Discipline Courses-I
Semester-I
Paper: Calculus-I
Lesson: Curvature
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1. Learning Outcomes

After you have read this chapter, you should be able to

- Define the curvature,
- find the radius of curvature,
- interpretation of curvature in two dimensional space,
- find the tangential and normal components of acceleration.

2. Introduction

The curvature of a curve at a given point is a measure of the rate of change of bending of the curve at that point. You can imagine that the turn on a 25 paise coin indicates a more rapid change of direction per unit length of arc length than that on a one rupee coin. Mathematically speaking, we say that the curvature at a point on a 25 paise coin is greater than that at the corresponding point on a one rupee coin. Now we give the proper definition of the curvature of a curve at a point.

3. Curvature:

If C is a space curve defined by the function $\vec{r}(\mathrm{u})$, then we have seen that $\frac{d\vec{r}}{du}$ is a vector in the direction of tangent to C. If the scalar u is taken as the arc length S measured from some fixed point on C, then $\frac{d\vec{r}}{ds}$ is a unit tangent vector to C and is denoted by $\hat{T}$. The rate at which $\hat{T}$ change with respect to S is a measure of the Curvature of C and is given by $\frac{d\hat{T}}{ds}$. The direction of $\frac{d\hat{T}}{ds}$ at any given point on C is normal to the curve at that point. If $\hat{N}$ is a unit vector in this normal direction, it is called the principal normal to the curve. Then $\frac{d\hat{T}}{ds} = \kappa \hat{N}$, where κ is called the Curvature of C at the specified point

3.1. Radius of Curvature:- The quantity $\rho = \frac{1}{\kappa}$ is called the radius of curvature

Definition: If C is a smooth cure in two dimensional space or three dimensional space that is parameterized by arc length then the curvature of C, denoted by $\kappa = \kappa(s) =$ Greek "kappa" is defined by

$$\kappa(s) = \left|\frac{d\hat{T}}{ds}\right| = \left|\vec{r}''(s)\right|$$

Value addition: Alternative formula for calculating Curvature
If $\hat{r}(t)$ is a smooth curve, then curvature is $k = \frac{1}{\vec{v}}\left|\frac{d\hat{T}}{dt}\right|$, where $\hat{T} = \frac{\vec{v}}{\left|\vec{v}\right|}$ is the unit tangent vector and $\left|\hat{v}\right| = \frac{ds}{dt}$.

Example 1: Show that a circle of radius a centred at the origin has constant curvature $\frac{1}{a}$.

Solution: The equation of circle of radius a centred at origin in terms of arc length can be written as

$$\vec{r}(s) = a\cos\left(s/a\right)\hat{i} + a\sin\left(s/a\right)\hat{j},\ o \le s \le 2\pi a$$

On differentiating above equation w.r.t.s

$$\vec{r}'(s) = -a\sin\left(s/a\right).\frac{1}{a}\hat{i} + a\cos\left(s/a\right)\frac{1}{a}\hat{j}$$

$$= -\sin\left(s/a\right)\hat{i} + \cos\left(s/a\right)\hat{j}$$

Again differentiate w.r.t.s

$$\vec{r}''(s) = -\cos\left(s/a\right).\frac{1}{a}\hat{i} - \sin\left(s/a\right).\frac{1}{a}\hat{j}$$

$$= -\frac{1}{a}\cos\left(s/a\right)\hat{i} - \frac{1}{a}\sin\left(s/a\right)\hat{j}$$

$$\Rightarrow K(s) = \left|\vec{r}''(s)\right| = \sqrt{\left(-\frac{1}{a}\cos\left(s/a\right)\right)^2 + \left(-\frac{1}{a}\sin\left(s/a\right)\right)^2}$$

$$= \frac{1}{a}\sqrt{\cos^2\left(s/a\right) + \sin^2\left(s/a\right)} = \frac{1}{a}$$

Hence, circle has constant curvature $\frac{1}{a}$.

