Conics

Discipline Courses-I
Semester-I
Paper: Calculus-I
Lesson: Conics
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Conics

Table of Contents:

1. Learning Outcomes:

After you have read this chapter, you should be able to

- Define and understand the conics,
- Define the circle and understand the equation of circle,
- Define the parabola, understand the equation of parabola and learn the technique of tracing the parabola,
- Define the ellipse, understand the equation of ellipse and learn the technique of tracing the ellipse,
- Define the hyperbola, understand the equation of hyperbola and learn the technique of tracing the hyperbola.

Conics

"God ever geometrizes" – PLATO

"[Analytic Geometry], far more than any of his meta–physical speculation, immortalized the name of Descartes, and constitutes the greatest single step ever made in the progress of the exact sciences" – JOHN STUART MILL

"I desire only tranquility and repose" – RENE DESCARTES

2. Introduction:

In mathematics, one of the most important problem was that of finding equation of tangent to a curve at a specific point. For simple curves such as circles, parabolas, ellipses and hyperbolas, the formulae were available. But the real difficulty was seen with curves having complicated graphs like $y^3 = x^3 + a$, $y^3 + axy + x^3 = 0$ etc. There was one more problem that of the path in which planets were revolving round the sun. Kepler, successfully, (after many long years of watching sky) successfully concluded that the planets are moving around the sun in elliptic orbits. Although, it was quite a major breakthrough, Kepler could not give a logical reasoning for his claim. It was Newton who settled the problem of orbits of planets mathematically while it was Leibnitz's who settled the problem of tangents. So both of them share the credit for inventing "calculus" independently in the seventeenth century, one using physical approach and the other using geometrical approach. Indeed, calculus grows from a quite simple idea–the idea of speed that is the rate of change which in turn describes the calculus as the mathematics of motion and change.

The purpose of this chapter is to study the basic theory of conic sections which is broadly classified as degenerate conic which includes a point, a line and a pair of intersecting lines or non–degenerate conic which includes circle, parabola, ellipse and hyperbola as these are the paths travelled by planets, satellites and other bodies whose motions are driven by inverse square forces. The chapter begins with the formal definition of circle, ellipse, parabola and hyperbola. Then we have studied various properties of these conics in detail. A technique for tracing of the conics is discussed which is further explained with the help of examples.

3. Conics: A conic section or simply conic is a curve obtained as the intersection of a double-napped right circular cone with a plane.

Value Additions: Remarks

a) We observe that a conic is just a section or a slice through a cone.

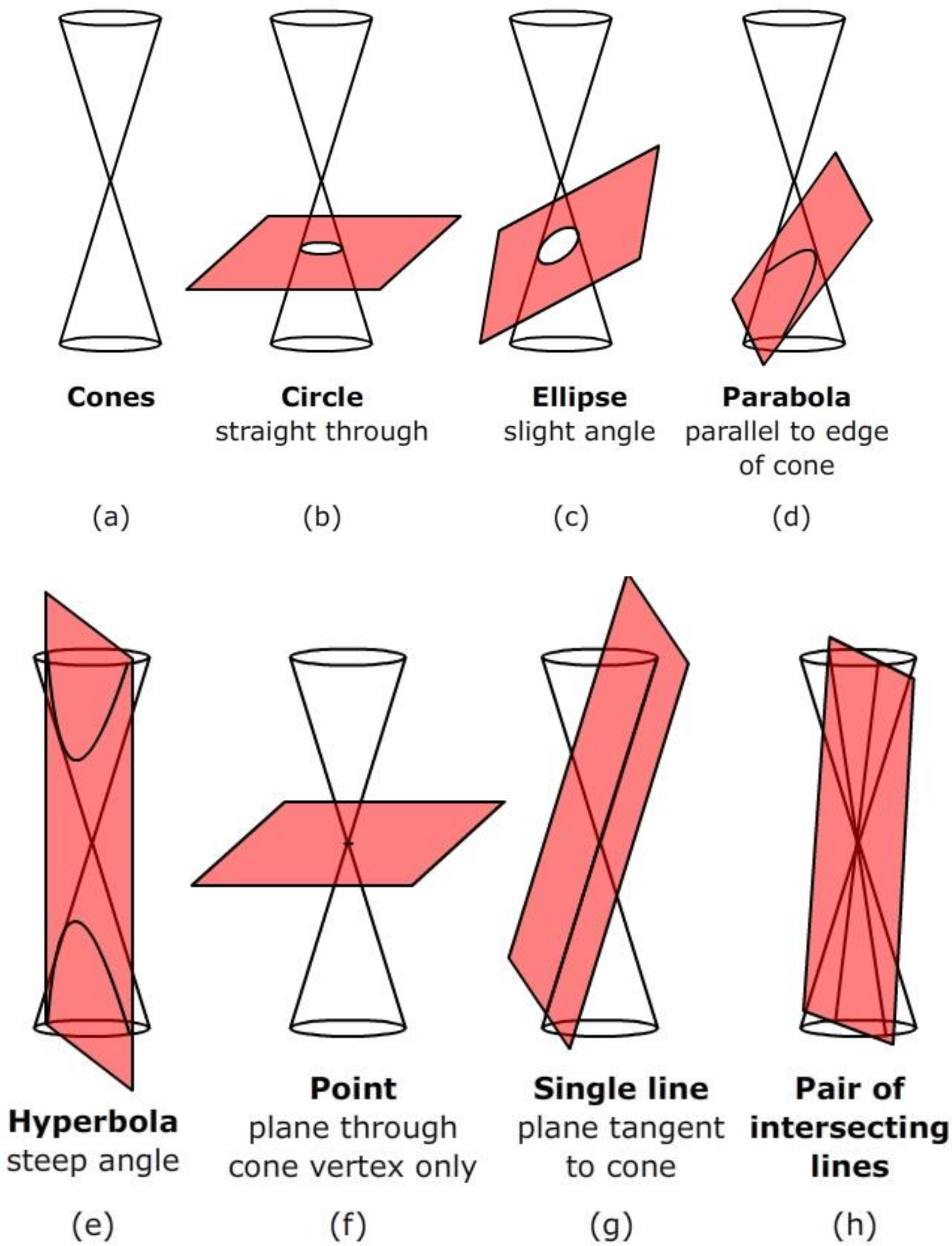


Figure 1

b) We can say that all the curves of fig. 1 are related. Circles, ellipses, parabolas and hyperbolas are four types of conics. By changing the angle and location of intersection, we can produce a circle, ellipse, parabola or hyperbola. These conics are called **non-degenerate**

c) In the special cases, when the plane passes through the vertex of the double-napped cone, then the intersection may be a point, a single line or a pair of intersecting lines. These are called **degenerate conics**.

d) The circle and the ellipse arise when the intersection of cone and plane is a closed curve. The circle is obtained when the cutting plane is perpendicular to the symmetry axis of the cone. If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a parabola. In the remaining case the plane will intersect both halves (nappes) of the cone and we get a hyperbola. We can use parallelogram method that is useful in engineering applications in which a conic is constructed point by point by means of connecting certain equally spaced points on horizontal line and vertical line.

e) In analytic geometry, a conic may be defined as a plane algebraic curve of degree two.

