In fact, $\lim_{x \to c} g(x)$ does not exist. The function g is not continuous at the point c.



$y = f(x)$



$y = g(x)$

I.Q. 1

I.Q. 2

3. Continuous Functions:

Definition: Let f be a function defined in a neighbourhood of the point c. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

Algebra of Continuous Functions

In practice, most functions of a real variable have domains that are intervals or unions of separate intervals, thus we have to consider three type of points; interior points, left end points and right end points.

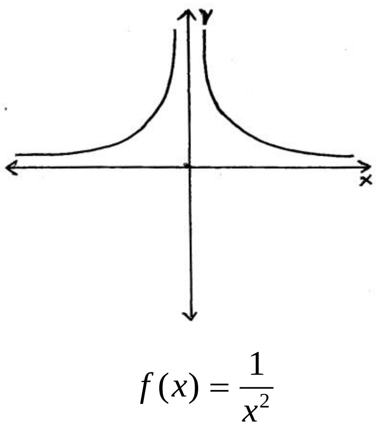
In general, a function *f* is continuous at *a* left end point *a* of its domain if it is right continuous at *a* and continuous at *a* right end point *b* of its domain if it is left continuous at *b*. A function is continuous at an interior point *c* of its domain if and only if it is both right continuous and left continuous at *c*.

Value addition : Example

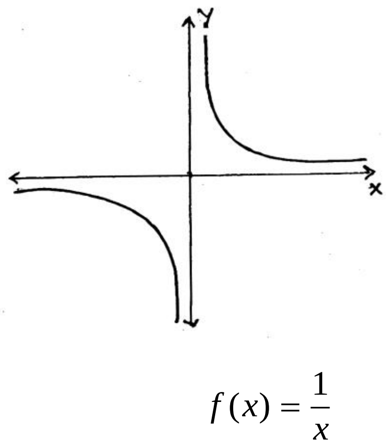
(1) The function $f(x)=\frac{1}{x^2}$ is not continuous at *x* = 0, because $f(x)$ is not defined at *x* = 0. Moreover, this point of discontinuity cannot be removed, since we have

$$\lim_{x\to c}\frac{1}{x^2}=+\infty$$

(2) The function $f(x)=\frac{1}{x}$ is not continuous at *x* = 0 and its discontinuity at *x* = 0 cannot be removed, because $\lim_{x\to 0^-}\frac{1}{x}=-\infty$ and $\lim_{x\to 0^+}\frac{1}{x}=-\infty$.



$f(x)=\frac{1}{x^2}$



$f(x)=\frac{1}{x}$

I.Q. 3

Algebra of Continuous Functions

3.1 Continuity of a Function at a Point

The notion of continuity is one of the central concepts of Mathematical Analysis. We will define what it means to say that a function is continuous at a point. It is known as an $\varepsilon - \delta$ criterion.

Definition: Let $A \subseteq R, f : A \to R$ and $c \in A$. Then function *f* is said to be continuous at *c* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

for $x \in A$ and $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

If *f* fails to be continuous at *c*, then we say that *f* is discontinuous at *c*.

Value Addition: Note
(1) In the above definition *c* must be in *A* but need not be a cluster point of *A*. This $\lim_{x \to c} f(x)$ need not exist, even when *f* is continuous at *c*. Consider the function $f : A \to R$ defined by $f(x) = \sqrt{x^3 - x^2}$ where $A = \{0\} \cup [1, \infty)$ is the domain of *f. f* is continuous at 0, but $\lim_{x \to 0} f(x)$ does not exist. (2) In case *c* is a cluster point of *A*, then definition 2.2 is equivalent to (i) $f(c)$ exists (ii) $\lim_{x \to c} f(x)$ exists and (iii) $\lim_{x \to c} f(x) = f(c)$

I.Q. 4

I.Q. 5

In fact we say that

If *c* is a cluster point of *A*, then *f* is continuous at *c* iff

$$\lim_{x \to c} f(x) = f(c)$$

3.2 Another Definition of Continuity

The definition of continuity at a point can be reformulated in terms of neighbourhoods.

The function *f* is continuous at *c* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$x \in A$ and $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

i.e., $x \in A$ and $c-\delta < x < c+\delta \Rightarrow f(c)-\varepsilon < f(x) < f(c)+\varepsilon$

i.e., $x \in A$ and $x \in]c-\delta, c+\delta[\Rightarrow f(x) \in]f(c)-\varepsilon, f(c)+\varepsilon[$

i.e., $x \in A \cap]\ c-\delta, c+\delta[\Rightarrow f(x) \in]f(c)-\varepsilon, f(c)+\varepsilon[$

Thus in other words,

Let $A \subseteq R$, $f : A \to R$ and $c \in A$. Then function *f* is continuous at c iff for every $\varepsilon-nbd\ V_\varepsilon(f(c))$ of $f(c)$, there exists $\delta-nbd\ V_\delta(c)$ of c such that

$$x \in A \cap V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$$

i.e., $f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$

Value Addition : Note
If $c \in A$ is not a cluster point of *A*, then there exists a neighbourhood $V_\delta(c)$ of *c* such that $A \cap V_\delta(c) = \{c\}$. Then the function will be automatically continuous at $c \in A$. Such points are called Isolated Points. For the function $f : A \to R$ defined by $f(x) = \sqrt{x^3 - x^2}$. where $A = \{0\} \cup [1, \infty)$ is the domain of *f*. Here, 0 is an isolated point of the domain of *f*.

Example 1: Consider the function *f* : *R* → *R*, defined by

$$f(x)=\begin{cases} x\sin\dfrac{1}{x} & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$$

Solution: Here $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} x\sin\dfrac{1}{x}=0=f(0)$

∴ the function *f* is continuous at 0.