Example 2: Show that a line in two dimensional space or three dimensional space has zero curvature.

Solution: The equation of a line in two dimensional space and three dimensional space in terms of arc length is given by

$$\vec{r} = \vec{r}_0 + s\hat{u} \qquad (1)$$

Where $\vec{r}_0$ is the term in of point on the line and $\hat{u}$ is a unit vector parallel to the line.

On differentiating equation (1) w.r.t.s

$$\vec{r}''(s) = \frac{d\vec{r}'}{ds} = \frac{d}{ds}\left(\vec{r}_o + s\hat{u}\right) = O + \hat{u} = \hat{u} \quad \left(\because \vec{r}_O \ is \ \cos\tan t\right)$$

Again Differentiating w.r.t.s, we get

$$\vec{r}''(s) = \frac{d\vec{r}'}{ds} = \frac{d}{ds}\left(\hat{u}\right) = 0 \quad \left(\text{since } \hat{u} \text{ is constant}\right)$$

$$\Rightarrow \qquad k(s) = \left|\vec{r}''(s)\right| = 0 .$$

3.2. Formulas for Curvature:

We state the theorem which provides two formulas for curvature in terms of a general parameter t.

Theorem 1: If $\vec{r}(t)$ is a smooth vector-valued function in two dimensional space or three dimensional space, then for each value of t at which $\hat{T}'(t)$ and $\vec{r}''(t)$ exist, the Curvature k can be expressed as

(i) $$k(t) = \frac{\left|\hat{T}'(t)\right|}{\left|\vec{r}'(t)\right|} \qquad (2)$$

(ii) $$k(t) = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3} \qquad (3)$$

Where $\hat{T}(t)$ is unit tangent vector at t.

Value Addition: Note
Formula (2) will be applicable, when $\hat{T}(t)$ is known, however formula (3) will be applicable when $\vec{r}(t)$ and its derivatives are known.

Example 3: Find the Curvature for the circular helix defined by x = a cos t, y = asin t , z = ct, where a > 0.

Solution: The radius vector for the helix is given by

$$\vec{r}(\mathrm{t}) = x(\mathrm{t})\,\hat{i} + y(\mathrm{t})\,\hat{j} + z(\mathrm{t})\,\hat{k}$$

$$= a\cos t\,\hat{i} + a\sin t\;\hat{j} + ct\,\hat{k}$$

On differentiating above w.r.t.t

$$\vec{r}'(\mathrm{t}) = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + c\,\hat{k}$$

Again Differentiating w.r.t, we get

$$\vec{r}''(\mathrm{t}) = -a\cos t\,\hat{i} + a\sin t\,\hat{j}$$

Now,

$$\vec{r}'(\mathrm{t}) \times \vec{r}''(\mathrm{t}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & o \end{vmatrix}$$

$$= ac\sin t\,\hat{i} - \hat{j}\;ac\cos t + \hat{k}\,a^2$$

Hence,

$$\left|\vec{r}'(t)\right| = \sqrt{(-a\sin t)^2 + (a\,\cos\,t)^2 + c^2}$$

$$= \sqrt{a^2 + c^2}$$

$$\left|\vec{r}'(t) \times \vec{r}''(t)\right| = \sqrt{(ac\sin t)^2 + (-ac\,\cos\,t)^2 + \left(a^2\right)^2}$$

$$= \sqrt{a^2c^2 + a^4}$$

$$= a\sqrt{c^2 + a^2}$$

Hence,

$$k(\text{t}) = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$$

$$= \frac{a\sqrt{a^2 + c^2}}{\left(\sqrt{a^2 + c^2}\right)^3} = \frac{a}{a^2 + c^2}$$

Which is independent to t

Therefore, helix has constant curvature.

Value Addition: Note
(1) if c = 0, the helix reduces to a circle of radius a and its curvature reduces to $\frac{1}{a}$ (ii) If a = 0, the helix becomes the z axis and its curvature reduces to 0.