Geometrically, a conic consists of those points whose distances to some point, called a focus and some line, called a directrix, are in a fixed ratio, called the eccentricity. The type of a conic corresponds to its eccentricity, those with eccentricity less than 1 being ellipses, those with eccentricity equal to 1 being parabolas and those with eccentricity greater than 1 being hyperbolas. In this focus- directrix definition of a conic, the circle in a limiting case with eccentricity 0.

Value Addition: Do you know?
Just as two distinct points determine a line, five points determine a conic if no three are collinear. Formally, given any five points in the plane in general linear position, meaning no three collinear, there is a unique conic passing through them, which will be non-degenerate, but if three of the points are collinear, the conic will be degenerate and may not be unique.

Now, we first define circles, parabolas, ellipses and hyperbola and then discuss their basic properties.

3.1. Circles:

A circle is the locus (set of points) of a point in a plane whose distance from a given fixed point in the plane is constant. The fixed point is called the centre of the circle and the constant distance is called the radius.

3.1.1. Equation of circle:

Let C(h, k) be the coordinates of the centre of the circle and let *a* be the radius

We take any point P (x', y') on the circle.

Then CP = a or $CP^2 = a^2$

From the right angled ∆CRP, we have

$$(CP)^2 = (x' - h)^2 + (y' - k)^2$$

By using $CP^2 = a^2$, we conclude

$$(x - h)^2 + (y - k)^2 = a^2$$

Thus the locus of P is given by

$$(x - h)^2 + (y - k)^2 = a^2 \quad \text{---------------} \quad (1)$$

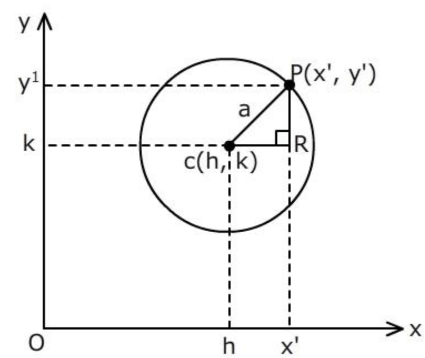


Figure 2

which is the required equation of the circle.

Value Addition: Deduction
We can deduce from equation (1) that the equation of the circle of radius *a* centred at the origin is $x^2 + y^2 = a^2$

Example 1: Show that the equation

$$x^2 + y^2 + 2gx + 2fg + c = 0 \quad \text{----------} \quad (2)$$

represents a circle.

Solution: Consider $x^2 + y^2 + 2gx + 2fg + c = 0$

i.e. $x^2 + y^2 + 2gx + 2fg + g^2 + f^2 = g^2 + f^2 - c$

i.e. $(x^2 + 2gx + g^2) + (y^2 + 2fg + f^2) = g^2 + f^2 - c$

i.e. $$[x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2+f^2-c}\right)^2 \quad \text{-----------} \quad (3)$$

Comparing equation (3) with eqn. (1), we conclude that equation (2) represents a circle with centre (-g, -f) and radius as $\sqrt{g^2+f^2-c}$.

Furthermore, we notice that the locus of the general equation (2) of the second degree will be

(a) a circle if the coefficients of x^2 and y^2 are equal, the term xy is lacking and the condition $g^2 + f^2 - c > 0$.

(b) a point if $g^2 + f^2 - c = 0$.

(c) represents no graph if $g^2 + f^2 - c < 0$.

Example 2: Find the centre and radius of the circle

$$x^2 + y^2 - 6x - 6y - 8 = 0$$

Solution: First we convert the given equation to equation (1) by completing the squares in x and y.

Consider $x^2 + y^2 - 6x - 6y - 8 = 0$

i.e. $x^2 + y^2 - 6x - 6y + 9 + 9 = 8 + 9 + 9$

i.e. $(x^2 - 6x + 9) + (y^2 - 6y + 9) = 26$

i.e. $(x-3)^2 + (y-3)^2 = \left(\sqrt{26}\right)^2$

which represents a circle with centre (3,3) and radius as $\sqrt{26}$.

Example 3: Find the equation of the circle which passes through the point (7, 9), is tangent to the x–axis, has its centre on the line x–y+1 = 0. Also trace the graph.

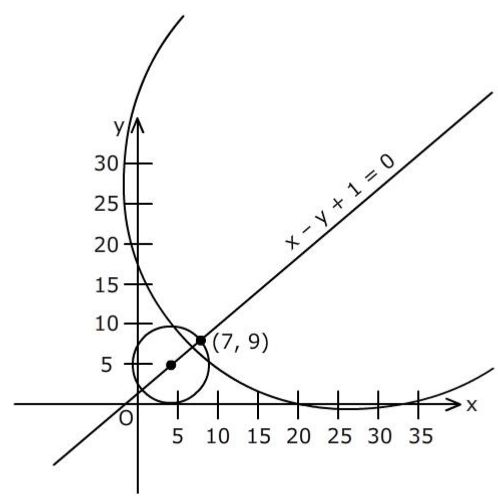


Figure 3

Solution: Let the equation of the circle be

$$(x - h)^2 + (y - k)^2 = r^2$$

According to the given condition, we have

$$(7 - h)^2 + (9 - k)^2 = r^2$$

$$k = r$$

and $$h - k + 1 = 0.$$

On solving the above equations, , we obtain two sets of solutions h = 4, k = 5, r = 5 and h = 28, k = 29, r = 29. Therefore, the two sets of solutions, i.e., two circles are given by

$$x^2 + y^2 - 8x - 10y + 16 = 0$$

and $$x^2 + y^2 - 56x - 58y + 784 = 0$$

3.1.2. Definition:

(a) Interior of the circle is the set of points that lie inside the circle

$$(x - h)^2 + (y - k)^2 = a^2$$

i.e. the points less than 'a' unit from (h, k).

They satisfy the inequality

$$(x - h)^2 + (y - k)^2 < a^2$$

(b) Exterior of the circle in the set of points that lie outside the circle

$$(x - h)^2 + (y - k)^2 = a^2$$

i.e. the points more than a units from (h, k). They satisfy the inequality $(x - h)^2 + (y - k)^2 > a^2$

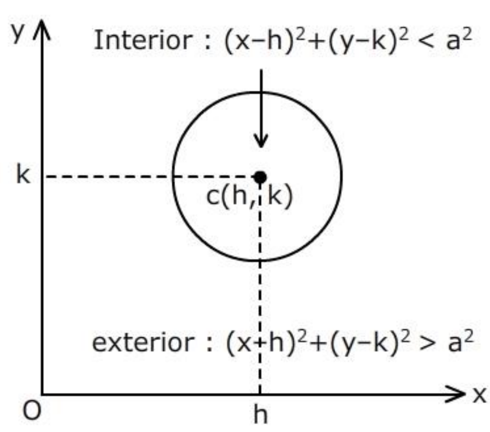


Figure 4

Example 4: Find an equation for the circle centred at (–2, 1) that passes through the point (1, 3). Is the point (1.1, 2.8) lie inside, outside, or on the circle?