Example 2: Let $f(x)=\operatorname{sgn}(x)\ \forall\ x\in R$. Where sgn is the signum function

Show that *f* is not continuous at 0.

Solution: We have

$$f(x)=\begin{cases} 1 & , \quad x>0 \\ 0 & , \quad x=0 \\ -1 & , \quad x<0 \end{cases}$$

Here $\lim\limits_{x\to 0^-} f(x)=-1$

$\lim\limits_{x\to 0^+} f(x)=1$

∴ $\lim\limits_{x\to 0^+} f(x)\neq \lim\limits_{x\to 0^-} f(x)$

∴ $\lim\limits_{x\to 0} f(x)$ does not exist.

⇒ *f* is not continuous at *x* = 0.

Example 3: The function *f* : *R* → *R*, defined by *f* (*x*) = [*x*] is called the greatest integer function. Check the continuity of *f* (*x*).

Solution: We define [*x*] to be the greatest integer *n* ∈ *Z* such that *n* ≤ *x*.

Let z be any integer.

$$\lim_{x \to z^-} f(x) = z - 1$$

Also $\lim_{x \to z^+} f(x) = z$

$\Rightarrow \quad \lim_{x \to z^-} f(x) \neq \lim_{x \to z^+} f(x)$

$\Rightarrow \quad \lim_{x \to z} f(x)$ does not exist

$\therefore \quad$ *f* is not continuous at any integer. But when *x* is not an integer, i.e., *x* is a real number then $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

I.Q. 6

Example 4: Determine the points of discontinuity of $f(x) = [\sin x], 0 \leq x \leq 2\pi$

Solution:

Here $f(x) = \begin{cases} 0 \ , & 0 \leq x < \pi/2 \\ 1 \ , & x = \pi/2 \\ 0 \ , & \pi/2 < x \leq \pi \\ -1 \ , & \pi < x \leq 3\pi/2 \\ -1 \ , & 3\pi/2 < x < 2\pi \end{cases}$

This function is discontinuity at 0, $\pi/2$, π, 2π, $5\pi/2$, ……

Example 5: Determine the points of continuity of the following functions:

(i) $\quad f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 1} \ \forall \ x \in R$

$$= \frac{(x+1)^2}{x^2 + 1}$$

It is continuous everywhere on *R*.

(ii) $h(x) = \sqrt{1 + \frac{|\sin x|}{x}},\ x \neq 0$

Solution: $h(x) = \frac{\sqrt{1+|\sin x|}}{x} = \frac{f(x)}{g(x)}$ where $x \neq 0$.

$$|\sin x| = \begin{cases} \sin x & ; \quad x \geq 0 \\ -\sin x & ; \quad x < 0 \end{cases}$$

Here $g(x) = 0$ for $x = 0$ and $g(x) \neq 0$; $x \neq 0$

$\frac{f(x)}{g(x)}$ is defined $\forall \quad x \neq 0$

sin x is continuous $\Rightarrow$ |sin x| is continuous

$\Rightarrow \qquad 1 + |\sin x|$ is continuous.

Also $g(x) = x$ is continuous.

Therefore, $h(x) = \frac{f(x)}{g(x)}$ is continuous $\forall \quad x \neq 0$.

(iii) $k(x) = \cos\sqrt{1+x^2}\ (x \in R)$

Solution: cos and $\sqrt{1+x^2}$ are continuous functions.

Therefore, $\cos\sqrt{1+x^2}$ is also a continuous function.

Example 6: Show that if $f : A \rightarrow R$ is continuous on $A \subseteq R$ and if $n \in N$, then the function f^n defined by $f^n(x) = (f(x))^n$ for $x \in A$, is continuous on A.

Solution: Here f is continuous on A

For $\quad f^2(x) = (f\ f)(x)$

$$= f(x)f(x) = (f(x))^2$$

If *f* is continuous so f^2 will also be continuous.

Similarly, $f^3, f^4, \ldots$ will also be continuous.

$\Rightarrow$ $f^n(x) = (f(x))^n$ will also be continuous.

I.Q. 7

3.3 Sequential Criteria for continuity:

Theorem 1: A function $f : A \to R$ is continuous at a point c $\in$ A iff for every sequence (x_n) in *A* that converges to *c*, the sequence $f(x_n)$ converges to $f(c)$.

Proof: Necessary part

Suppose *f* is continuous at *c* and let the sequence (x_n) in *A* be such that it converges to *c*.

Since *f* is continuous at *c*, therefore for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in A$ and $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

(1)

Now the sequence (x_n) converges to *c*, therefore for $\delta > 0$, there exists a +ve integer *m* such that $|x_n - c| < \delta \ \forall \ n \geq m$

(2)

From (1) and (2) we have

$|f(x_n) - f(c)| < \varepsilon \ \forall \ n \geq m$ (replacing *x* by x_n in (1))

$\Rightarrow$ The sequence $(f(x_n))$ converges to $f(c)$.

Sufficient part

Suppose that for every sequence (x_n) in *A* that converges to *c*, the sequence $f(x_n)$ converges to $f(c)$.

We have to prove that *f* is continuous. On contrary, assume that *f* is not continuous at *c*.

Then there exist an $\varepsilon > 0$ for which no δ works.

i.e., for $\delta_n = \frac{1}{n},\ n \in N,\ \exists\ y_n \in A$ s.t

$$|y_n - c| < \frac{1}{n} \text{ but } |f(y_n) - f(c)| \geq \varepsilon \quad \forall\ n$$

This implies that the sequence (y_n) in *A* is such that $\lim_{n\to\infty} y_n = c$ $\left[\text{as} \lim_{n\to\infty} \frac{1}{n} = 0\right]$ but the sequence $(f(y_n))$ does not converge to $f(c)$ which is a contradiction.

Therefore *f* is continuous.