Example 4 : Find the curvature of the ellipse at the end point of the major and minor axes defined by $\vec{r} = 2\cos t\,\hat{i} + 3\sin t\,\hat{j},\ 0 \le t \le 2\pi$ and use a graphing utility to generate the graph of k(t).

Solution: Given, $\vec{r}(t) = 2\cos t\,\hat{i} + 3\sin t\,\hat{j} + o\,\hat{k}$

On differentiating above w.r.t. we have

$$\vec{r}'(t) = -2\sin t\,\hat{i} + 3\cos t\,\hat{j}$$

Again differentiate w.r.t.t, we get

$$\vec{r}''(t) = -2\cos t\,\hat{i} - 3\sin t\,\hat{j}$$

Now,

$$\vec{r}'(\text{t}) \times \vec{r}''(\text{t}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin t & 3\cos t & o \\ -2\cos t & -3\sin t & o \end{vmatrix}$$

$$= \hat{i}\,(o) - \hat{j}\,(o) + \hat{k}\,\left(6\sin^2 t + 6\cos^2 t\right) = 6\hat{k}$$

Hence,

$$\left|\vec{r}'(\text{t})\right| = \sqrt{\left(-2\cos t\right)^2 + \left(-3\sin t\right)^2}$$

$$= \sqrt{4\cos^2 t + 9\sin^2 t}$$

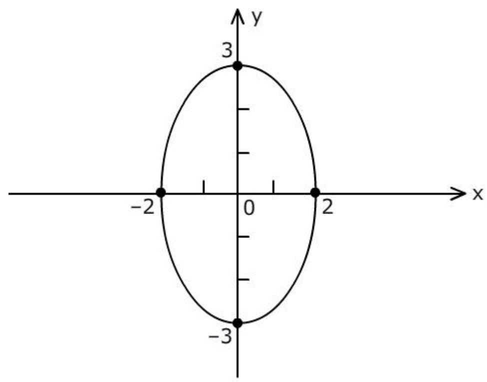
$$\vec{r}'(\text{t}) \times \vec{r}''(\text{t}) = \sqrt{\left(6\right)^2} = 6$$

We know that

$$k(\text{t}) = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$$

$$= \frac{6}{\left(4\sin^2 t + 9\cos^2 t\right)^{3/2}} \qquad (4)$$

Since, the end point of the minor axis are (2, 0) and (-2, 0) which correspond to t = 0 and t= π respectively



r = 2 cos ti + 3 sin tj

Figure 1

Substituting these values in equation (4), we get

$$k = k(o) = \frac{6}{\left(4\sin^2 0 + 9\cos^2 0\right)^{3/2}} = \frac{6}{27} = 2/9$$

$$k = k(\pi) = \frac{6}{\left(4\sin^2 \pi + 9\cos^2 \pi\right)^{3/2}} = \frac{6}{27} = 2/9$$

Since the end points of the major axis are (0, 3)and (0, -3) which correspond to $t = \pi/2$ *and* $t = 3\pi/2$ respectively. Substituting these values in (4), we get

$$k = k(\pi/2) = \frac{6}{\left(4\sin^2 \pi/2 + 9\cos^2 \pi/2\right)^{3/2}} = \frac{6}{4^{3/2}} = 3/4$$

$$k = k\left(3\pi/2\right) = \frac{6}{\left(4\sin^2 3\pi/2 + 9\cos^2 3\pi/2\right)^{3/2}} = \frac{6}{4^{3/2}} = 3/4$$