Solution. The equation of the circle centred at (–2, 1) is given by

$$(x + 2)^2 + (y - 1)^2 = a^2 \qquad \text{(by using equation (1))}$$

Now, the point (1, 3) lie on the circle so that

$$(1 + 2)^2 + (3 - 1)^2 = a^2,$$

i.e., $a^2 = 13$

Thus, $(x + 2)^2 + (y - 1)^2 = 13$

is the required equation of the circle.

Since $(1.1 + 2)^2 + (2.8 - 1)^2 = 12.85 < 13$, therefore, by using definition 1.9 the point lie inside the circle.

Example 5: Find an equation of the circle which passes through the points (1, 0), (0, 1) and (2, 2).

Solution: Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fg + c = 0$$

Since the circle passes through the points (1, 0), (0, 1) and (2, 2), therefore we obtain

$$1 + 2g + c = 0$$

$$1 + 2f + c = 0$$

$$8 + 4g + 4f + c = 0$$

On solving the above equations, we have

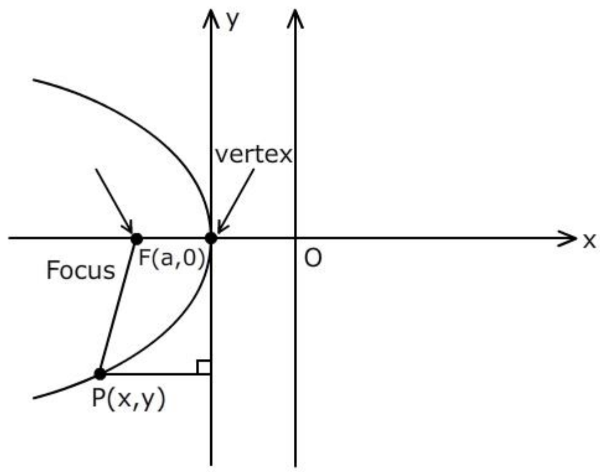
$$f = g = {-7}/{6} \quad and \quad c = 4/3.$$

Thus, the required equation circle is given by

$$3x^2 + 3y^2 - 7x - 7y + 4 = 0$$

3.2. Parabola:

A parabola is the locus of a point in a plane that is equidistant from a given fixed point and a given fixed line in the plane. The fixed point is called the focus of the parabola and the fixed line is called the directrix (see figure 5).



directrix : x = a
Figure 5

Value Addition: Remarks
a) A parabola can also be defined as the locus of all points whose distance to the focus is equal to the distance to the directrix. b) If the focus lies on the directrix, then the parabola is the line through the focus perpendicular to the directrix which in turn considered to be a degenerate case. Hence, we will assume that focus does not lie on the directrix in the definition 1.12 of the parabola. c) A parabola is symmetric about the line which passes through the focus and is perpendicular to the directrix. This line, known as the axis or the axis of symmetry of the parabola, intersects the parabola at a point called vertex.

3.2.1. Equation of Parabola:

Let F(a, 0) be the focus and L be the directrix. From F draw a line FOX' perpendicular to the directrix L. Take O as the mid-point of FQ. From O draw a line OY perpendicular to FQ. Now take OX and OY as the x-axis and y-axis respectively. Let OF = a. Then OQ = a and O lies on the locus. Let P(h, k) be any point on the locus and let PM be perpendicular to L. Then FP = PM (see figure (6)).

Now, PM = PN + NM = h + a (using NM = OQ)

Also, FP $= \sqrt{(h-a)^2 + k^2}$ (by using distance formula)

By using above relations, we obtain

$$\sqrt{(h-a)^2 + k^2} = h + a$$

i.e. $h^2 - 2ah + a^2 + k^2 = h^2 + a^2 + 2ah$

(squaring both the sides)

i.e. $k^2 = 4ah$

Thus, the locus of P is given by

$$y^2 = 4ax \qquad \text{---------} \ (4)$$

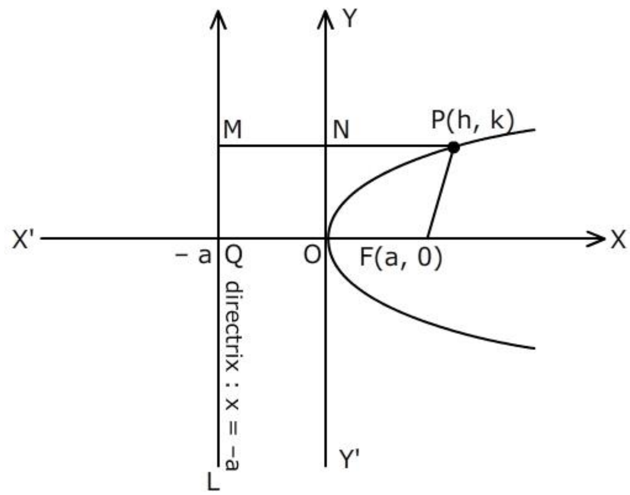
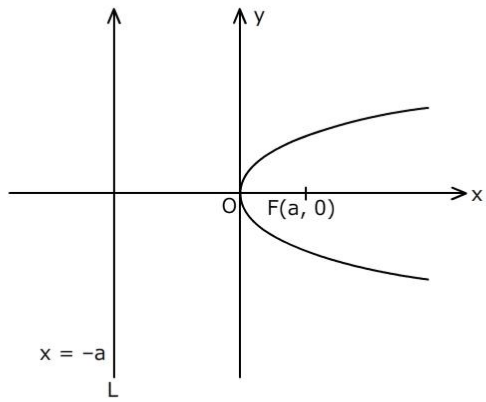


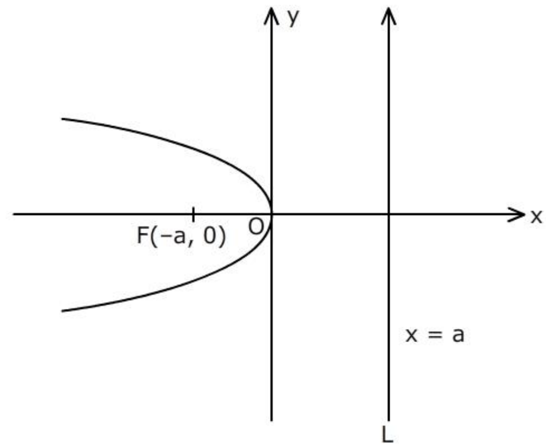
Figure 6

Value Addition: Remarks	
(a)	The equation (4) represents parabola with origin as the vertex, x-axis as the axis of symmetry of the parabola and y-axis as the tangent at the vertex. The equation (4) is called the standard form of the equation of the parabola. The positive number *a* is called the focal length of the parabola.
(b)	We notice that the second degree equations $Ax^2 + Dx + Ey + F = 0, \qquad Cy^2 + Dx + Ey + F = 0$ in each of which the xy–term is lacking and only one of the variables occurs squared while the other appears to the first power, represents parabolas whose axes are parallel to the x and y–axis respectively