Example 7: Show that the Dirichlet's functions is discontinuous at every real number.

Solution: Let $f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$

Let c be a real number.

Case-I: Let *c* is a rational number then

$$f(c) = 1$$

By density theorem, there exists a sequence (x_n) of irrational numbers such that $\lim_{n\to\infty} x_n = c$

Here $f(x_n) = 0 \quad \forall\ n$

Thus, $\exists$ a sequence (x_n) such that

$$\lim_{n\to\infty} x_n = c$$

but $\lim_{n\to\infty} f(x_n) \neq f(c)$

By sequential criteria, we have that the function *f* is not continuous at *c*.

Case-II: Let *c* be an irrational number. Then *f* (*c*) = 0

By density theorem, $\exists$ a sequence (y_n) of rational numbers such that

$$\lim_{n\to\infty} y_n = c$$

Here $f(y_n) = 1 \ \forall \ n$

$$\lim_{n\to\infty} f(y_n) = 1 \neq f(c)$$

Thus, $\exists$ a sequence (y_n) s.t

$$\lim_{n\to\infty} y_n = c$$

but $\lim_{n\to\infty} f(y_n) \neq f(c)$

By sequential criteria, the function is not continuous at *c*.

Thus from Case I and Case II, we have that function *f* is not continuous at any real number.

Example 8: Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

Use the $\varepsilon - \delta$ definition to show that *f* is continuous at 0.

Solution: Let $\varepsilon > 0$

Now $\left|f(x)-f(0)\right|=\left|f(x)\right|\le x^2 \ \forall \ x$

We want this to be less than ε, so we let $\delta=\sqrt{\varepsilon}$.

Then $|x-0|<\delta$ implies $x^2<\delta^2=\varepsilon$

$\therefore \quad |x-0|<\delta \Rightarrow \left|f(x)-f(0)\right|<\varepsilon$

Hence ε – δ property holds and *f* is continuous at 0.

We can also show that if $\lim(x_n)=0$ for every seq (x_n) then lim $f(x_n)$ = 0

Thus sequential criteria can also be used to prove that *f* is continuous at 0.

Example 9: Prove that the function $\sqrt{x}$ is continuous on its domain $[0,\infty)$.

Solution: Let $f(x)=\sqrt{x}$, $x\ge 0$

The function *f* is continuous at *c*

$\therefore$ $c \ge 0$.

Let $\varepsilon > 0$ be arbitrary.

Case-I: When $c = 0$,

then $\left|f(x)-f(c)\right|=\left|\sqrt{x}-0\right|$

$\left|\sqrt{x}-0\right|<\varepsilon$ if $0\le x<\varepsilon^2$

Let $\delta = \varepsilon^2$

Then $0\le x<\varepsilon^2 \Rightarrow \left|\sqrt{x}-0\right|<\varepsilon$

This shows that the function *f* is continuous at 0.

Case-II: When $c \neq 0$

then $\left|f(x) - f(c)\right| = \left|\sqrt{x} - \sqrt{c}\right| = \left|\dfrac{x-c}{\sqrt{x}+\sqrt{c}}\right| \leq \dfrac{|x-c|}{\sqrt{c}}$

$\Rightarrow$ $\left|f(x) - f(c)\right| < \varepsilon$ if $|x-c| < \sqrt{c}\ \varepsilon$

take $\delta = \sqrt{c}\ \varepsilon$

then $|x-c| < \delta \Rightarrow \left|\sqrt{x} - \sqrt{c}\right| < \varepsilon$

This implies that the function $\sqrt{x}$ is continuous at *c*.

4. Algebra of Continuous Functions

Let $A \subseteq R$ and let *f* and *g* be functions defined on *A* to *R*. We define the sum *f* + *g*, the difference *f* – *g*, and the product *fg* on *A* to *R* to be the functions given by

$$(f+g)(x) = f(x) + g(x), \text{ for all } x \in A$$

$$(f-g)(x) = f(x) - g(x), \text{ for all } x \in A$$

$$(fg)(x) = f(x)g(x), \text{ for all } x \in A$$

Further, if $b \in R$, we define the multiple *bf* to be the function given by

$$(bf)(x) = bf(x) \text{ for all } x \in A.$$

Also, if $h(x) \neq 0$ for $x \in A$, we define the quotient $\dfrac{f}{h}$ to be the function given by $\left(\dfrac{f}{h}\right)(x) = \dfrac{f(x)}{h(x)}$ for all $x \in A$

Value addition
If *f* and *g* are functions, both bounded on *A* and *c* is any real

number, then the function $f + g$, $f - g$, cf and $f\,g$ are each bounded on A.

I.Q. 8

Theorem 2: Suppose the function f and g are continuous at the point c. Then the functions $f + g$, $f - g$, $f\,g$ are continuous at c.

Proof: If c $\in$ A is not a cluster point of A, then the conclusion is obvious. Hence we assume that c is a cluster point of A. Since f and g are continuous at c, then

$$f(c) = \lim_{x \to c} f(x)$$

and $$g(c) = \lim_{x \to c} g(x)$$

We have $\lim_{x \to c} (f + g)(x) = \lim_{x \to c} \left[f(x) + g(x) \right]$

$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$= f(c) + g(c)$$

$$= (f + g)(c)$$

$$\therefore \quad \lim_{x \to c} (f + g)x = (f + g)(c)$$

Thus $f + g$ is continuous at c.

In a similar manner we can show that $f - g$ is continuous at c.

Again, $\lim_{x \to c} (f\,g)(x) = \lim_{x \to c} f(x) g(x)$

$$= \lim_{x \to c} f(x) \lim_{x \to c} g(x)$$

$$= f(c)\; g(c)$$

$= f\,g\,(c)$

Thus we have $\lim_{x \to c} (f\,g)(x) = f\,g(c)$

$\Rightarrow$ The function $f\,g$ is continuous at c.

Converse of the above theorem need not be true.

For Example:

(i) Let $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

and

$$g(x) = \begin{cases} \dfrac{1 - e^{\frac{1}{x}}}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then $(f + g)(x) = 0$ for all x

$\therefore$ $f + g$ is continuous at 0 (being a constant function)

but the functions f and g are discontinuous at $x = 0$

(ii) Let $f(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$

and $g(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$

then $fg\,(x) = 1$ for all $x \in R$.

Thus the function $f\,g$ is continuous at every point of R, but the functions f and g are discontinuous at every point of R.

I.Q. 9

I.Q. 10

Theorem 3: If *f* is continuous at a point *c* and *b* $\in$ *R* then *bf* is continuous at *c*.

Proof: As *f* is continuous at *c*, therefore we have $\lim_{x \to c} f(x) = f(c)$

Now $\lim_{x \to c}(b\,f)(x) = b \lim_{x \to c} f(x) = bf(c)$

$\Rightarrow$ *b f* is continuous at *c*.

Theorem 4: Let *f* : *A* $\to$ *R* is continuous at *c* $\in$ *A* and *h* : *A* $\to$ *R* is continuous at *c* $\in$ *A* and if $h(x) \neq 0$ for all *x* $\in$ *A*, then the quotient $\frac{f}{h}$ is continuous at *c*.

Proof: Since *c* $\in$ *A*, then $h(c) \neq 0$

As *h* is continuous at c, therefore we have $\lim_{x \to c} h(x) = h(c)$

Now $\lim_{x \to c} \left(\frac{f}{h}\right)(x) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} h(x)}$

$= \frac{f(c)}{h(c)}$

$= \frac{f(c)}{h}$

Therefore $\frac{f}{h}$ is continuous at *c*.

Converse of the above theorem need not be true.

For Example:

Let $f(x) = h(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$

then $\frac{f}{h}(x) = 1 \quad \forall x \in R$. Hence $\frac{f}{h}$ is continuous at every point of *R*, but *f* and *g* are discontinuous at every point of *R*.

The next result is an immediate consequence of the above theorems, applied to every point of A. however, since it is an extremely important result, we will state it formally.

Theorem 5: Let $A \subseteq R$, let *f* and *g* be continuous on *A* to *R*, and let $b \in R$.

(i) The functions *f* + *g*, *f* – *g*, *f g* and *bf* are continuous on *A*.

(ii) If h : A → R is continuous on A and $h(x) \neq 0$ for $x \in A$, then the quotient $\frac{f}{h}$ is continuous on A.

I.Q. 11

I.Q. 12

Example 10: All polynomials $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ are continuous at each point of *R*.

Solution: Every polynomial *p* is a function on *R* into *R*.

Now $\lim_{x \to c} p(x) = \lim_{x \to c} \left[a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \right]$

$$= \lim_{x \to c} \left(a_nx^n \right) + \lim_{x \to c} \left(a_{n-1}x^{n-1} \right) + \ldots + \lim_{x \to c} \left(a_1x \right) + \lim_{x \to c} a_0$$

We know that constant functions are continuous everywhere and the identity function is also continuous everywhere

Therefore if $f(x) = k$ where *k* is a constant and $g(x) = x$, then $(fg)(x) =$ *kx* and $h(x) = k_1x + k_0$ are continuous at every point of *R*. By repeatedly using the theorem, we conclude that $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ is continuous on *R*.

Example 11: If *p* and *q* are polynomials functions of *R*, then there are at most a finite number, α_1, …, α_m of real roots of *q*.

Solution: If $x \notin \{\alpha_1, \ldots, \alpha_m\}$ then $q(x) \neq 0$.

So that we can define the rational function r by $r(x) = \frac{p(x)}{q(x)}$ for $x \notin \{\alpha_1, \ldots, \alpha_m\}$

If *c* is not a zero of $q(x)$, then $q(c) \neq 0$, and it follows that $\lim_{x \to c} q(x) = q(c) \neq 0$.

Therefore $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)}$ = *r*(*c*)

⇒ *r* is continuous at *c*.

Since *c* is any real number that is not a root of *q*, we conclude that a rational function is continuous at every real number for which it is defined.

I.Q. 13

Example 12: Show that sine function is continuous on *R*.

Solution: For all *x*, *y*, *z* ∈ *R* we have

$$|\sin z| \leq |z| \text{ and } |\cos z| \leq |1|$$

and $\sin x - \sin y = 2\sin\left[\frac{1}{2}(x-y)\right]\cos\left[\frac{1}{2}(x+y)\right]$

Hence if *c* ∈ *R*, then we have

$$|\sin x - \sin c| \le 2.\frac{1}{2}|x-c|.1$$

$$= |x-c|$$

Therefore sine function is continuous at *c*.

Since $c \in R$ is arbitrary, it follows that sine function is continuous on *R*.

Example 13: Show that the cosine function is continuous on *R*.

Solution: For all *x*, *y*, $z \in R$ we have

$$|\sin z| \le |z| \text{ and } |\sin z| \le 1$$

and $$\cos x - \cos y = -2\sin\left[\frac{1}{2}(x+y)\right]\sin\left[\frac{1}{2}(x-y)\right]$$

Hence if $c \in R$, the we have

$$|\cos x - \cos c| \le 2.1.\frac{1}{2}|c-x|$$

$$= |x-c|$$

Therefore cosine function is continuous at *c*.

Since $c \in R$ is arbitrary, it follows that cosine function is continuous on *R*.