Value Addition: Standard forms of the parabolas



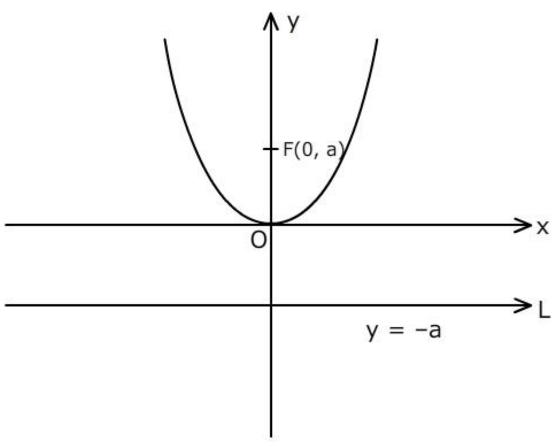
(i)



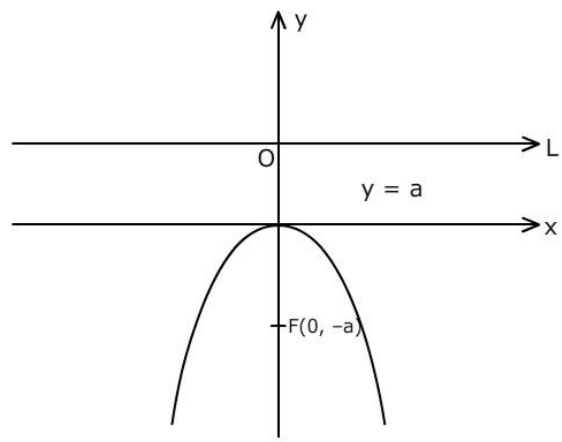
(ii)

Right handed parabola: $y^2 = 4ax$
Vertex : (0, 0)
Focus : (a, 0)
Directrix : $x + a = 0$
x-axis as the axis of symmetry

Left handed parabola : $y^2 = -4ax$
Vertex : (0, 0)
Focus : (-a, 0)
Directrix : $x - a = 0$
x-axis as the axis of symmetry



(iii)



(iv)

Upward parabola : $x^2 = 4ay$
Vertex : (0, 0)
Focus : (a, 0) (0, a)
Directrix : $y + a = 0$
y-axis as the axis of symmetry

Downward parabola : $x^2 = -4ay$
Vertex : (0, 0)
Focus : (0, -a)
Directrix : $y - a = 0$
y-axis as the axis of symmetry

Figure 7

3.2.2. Definitions:

The latus rectum is a chord of the parabola passing through the focus and perpendicular to the axis of symmetry.

By definition, in the figure 8 , MFM' is the latus rectum

But MFM' = 2FM

The coordinates of M is (a, FM)

Since M lies on the parabola, therefore, we have

$FM^2 = 4a.\ a$

i.e. $FM = 2a$

Thus, MFM' = 4a,

which is the length of the latus rectum of the parabola. Thus the length of the latus rectum is twice the distance between directrix and focus.

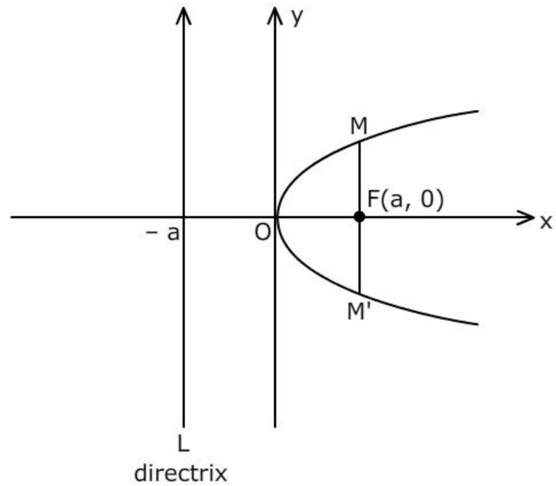


Figure 8

Example 6: Find the equation to the parabola whose focus is (3, -4) and directrix is the line x + y - 2 = 0.

Solution: Let P(x, y) be any point on the parabola.

Join FP and draw PM perpendicular to the dircetrix L (in figure 9)

Now, $FP = \sqrt{(x-3)^2 + (y+4)^2}$

Also $PM = \dfrac{|x+y-2|}{\sqrt{1^2+1^2}}$

Since FP = PM, therefore

$$\sqrt{(x-3)^2 + (y+4)^2} = \frac{|x+y-2|}{\sqrt{2}}$$

which simplifies to

$x^2 - 2xy + y^2 - 8x + 20y + 46 = 0$

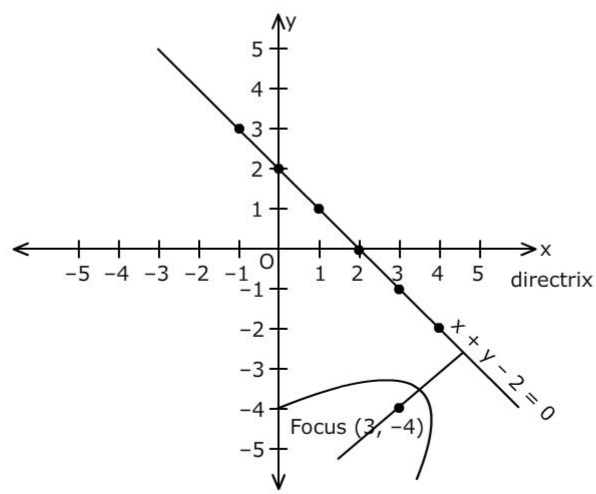


Figure 9

Example 7: Find the vertex, axis, latus rectum and focus of the parabola

$y^2 + 6x - 2y + 13 = 0$

Solution : Rewrite the given equation as

$y^2 - 2y = -6x - 13$

i.e. $y^2 - 2y + 1 = -6(x + 2)$ (Competing the square on L.H.S) i.e. $(y - 1)^2 = -6(x + 2)$ ---------------------- (5)

Put $Y = y - 1$, $X = x + 2$.

Then the origin shifts to (–2, 1) and the equation (5) reduces to

$$Y^2 = -6X$$

which represents a parabola with vertex (–2, 1) and latus rectum as 6.

Now, latus rectum = 4 AF

Then, $AF = \frac{6}{4} = \frac{3}{2}$

Also, OM + MN = $2 + \frac{3}{2} = \frac{7}{2}$

Thus, the coordinates of F are $\left(-\frac{7}{2}, 1\right)$

Further, the equation of axis of symmetry is y =1

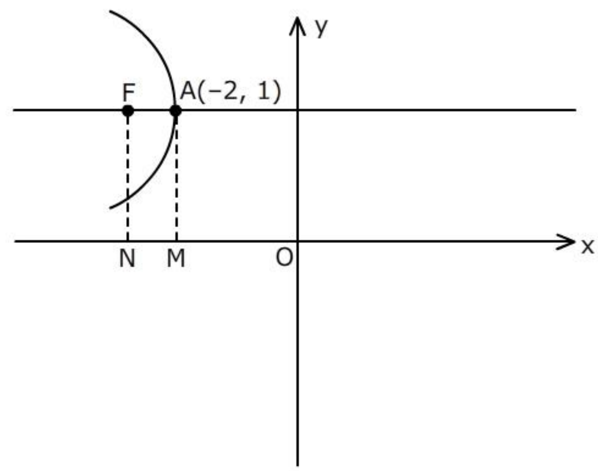


Figure 10

3.2.3. A technique for tracing a parabola:

Parabolas from their standard equations can be traced according to the following steps.