I.Q. 14

Value Addition:
The functions tan, cot, sec, cosec are continuous where they are defined. For example, $\cot x = \frac{\cos x}{\sin x}$, provided sin $x \ne 0$ (i.e., provided $x \ne n\pi, n \in Z$)

Since sin *x* and cos *x* are continuous on *R*, it follows from Theorem 4 that the function cot *x* is continuous on its domain.

Example 14: Check the continuity of $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$.

This function will be continuous at all nonzero *x* in *R*. (by Theorem 2).

Theorem 6: Let $A \subseteq R$, let *f* : *A* → *R*, and let $|f|$ be defined by $|f|x = |f(x)|$ for *x* ∈ *A*. If *f* is continuous at a point *c* ∈ *A*, then | *f* | is continuous at *c*.

Proof: Since *f* is continuous at *x* = *c*, therefore for every ε > 0, there exists a δ > 0 such that *x* ∈ *A* and $|x-c| < \delta$

$$\Rightarrow \quad |f(x) - f(c)| < \delta \qquad \ldots \quad (1)$$

Also, we have that

$$\big||f|(x) - |f|(c)\big| = \big||f(x)| - |f(c)|\big| \leq |f(x) - f(c)| \qquad \ldots \quad (2)$$

From (1) and (2) we have

$$x \in A \text{ and } |x-c| < \delta \Rightarrow \big||f|(x) - |f|(c)\big| \leq |f(x) - f(c)| < \varepsilon$$

Thus, for every ε > 0, there exists a δ > 0

s.t $x \in A$ and $|x-c| < \delta \Rightarrow \big||f|(x) - |f|(c)\big| < \varepsilon$

which shows that the function $|f|$ is continuous at *c*.

Converse of the above theorem is not true in general,

Consider a function *f* defined by

$$f(x) = \begin{cases} -1, & x < c \\ 1, & x \geq c \end{cases}$$

Then $|f(x)| = 1$

$\therefore \quad \lim_{x \to c} |f(x)| = 1 = |f(c)|$

but $\lim_{x \to c} f(x)$ does not exist at $x = c$

This shows that $|f|$ is continuous at *c* but *f* is not continuous at *c*.

Note: If *f* is continuous on *A*, then $|f|$ is continuous on *A*.

I.Q. 15

I.Q. 16

Theorem 7: Let $A \subseteq R$, *f* : *A* → *R* and $f(x) \geq 0$ for all *x* ∈ *A*. Let $\sqrt{f}$ be defined by $\left(\sqrt{f}\right)(x) = \sqrt{f(x)}$ for *x* ∈ *A*. If f is continuous at a point *c* ∈ *A*, then $\sqrt{f}$ is continuous at *c*.

Proof: Since f is continuous at c,

then $\quad f(c) = \lim_{x \to c} f(x)$

we have $\quad \lim_{x \to c} \sqrt{f}(x) = \lim_{x \to c} \sqrt{f(x)} = \sqrt{f(c)}$

$\therefore \quad \lim_{x \to c} (\sqrt{f})(x) = \sqrt{f(c)}$

Thus $\sqrt{f}$ is continuous at *c*.

Value Addition:
All these results can also be proved using sequential criteria for continuity.

I.Q. 17

Now we combine the real valued functions *f* and *g* to obtain

$$\max(f,g)(x)=\max\{f(x),g(x)\}$$

and $$\min(f,g)(x)=\min\{f(x),g(x)\}$$

The next theorem shows that these new functions are continuous if *f* and *g* are continuous.

Theorem 8: If the functions *f* and *g* are continuous at the point *c*, then $\max(f,g)$ and min (f,g) are also continuous at *c*.

Proof: We know that

$$\max(f,g)=\frac{1}{2}(f+g)+\frac{1}{2}|f-g|$$

and $$\min(f,g)=\frac{1}{2}(f+g)-\frac{1}{2}|f-g|$$

so the proof follows from theorem 2 and theorem 6.

Example 15: Let $f(x)=2x+1$ and $g(x)=x^2-2$.

Find $(f+g)(x),(f-g)(x),f(x+2),f(x)+2,g(x+2),g(x)+2,\,3f(x),$

$f(3x),3g(x),g(3x),(fg)(x),(f/g)(x),|f|(x),\max\{f,g\}(x)$ and $\min\{f,g\}(x)$

Solution: $f(x)=2x+1$

and $g(x)=x^2-2$

$$(f+g)(x) = x^2+2x-1$$

$$(f-g)(x) = -x^2+2x+3$$

$f(x+2)$ = $2x + 5$

$f(x)+2$ = $2x + 3$

$g(x+2)$ = x^2+4x+2

$g(x)+2$ = x^2

$3f(x)$ = $6x + 3$

$f(3x)$ = $6x + 1$

$3g(x)$ = $3x^2 - 6$

$g(3x)$ = $9x^2 - 2$

$(fg)(x)$ = $2x^3+x^2-4x-2$

$(f/g)(x)$ = $\dfrac{2x+1}{x^2-2}$

$|f|(x)=|2x+1|$

$$\max\{f,g\}(x)=\begin{cases} x^2-2 \text{ if } x\le -1 \text{ or } x\ge 3 \\ 2x+1 \text{ if } -1\le x\le 3 \end{cases}$$

$$\min\{f,g\}(x)=\begin{cases} 2x+1 \text{ if } x\le -1 \text{ or } x\ge 3 \\ x^2-2 \text{ if } -1\le x\le 3 \end{cases}$$