(a) Rewrite the given equation in one of the standard forms (see fig. 6).

(b) If the equation of curve contain even power of y, then the parabola is symmetrical about x–axis. Likewise, if the curve contain even power of x, then the parabola is symmetrical about the y-axis.

(c) Since vertex lies at the origin, therefore parabola pass through the origin.

(d) If the negative value of x makes the value of y imaginary, then no part of the curve lie to the left of the x-axis.

(e) Find the value of 'a' and hence obtain the value of latus rectum.

By using the above criteria, trace the parabola by taking origin as the vertex of the parabola.

Example 8: Sketch the graph of the parabola

$$y^2 + 16x = 0.$$

Solution: Rewriting the given equation in the standard form as

$$y^2 = -16x = -4(4x)$$

Since the given equation contains the even power of y, therefore x-axis is the axis of symmetry of the parabola.

Now, 4a = 16 (coefficient of x), i.e., a = 4

We notice that the positive values of x makes the value of y imaginary. Thus, no part of the curve lies to the right of the y-axis.

By using the above criteria, we can trace the parabola as follows:

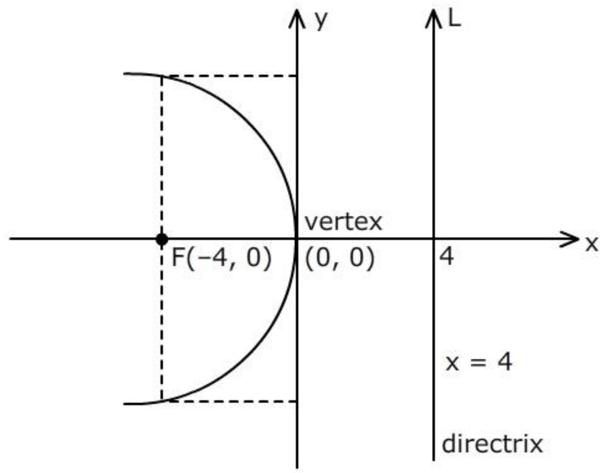


Figure 11

Example 9: Find the vertex and focus of the parabola $2y^2–9x–12y = 0$, and also find the equation of the directrix and trace the graph.

Solution: The equation of the parabola is given by

$$2y^2 - 9x - 12y = 0$$

i.e., $y^2 - \frac{9}{2}x - 6y = 0$

i.e., $(y-3)^2 - 9 - \frac{9}{2}x = 0$

i.e., $(y-3)^2 = \frac{9}{2}(x+2)$

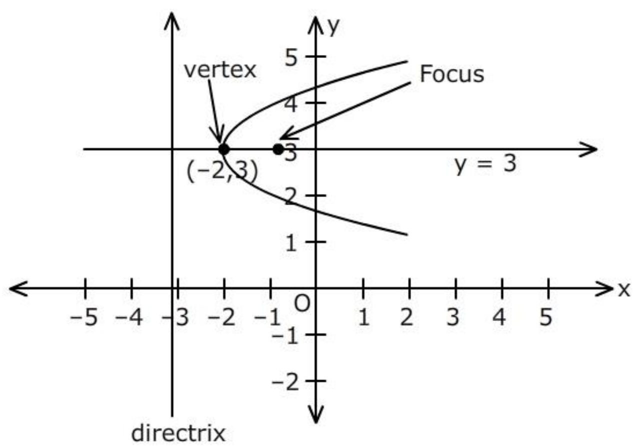


Figure 12

Let Y = y-3 and X = x + 2. Then, we have

$$Y^2 = \frac{9}{2}X$$

so that vertex of parabola is (-2, 3), its axis along the line y = 3 and its focus is at $\left(\frac{-7}{8}, 3\right)$.

The equation of the directrix is $x = \frac{-25}{8}$.

Value Addition: Do you know?
A point $P(x^{'} y^{'})$ lies inside, on or outside the parabola $y^2=4ax$ according as the expression $y^{'2} - 4ax^{'} < 0, = 0$ or > 0 respectively.

3.3. Ellipse:

An ellipse is the locus of the point in a plane whose distance from two fixed points in the plane have a constant sum. The two fixed points are called the foci of the ellipse, the line through the foci of the ellipse is called the ellipse's focal axis. The point on the axis midway between the foci is called the centre. The points where the focal axis and ellipse cross are called the vertices of the ellipse.

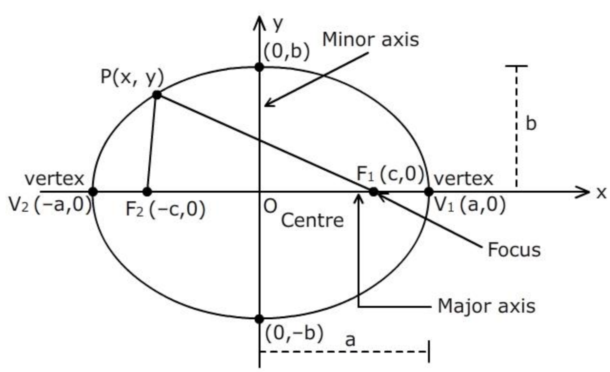


Figure 13

Value Addition: Remarks	
a)	If the foci of an ellipse coincide, then the ellipse reduces to a circle.
b)	The line segment through the foci and across the ellipse is called the major axis and the line segment across the ellipse through the centre and perpendicular to the major axis is called the minor axis. We also notice that the vertices of the ellipse are the end points of the major axis (see figure 13).
c)	A parabola may be regarded as the limiting case of an ellipse with its centre at infinity.

3.3.1. Equation of ellipse:

Let $F_1(c, 0)$ and $F_2(-c, 0)$ be the foci of the ellipse and let P(h,k) be any point on the ellipse.