5. Composition of Continuous Functions

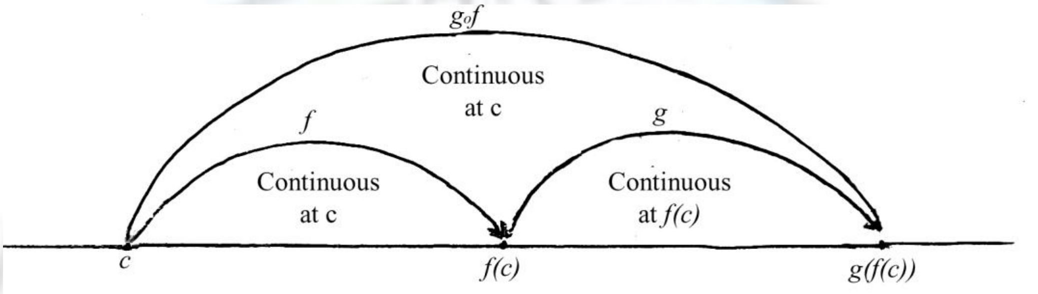
If f and g are functions, the composite function $g\circ f$ is defined by

$$(g\circ f)(x)=g\big(f(x)\big)$$

Value addition
Two functions can be composed when the range of the first lies in

the domain of the second. To find $(g\circ f)(x)$, we first find $f(x)$ and then $g(f(x))$.

We now show that the composition $g\circ f$ is continuous at c. In order to assure that $g\circ f$ is defined on all of A, we also need to assume that $f(A)\subseteq B$.



Theorem 9: Let A, $B\subseteq R$ and let $f : A \to R$, $g : B \to R$ be such that $f(A)\subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $f(c)(\in B)$. Then the composition $g\circ f : A \to R$ is continuous at c.

Proof: Let $\varepsilon > 0$ be arbitrary. Since g is continuous at $f(c)$, therefore there exists $\eta > 0$ s.t

$$y\in B, |y-f(c)|<\eta \Rightarrow |g(y)-g(f(c))|<\varepsilon \qquad \ldots \quad (1)$$

Again, as f is continuous at c, therefore for $\eta > 0$, there exists $\delta > 0$ s.t

$$x\in A, |x-c|<\delta \Rightarrow |f(x)-f(c)<\eta| \qquad \ldots \quad (2)$$

Also as $f(A)\subseteq B$, therefore $x \in A \Rightarrow f(x)\in B$ $\qquad \ldots \quad (3)$

Now, from (1), (2) and (3) we have

$$x\in A, |x-c|<\delta \Rightarrow f(x)\in B, |f(x)-f(c)|<\eta$$

$$\Rightarrow |g(f(x))-g(f(c))|<\varepsilon$$

$$\Rightarrow \left|(g \circ f)(x) - (g \circ f)(c)\right| < \varepsilon$$

Thus, $x \in A, |x - c| < \delta \Rightarrow \left|(g \circ f)(x) - (g \circ f)(c)\right| < \varepsilon$

This shows that $g \circ f$ is continuous at *c*.

Another Proof:

The above theorem can be proved by using the sequential criterion for continuity.

Let (x_n) be a sequence in *A* that converges to *c*.

Since *f* is continuous at *c*,

$$\lim_{n\to\infty} x_n = c$$

$$\Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(c)$$

Since $f(A) \subseteq B$

$$\therefore \quad f(x_n) \in B \ \forall \ n.$$

So, the sequence $(f(x_n))$ in *B* converges at *f* (*c*).

Since *g* is continuous at *f* (*c*),

$\therefore$ the sequence $(g(f(x_n)))$ converges to $g(f(c))$.

i.e., $\lim_{n\to\infty} g(f(x_n)) = g(f(c))$

$$\Rightarrow \quad \lim_{n\to\infty} (g \circ f)(x_n) = (g \circ f)(c)$$

Thus, if (x_n) is a sequence in A such that $\lim_{n\to\infty} x_n = c$, then

$$\lim_{n\to\infty} (g \circ f)(x_n) = (g \circ f)(c)$$

This shows that $g \circ f$ is continuous at *c*.

Converse of the above theorem need not be true, as can be seen from the following example.

Example 16: Let $f(x) = \begin{cases} \frac{1}{x} & , \text{ for } x \neq 0 \\ 0 & , \text{ for } x = 0 \end{cases}$

and $g(x) = f(x) \ \forall \ x \in R$

Here $(g \circ f)(x) = x$ for all $x \in R$ and hence $g \circ f$ is continuous at all $x \in R$ but *f* and *g* are not continuous at *x* = 0.

I.Q. 18

Theorem 10: Let $A, B \subseteq R$, let $f : A \to R$ be continuous on *A* and let $g : B \to R$ be continuous on B. If $f(A) \subseteq B$, then the composite function $g \circ f : A \to R$ is continuous on *A*.

Proof: The theorem follows immediately from the preceding result, if *f* and *g* are continuous at every point of *A* and *B* respectively.

Note: In general $f \circ g \neq g \circ f$ although sometimes they are equal.

Example 17: Consider the functions $f : R \to R$ and $g : R \to R$ defined by $f(x) = 3x + 2$ and $g(x) = \frac{1}{x-2}$.

here $(g \circ f)(x) = g(3x-2)$

$$= \frac{1}{(3x+2)-2}$$

$$= \frac{1}{3x}, \text{ whereas}$$

$$(fog)(x) = f\left(\frac{1}{x-2}\right)$$

$$= 3\left(\frac{1}{x-2}\right) + 2$$

$$= \frac{2x-1}{x-2}$$

which shows that $f \circ g \neq g \circ f$

Value Addition:
Composite functions obey the associative law. i.e., $h \circ (g \circ f) = (h \circ g) \circ f$

Example 18: Show that the function $f(x) = |x| \ \forall \ x \in R$ is continuous on *R*.

Solution: Let $\varepsilon > 0$ be arbitrary

Let *c* be any real number, then

$$|f(x) - f(c)| = \big|\,|x| - |c|\,\big| \leq |x - c|$$

Take $\delta = \varepsilon$, then $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$

Thus for every $\varepsilon > 0$, there exist $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

This shows that *f* is continuous at every real no. *c*. As c is any real no, therefore *f* is continuous on *R*.

Using result of above question, we prove that if *f* is continuous, then $|f|$ is continuous on *R*.

Proof: Let $g(x) = |x| \ \forall \ x \in R$

$$(g \circ f)(x) = g(f(x))$$

$$= |f(x)| = |f|(x)$$

$\Rightarrow$ $g \circ f = |f|$

Since *f* an *g* both are continuous on *R*.

$\therefore$ $g \circ f$ is continuous.

$\Rightarrow$ $|f|$ is continuous.

Example 19: Let $g(x) = \sin x \ \ \forall \ x \in R$.

We have already shown that *g*(*x*) is continuous on *R*. If *f* is continuous on *R* then it follows that $g \circ f$ is continuous on *R*. In particular, if $f(x) = \frac{1}{x}$ for $x \neq 0$,

then the function $(g \circ f)(x) = g(f(x))$

$$= g\left(\frac{1}{x}\right)$$

$$= \sin\left(\frac{1}{x}\right)$$

$\Rightarrow$ $g \circ f$ is continuous at every point $x \neq 0$.

Example 20: Let *g* be defined on *R* by *g*(1) = 0 and *g*(*x*) = 2 if $x \neq 1$ and let $f(x) = x + 1$ for all $x \in R$.

Show that $\lim_{x \to 0} g \circ f \neq (g \circ f)(0)$

Solution: $g(x) = \begin{cases} 0 & \text{if} \quad x = 1 \\ 2 & \text{if} \quad x \neq 1 \end{cases}$

and $f(x) = x+1 \;\forall\; x \in R$

$$\lim_{x\to 0}(g\circ f)(x) = \lim_{x\to 0} g(f(x)) = \lim_{x\to 0} g(x+1) = \lim_{x\to 0} 2 = 2$$

$$(g\circ f)(0) = g(f(0)) = g(0+1) = g(1) = 0$$

Here *g* must be continuous at *f* (0) where *f* (0) = 1 and *g* is not continuous at
x = 1 = *f* (0).

Therefore, $\lim_{x\to 0}(g\circ f)(x) \neq (g\circ f)(0)$.

I.Q. 19

Example 21: Let *f*, *g* be continuous from *R* to *R*, and suppose that $f(r) = g(r)$ for all rational numbers *r*.

Is it true that $f(x) = g(x)$ for all $x \in R$?

Solution: Given *f* and *g* are continuous on *R*.

⇒ $f(r) = g(r)$ for $r \in Q$.

Let $c \in R \sim Q$ i.e., c is irrational $\forall\; n \in N$

⇒ There exists sequence $< x_n >$ of rational numbers in $]c-\frac{1}{n}, c+\frac{1}{n}[$

Therefore, $|x_n - c| < \frac{1}{n}$

⇒ $< x_n >$ converges to '*c*'.

Since *f* is continuous.

⇒ Sequence $\langle f(x_n)\rangle$ converges to *f* (*c*) and *g* is continuous.

⇒ $\langle g(x_n)\rangle$ converges to *g*(*c*)

Since $\langle x_n \rangle$ are rational numbers.

$\Rightarrow \quad f(x_n) = g(x_n) \ \forall\, n.$

$\Rightarrow \quad \lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} g(x_n) \Rightarrow f(c) = g(c)$

So it is true that $f(x) = g(x)$ for all $x \in R$.

I.Q. 20

Example 22: Let $f : R \to R$ be continuous on R, and let $P = \{x \in R : f(x) > 0\}$. If $c \in P$, show that there exists a neighbourhood $V_\delta(c) \subseteq P$.

Solution: Given f is continuous on R.

$$P = \{x \in R : f(x) > 0\}$$

Since $c \in P \Rightarrow f(c) > 0$

f is continuous on $R \Rightarrow f$ is continuous at c.

For $\varepsilon = \frac{1}{2} f(c) > 0$

There exist $\delta > 0$ such that

$|f(x) - f(c)| < \varepsilon$ whenever $0 < |x - c| < \delta$.

$\Rightarrow \quad -\varepsilon < f(x) - f(c) < \varepsilon$

$\Rightarrow \quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ when $0 < |x - c| < \delta$.

$\Rightarrow \quad f(c) - \frac{1}{2} f(c) < f(x) < f(c) + \frac{1}{2} f(c)$

$\Rightarrow \quad \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c)$

Here $f(c) > 0$

Therefore $0 < \frac{1}{2} f(c) > f(x) < \frac{3}{2} f(c)$ when $0 < |x - c| < \delta$

$\Rightarrow \quad f(x) > 0$ when $0 < |x - c| < \delta (= V_\delta(c))$.

Example 23: If *f* and *g* are continuous on *R*, let $S = \{x \in R : f(x) \geq g(x)\}$. If $(s_n) \subseteq S$ and lim (s_n) = s show that $s \in S$.

Solution: Here *f* and *g* are continuous on *R*.

$$S = \{x \in R : f(x) \geq g(x)\}$$

Now $s_n \in S \Rightarrow f(s_n) \geq g(s_n) \ \forall \ n \in N$

Again (s_n) converges to s and f and g are continuous at *R* i.e., continuous at *s*.

$\Rightarrow \quad (f(s_n))$ converges to *f* (*s*)

and $(g(s_n))$ converges to *g*(*s*)

But $f(s_n) \geq g(s_n) \ \forall \ n$.

$\Rightarrow \quad \lim_{n \to \infty} f(s_n) \geq \lim_{n \to \infty} g(s_n)$

$\Rightarrow \quad f(s) \geq g(s) \Rightarrow s \in S$.

Example 24: A function *f* : *R* → *R* is said to be additive if $f(x + y) = f(x) + f(y)$ for all *x*, *y* in *R*. Prove that if *f* is continuous at some point x_0, then it is continuous at every point of *R*.