If the sum of the distances from P to the foci, that is, PF_1+PF_2 is denoted by 2a, then the coordinates of a point P on the ellipse satisfy the equation

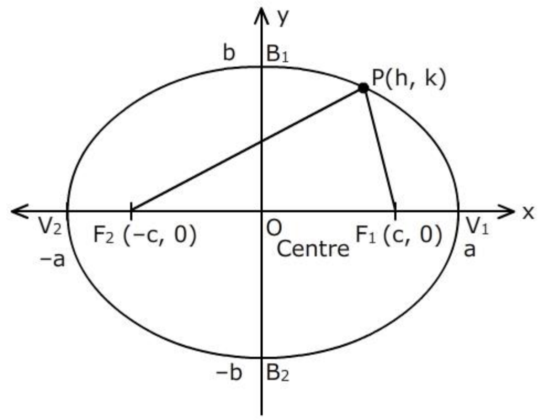


Figure 14

$$\sqrt{(h-c)^2+k^2}+\sqrt{(h+c)^2+k^2}=2a$$

i.e., $\sqrt{(h-c)^2+k^2}=2a-\sqrt{(h+c)^2+k^2}$

i.e., $(h-c)^2+k^2=4a^2+(h+c)^2+k^2-4a\sqrt{(h+c)^2+k^2}$ (squaring both sides)

i.e., $ch+a^2=a\sqrt{(h+c)^2+k^2}$

i.e., $c^2h^2+a^4+2cha^2=a^2\left(h^2+2ch+c^2+k^2\right)$ (agian squaring both sides)

i.e., $\left(c^2-a^2\right)h^2-a^2k^2=a^2\left(c^2-a^2\right)$

i.e., $\frac{h^2}{a^2}-\frac{k^2}{c^2-a^2}=1$ (Dividing both sides by $a^2\left(a^2-c^2\right)$)

Thus the locus of P(h, k) in given by

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1 \qquad \text{----------------} \qquad (6)$$

By using triangle inequality for triangle F_2PF_1, we observe that

$PF_1 + PF_2 > F_1F_2$, i.e. 2a > 2c which implies a > c

Thus, the expression $a^2 - c^2 > 0$

If $b=\sqrt{a^2-c^2}$, then $b^2=a^2-c^2$

Thus, equation (6) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{----------------} \qquad (7)$$

which is the standard form of the equation of the ellipse.

Value Addition: Remarks

a) The line segment V_1V_2 (in figure 14) of length 2a joining the points $(\pm a,0)$ is the major axis of the ellipse and the line segment B_1B_2 of length 2b joining the points $(0,\pm b)$ is the minor axis of the ellipse. The number *a* is the semi major axis and the number *b* is the semi minor axis. The centre to focus distance of the ellipse is given by

$$c = \sqrt{a^2 - b^2}$$

b) If the foci of an ellipse lie on the y-axis, then the standard equation of the ellipse is given by

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (See\ figure\ 15) \quad \text{--------------------} \quad (8)$$

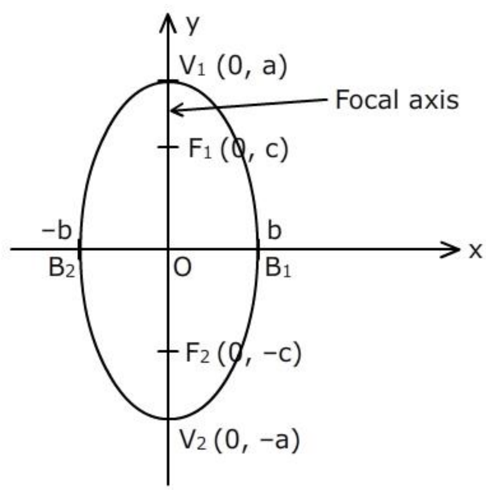


Figure 15

Value Addition: Remember

We observe that the second degree equation

$Ax^2 + Cy^2 + Dx + Ey + F = 0$

in which A and C are positive and unequal and the xy – term is lacking may be re-written as

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\frac{D^2C + E^2A - 4ACF}{4A^2C}} + \frac{\left(y + \frac{E}{2C}\right)^2}{\frac{D^2C + E^2A - 4ACF}{4AC^2}} = 1$$

Furthermore, we conclude that the given equation represents

(i) an ellipse if A and C are positive and $D^2C + E^2A - 4ACF > 0$.
The major axis is parallel to the x or y-axis according as A<C or A>C

(ii) a point ellipse $\left(\frac{-D}{2A}, \frac{-E}{2C}\right)$ if A and C are positive and $D^2C + E^2A - 4ACF = 0$

(iii) no locus if A and C are positive and $D^2C + E^2A - 4ACF < 0$.

3.3.2. A technique for tracing an ellipse:

Ellipses from their standard equations can be traced according to the given steps.

(i) Rewrite the given equation of an ellipse in the standard forms given in equation (7) or equation (8)

(ii) Since the equation of the ellipse contain even powers of x as well as of y, therefore the curve is symmetrical about both the major and minor axes.

(iii) The curve does not passes through the origin which is the centre of the ellipse.

(iv) The curve has no asymptotes.

(v) Find out the positions of major and minor axes. If the coefficient of x^2 is less than the coefficient of y^2, then the major axis is along the x-axis and the minor axis is along the y-axis.

(vi) Ellipse lies inside the rectangular box bounded by the lines $x=\pm a$ and $y=\pm b$ and it crosses the axes at the points $(\pm a,0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y} = \begin{cases} 0, if\ x = 0 \\ \infty, if\ y = 0 \end{cases}$$

By using the above criteria, trace the ellipse by taking origin as the centre of the ellipse.

Example 10: Sketch the graph of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \qquad ---------------- \qquad (9)$$

Solution: On comparing equation (9) with the equation (7), we conclude that $a^2=16$ and $b^2=9$ (by using the fact $a^2>b^2$).

Since the coefficient of x^2 is less than the coefficient of y^2,therefore, the major axis is along the x-axis and hence the minor axis is along the y-axis

The centre-to-focus distance is given by

$$c=\sqrt{a^2-b^2}=\sqrt{16-9}=\sqrt{7}$$

Also, the coordinates of the foci are $F_1\left(\sqrt{7},0\right)$ *and* $F_2\left(-\sqrt{7},0\right)$(foci on major axis and x-axis is the major axis)

Thus the ellipse lies inside the rectangular box bounded by the lines

$x = \pm 4$ and $y = \pm 3$

Now, $\dfrac{dy}{dx}=\dfrac{-9x}{16y}=\begin{cases}0, \text{ if } x=0\\ \infty, \text{if } y=0\end{cases}$

We conclude the tangents at (±4, 0) and (0, ±3) are perpendicular to the axes.

Now, we can trace the ellipse (in fig 16) by taking vertices at (±4, 0) and origin as the centre of the ellipse

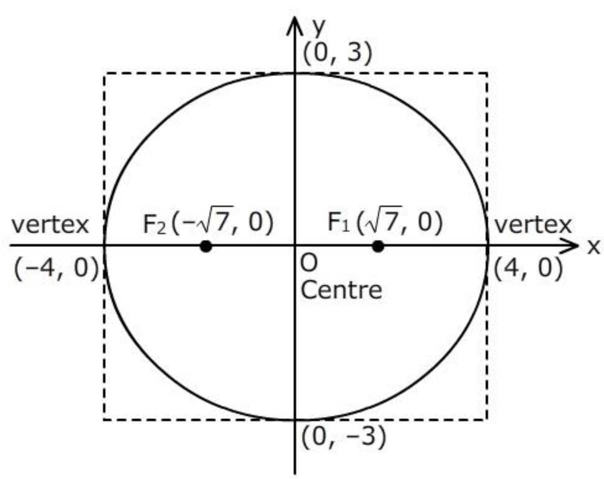


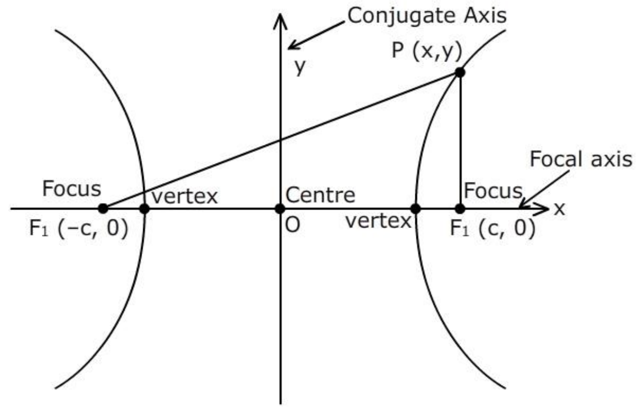
Figure 16

3.4. Hyperbola:

A hyperbola is the locus of the point in a plane whose distance from two fixed point in the plane have a constant difference.