Solution: Let *f* be continuous at x_0

It is given that $f(x + y) = f(x) + f(y) \ \forall \ x, y \in R$

Take $x = y = 0$

Therefore, $f(0) = f(0) + f(0)$

$\Rightarrow \quad f(0) = 0$

Take $y = -x$

$\Rightarrow \quad f(x - x) = f(x) + f(-x)$

$\Rightarrow \quad f(x) + f(-x) = f(0) = 0$

$\Rightarrow \quad f(-x) = -f(x) \ \forall \ x \in R$

Let (x_n) be a sequence in *R* such that $(x_n) \to 0$

$\Rightarrow \quad (x_n + x_0) \to 0 + x_0 = x_0$

f is continuous at x_0

$\Rightarrow \quad f(x_n + x_0) \to f(x_0)$

$\Rightarrow \quad (f(x_n) + f(x_0)) \to f(x_0)$

$\Rightarrow \quad (f(x_n)) \to 0$

$\Rightarrow \quad (x_n) \to 0 \Rightarrow (f(x_n)) \to 0$

$\Rightarrow \quad$ *f* is continuous at 0.

Let '*c*' be any point of *R*

We know *f* is continuous on *R*

$\Rightarrow \quad$ *f* is continuous at *x* = *c*.

Let a sequence (y_n) in *R* be such that $(y_n) \to c$.

i.e., $(y_n - c) \to 0$

$\Rightarrow$ $(f(y_n - c)) \to f(0)$ (As *f* is continuous at 0)

$\Rightarrow$ $(f(y_n) + f(-c)) \to f(0)$

$\Rightarrow$ $(f(y_n) - f(c)) \to f(0) = 0$

$\Rightarrow$ $(f(y_n) - f(c)) \to 0$

$\Rightarrow$ $(f(y_n)) \to f(c)$

$\Rightarrow$ *f* is continuous at '*c*'.

Since '*c*' is arbitrary

$\Rightarrow$ *f* is continuous on *R*.

Exercise

Q1. Let $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4} & , \ x \neq 2 \\ 3 & , \ x = 2 \end{cases}$

Show that *f* (*x*) is continuous at *x* = 2.

Q2. Examine the following function for continuity at *x* = 0

$$f(x) = \begin{cases} e^{1/x} & , \text{ when } x \neq 0 \\ 0 & , \text{ when } x = 0 \end{cases}$$

{Hint : $\lim_{x \to 0^-} f(x) = 0 = f(0)$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1/e^{-1/x} = \infty$ }.

Q3. Give examples of functions *f* and *g* such that *f*, *g* and *fg* are discontinuous while *f* + *g* is a continuous function

{Hint : Take $f(x) = [x]$ and $g(x) = x - [x] \ \forall \ x \geq 0$ }

Algebra of Continuous Functions

Q4. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{x+|x|}{|x|} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

at $x = 0$.

{Hint : $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \dfrac{x-x}{-x} = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \dfrac{x+x}{x} = 2$

So f is discontinuous at x = 0}

Q5. Let f : R → R be such that $f(x) = k$ (constant) for all $x \in R$. Show that f is continuous on R.

Q6. A function f is defined on [0, 1] by

$f(x) = \dfrac{1}{n}$ for $\dfrac{1}{n+1} < x \leq \dfrac{1}{n}$; n = 1, 2, 3, …

Show that the points of discontinuity of f are $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots$

Q7. Discuss continuity of the function

$$f(x) = \begin{cases} xe^{1/x} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

at $x = 0$

{Hint: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} xe^{1/x} = 0 \times 0 = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x / e^{-1/x} = \dfrac{0}{0}$}

Algebra of Continuous Functions

Q8. A function $f(x)$ is defined as follows:

$$f(x) = 1 + x \text{ if } x \le 2 \text{ and } f(x) = 5 - x \text{ if } x \ge 2.$$

Is the function continuous at $x = 2$?

Q9. Let $f(x) = \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} + \ldots$

If $f(x)$ continuous at the origin? Give reason for your answer.

{Hint : The function $f(x)$ will form an infinite G.P, whose first term is $\frac{x^2}{(1+x^2)}$ and common ratio is $\frac{1}{(1+x^2)}$ which is less than 1 when $x \neq 0$}

Q10. Let $f : R \to R$ be such that

$f(x) = x$ when x is irrational

$= -x$ when x is rational

Show that $f(x)$ is continuous only at $x = 0$.

Q11. Let $f(x) = \sqrt{4-x}$ for $x \le 4$ and $g(x) = x^2$ for all $x \in R$.

(a) Give the domains of $f + g, fg, f \circ g$ and $g \circ f$.

(b) Find the values of $f \circ g$ (0), $g \circ f$ (0), $f \circ g$ (1) and $g \circ f$ (2).

Q12. Use Algebra of continuous functions to prove that the following functions are continuous.

(i) $\left[\sin^2 x + \cos^6 x\right]^{\pi}$

(ii) $x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$.

Algebra of Continuous Functions

Q13. Let *f* and *g* be real valued functions. Show that min(*f*.*g*) = –max(–*f*, –*g*) and then prove min(*f*, *g*) is continuous at '*c*' where *f* and *g* are continuous at that point.

Q14. The following given functions is not continuous at 0. For each, construct a sequence $\{x_n\}$ such that $x_n \to 0$ and $f(x_n) \not\to 0$

a) $$f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

b) $$f(x) = \begin{cases} \dfrac{1}{x^2+2}, & x \leq 0 \\ \dfrac{1}{x^2+1}, & x > 0 \end{cases}$$

Q15. Use the algebra of continuous functions to prove that the functions sec *x* and cosec *x* are continuous everywhere on their domains.

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