The two fixed points are called the foci of the hyperbola. The line through the foci of the hyperbola is called the hyperbola's focal axis or transverse axis. The point on the axis midway between the foci is called the centre of the hyperbola.

The points where the focal axis and hyperbola cross are called the vertices. Conjugate axis is the line through the centre which is perpendicular to the focal axis (see figure 17)



Hyperbola in standard form
Figure 17

Value Addition: Remark
Every hyperbola has a pair of lines having the property that as a points Precedes to infinity along the hyperbola from the centre, the perpendicular distance of P from these line tends to zero. These lines are called the asymptotes of the hyperbola and they intersect at the centre of the hyperbola (see fig 18.)

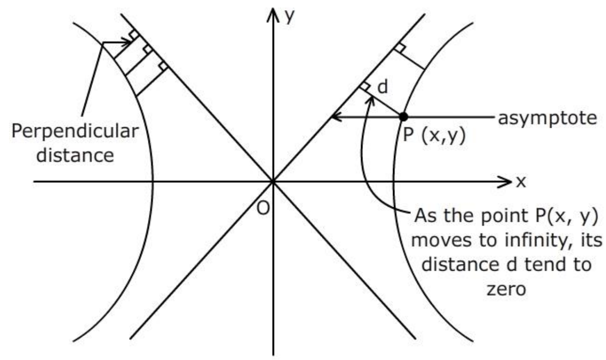


Figure 18

3.4.1. Equation of Hyperbola:

Let F_1 (c, 0) and F_2 (-c, 0) be the foci of the hyperbola and let P(h, k) be any point on the hyperbola.

If $PF_2 - PF_1$ (difference of the distance from P to the foci) is 2a, then the coordinates of a point P on the hyperbola satisfy the equation.

$$\sqrt{(h+c)^2 + k^2} - \sqrt{(h-c)^2 + k^2} = \pm 2a$$

i.e., $\sqrt{(h+c)^2 + k^2} = \pm 2a + \sqrt{(h-c)^2 + k^2}$

which on simplification (see equation of ellipse 1.25) reduces to

$$\frac{h^2}{a^2} - \frac{k^2}{a^2 - c^2} = 1$$

Therefore locus of P(h, k) is given by

$$\frac{x^2}{a^2} - \frac{y^2}{a^2 - c^2} = 1 \qquad \text{---------------------} \qquad (10)$$

We observe that 2a, being the difference of two sides of ΔF_1PF_2 (see fig 16) is less than 2c. This implies that $a^2 - c^2 < 0$.

If $b = \sqrt{(c^2 - a^2)}$, *then* $b^2 = c^2 - a^2$

Thus equation (10) becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ---------------------- \qquad (11)$$

which is the standard form of equation of the hyperbola.

Value Addition: Remarks

(a) The centre-to-focus distance of the hyperbola is given by

$$c = \sqrt{\left(a^2 + b^2\right)}$$

(b) The number *a* is the semi focal axis and the number *b* is the semi conjugate axis of the parabola.

(c) If the foci of the hyperbola lie on the y-axis, then the equation of the hyperbola is given by

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad -------------------- \qquad (12)$$

(d) If a = b, then the hyperbola is called equilateral or rectangular hyperbola. In this case, asymptotes are perpendicular to each other.

Value Addition: Remember

We notice that the second degree equation

$Ax^2 + Cy^2 + Dx + Ey + F = 0$

in which the xy – term is lacking may be re-written as

$$\frac{\left(x + \frac{D}{2A}\right)^2}{\frac{D^2C + E^2A - 4ACF}{C\left(4A^2\right)}} + \frac{\left(y + \frac{E}{2C}\right)^2}{\frac{D^2C + E^2A - 4ACF}{A\left(4C^2\right)}} = 1$$

Moreover, the given equation will represent

(i) a hyperbola if A & C are of opposite sign, (hence $C(4A^2)$ and $A(4C^2)$) are of opposite sign) and $DC^2 + E^2A - 4ACF \neq 0$

The hyperbola is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-h)^2}{b^2} = 1 \text{ or of the form}$$

$$\frac{-(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

according as $D^2C + E^2A - 4ACF$ has the sign as that of C or A.

(ii) a pair of straight lines intersecting in the point $\left(\frac{-D}{2A}, \frac{-E}{2C}\right)$ if A and C are of opposite signs and $D^2C + E^2A - 4ACF = 0$, therefore

$$A\left(x+\frac{D}{2A}\right)^2 + C\left(y+\frac{E}{2C}\right)^2 = 0$$

which is easily separable into linear factors because A and C have opposite signs.

Example 11: Find the centre and the length of the transverse or focal axis of the hyperbola

$$x^2 - 2y^2 - 2x + 8y - 1 = 0$$

Solution. Re-writing the given equation as

$$x^2 - 2x - 2\left(y^2 - 4y\right) = 1$$

i.e., $(x-1)^2 - 2(y-2)^2 = 1+1-8 = -6$

i.e., $\dfrac{(y-2)^2}{3} - \dfrac{(x-1)^2}{6} = 1$

Let X = x – 1 and Y = y – 2. Then the equation becomes

$$\frac{y^2}{3} - \frac{x^2}{6} = 1$$

which represents a hyperbola with centre (1, 2) and the transverse axis is of length $2\sqrt{3}$ which is parallel to the y–axis.

3.4.2. A technique for tracing a hyperbola:

(a) Rewrite the given equation of the hyperbola in the standard forms (11) or (12).

(b) Since the equation of the hyperbola contain even powers of both the variables x and y, therefore hyperbola is symmetrical about both the axes.

(c) The curve does not passes through the centre.

(d) Find out the position of focal axis and the conjugate axis of the hyperbola. In case of equation (11), it cuts the x–axis (focal axis) at the point (±a, o) but does not cut the y-axis (conjugate axis)

(e) Hyperbola lies outside the rectangular box bounded by the lines x = ±a and y = ±b.

(f) Find out the asymptotes of the hyperbola. These can be obtained by replacing 1 by 0 in equation (11). Then solve the new equation obtained for y in terms of x. Trace the asymptotes by extending the diagonals of the rectangular box.

On summing all the above points, trace the hyperbola by taking origin as the centre of the hyperbola.

Example 12: Find the equation of the hyperbola with vertices (0, ±4) and asymptotes y = ± 2x/3. Also trace the graph.

Solution: Since the vertices lie on the y–axis, therefore, y–axis is the focal axis of the hyperbola .Thus, the equation of the hyperbola is given by

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{-----------------} \qquad (13)$$

Now, vertices lie at (o, ±4), so that a = 4.
We notice that the asymptotes of (13) are

$$y = \pm\frac{a}{b}x = \pm\frac{4x}{b} = \pm\frac{2}{3}x$$

which gives b = 6.

It follows that $\frac{y^2}{16} - \frac{x^2}{36} = 1$ is the required equation of the hyperbola (see fig. 19).

Now, $c = \sqrt{a^2 + b^2} = \sqrt{16+36} = \sqrt{52}$

Thus, the coordinates of foci are

$$F_1\left(0, \sqrt{52}\right) \; and \; F_2\left(0, -\sqrt{52}\right)$$

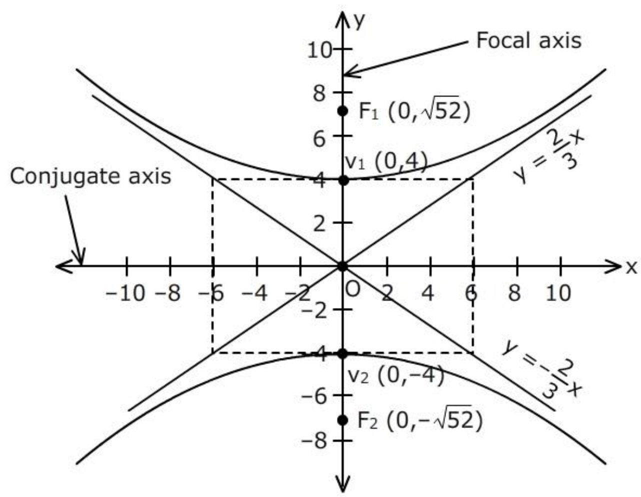


Figure 19

Example 13: Trace the graph of the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

Solution: We are given that

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \qquad \text{---------------------------------} \qquad (14)$$

We notice that the x–axis is the focal axis and y–axis is the conjugate axis.
By comparing equation (14) with equation (11), we obtain

$a^2 = 4$ and $b^2 = 5$.

Also, centre -to-focus distance $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$

In this case, coordinates of the foci are (3, 0) and (-3, 0).

Now, the asymptotes are given by the equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 0$$

i.e., $y = \pm\frac{\sqrt{5}}{2}x$

By definition, hyperbola lies outside the rectangular box bounded by the lines

$x = \pm 2$ and $y = \pm\sqrt{5}$.

Also, the hyperbola crosses the x–axis at the points (±2, 0), and it never crosses its conjugate axis.

Now, we can trace the hyperbola (see fig. 20) by taking vertices at (±2, 0) and origin as the centre of the hyperbola

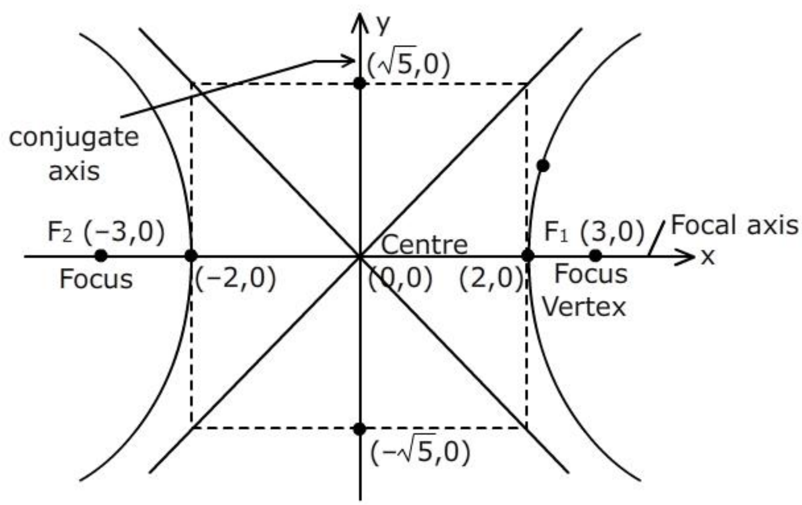


Figure 20

Exercise:

1. Find the equation of the circle which contain the point of intersection of the lines $x + 3y - 6 = 0$ and $x - 2y - 1 = 0$ and centred at the origin. Also trace the graph.

2. Identify the locus of the equation

   $x^2 + y^2 + 6x - 4y + 9 = 0,$

   and trace the graph also.

3. Find the vertex, focus and directrix of the parabola

   $y^2 + 4x + 2y - 8 = 0.$

4. Find the locus of a point which moves so that its distance from the point $\left(0, \frac{-7}{3}\right)$ is equal to its distance from the line $x = \frac{7}{3}$

5. Locate the vertex and focus of the parabola

   $x^2 - 6x - 12y - 15 = 0$

   Also, write the equation of the directrix, axis and tangent at the vertex

6. Find the vertex, focus, axis and directrix of the parabola

   $x^2 - 4x - 8y + 28 = 0$

7. Find the equation of the ellipse having a focus at the point (4, 0) and length of semi–major axis 5

8. Find the equation of the ellipse containing the points (–3, 2) and (2, –4)

9. Find the locus of a point which moves so that its distance from the point (–2, 0) is one–third its distance from the line x = –18.

10. Locate the foci and vertices of the hyperbola

    $49x^2 - 4y^2 = 196.$

    Find the lengths of the semi–transverse and semi–conjugate axes. Also write the equations of the asymptotes.

Solutions:

1. $x^2 + y^2 = 10$

2. represents a circle with centre (-3, 2) and 2 as the radius.

3. Vertex : $\left(\frac{9}{4}, -1\right)$

Focus : $\left(\frac{5}{4}, -1\right)$

Directrix : $4x - 13 = 0$

4. Locus is $3x^2 + 28y = 0$

5. Vertex : (3, –2)

   Focus : (3, 1)

   Directrix : y + 5 = 0

   axis : x – 3 = 0

   tangent at the vertex: y + 2 = 0

6. Vertex : (2, 3)

   Focus : (2, 5)

   Directrix : y = 1

   axis : x = 2

7. $9x^2 + 25y^2 = 225$

8. $12x^2 + 5y^2 = 128$

9. Locus is ellipse: $8x^2 + 9y^2 = 288$

10. Foci : $\left(\pm\sqrt{53}, 0\right)$

    Vertices : $\left(\pm 2, 0\right)$

    semi–transverse axis : 2

    semi–conjugate axis : 7

    Asymptotes : $y = \pm\frac{7}{2}x$

Summary:

Conics

In this chapter, we have focused on the followings and try to elaborate in an easy and understandable way

- Definition of the conics,
- Define the circle and the equation of circle,
- Define the parabola, the equation of parabola and technique to trace the parabola,
- Define the ellipse, the equation of ellipse and technique to trace the ellipse,
- Define the hyperbola, the equation of hyperbola and technique to trace the hyperbola.

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