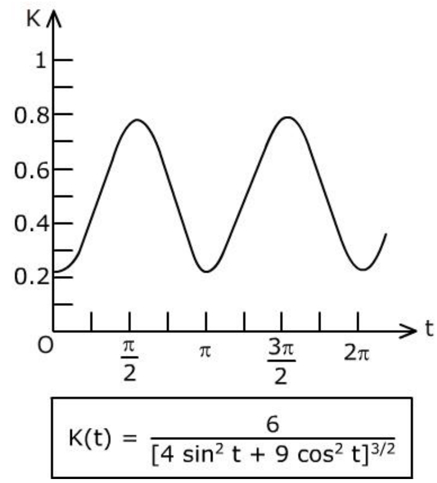


Figure 2

3.3. An interpretation of Curvature in 2 dimensional-space:

A useful geometric interpretation of curvature in two dimensional space can be obtained by considering the angel ϕ measured counter clock wise from the direction of positive x axis to the unit tangent vector $\hat{T}$. We can express $\hat{T}$ in terms of ϕ as follows

$$\hat{T}(\phi) = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

On differentiating above equation w.r.t. ϕ

$$\frac{d\hat{T}}{d\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\Rightarrow \qquad \frac{d\hat{T}}{ds} = \frac{d\hat{T}}{d\phi}.\frac{d\phi}{ds}$$

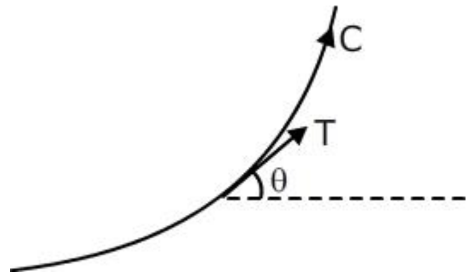


Figure 3

We known that

$$k(s) = \left|\frac{d\hat{T}}{ds}\right| = \left|\frac{d\phi}{ds}\right| \left|\frac{d\hat{T}}{d\phi}\right|$$

$$= \sqrt{(-\sin\phi)^2 + \cos^2\phi} \;\; \left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{ds}\right|$$

Which tells us that curvature in 2-dimensional space can be expressed as the magnitude of the rate of change of ϕ with respect to S.

Value addition : Formula Summary
(1) $k(s) = \left\|\frac{d\hat{T}}{ds}\right\| = \left\|\vec{r}''(s)\right\|$ (2) $k(t) = \frac{\left\|\hat{T}'(t)\right\|}{\left\|\vec{r}'(t)\right\|}$ (3) $k(t) = \frac{\left\|\vec{r}'(t) \times \vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|^3}$ (4) Formula for calculating $\hat{N}$: If $\vec{r}(t)$ is a smooth curve, then

principal unit normal is $\hat{N} = \dfrac{\dfrac{d\hat{T}}{dt}}{\left|\dfrac{d\hat{T}}{dt}\right|}$, where $\hat{T} = \dfrac{\vec{v}}{|\vec{v}|}$ is the unit tangent vector.

3.4. Exercise I:

Find curvature k(s) for the following curves

1. $\vec{r} = \sin\left(1 + s/2\right)\hat{i} + \cos\left(1 + s/2\right)\hat{j} + \sqrt{3}\left(1 + s/2\right)\hat{k}$

2. $\vec{r} = \left(1 - \frac{2}{3}s\right)^{3/2}\hat{i} + \left(\frac{2}{3}s\right)^{3/2}\hat{j}, \ 0 \le s \le 3/2$

Find $\hat{T}$, $\hat{N}$ and curvature κ for the following curves

3. $\vec{r}(t) = (3\sin t)\hat{i} + (4\cos t)\hat{j} + 4t\hat{k}$

4. $\vec{r}(t) = 2t\hat{i} + t^2\hat{j} + \frac{1}{3}t^3\hat{k}, \ t > 0$

5. $\vec{r}(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j} + 2\hat{k}$

6. $\vec{r}(t) = (6\sin 2t)\hat{i} + (6\cos 2t)\hat{j} + 5t\hat{k}$

7. $\vec{r}(t) = t^2\hat{i} + t^3\hat{j}$

8. $\vec{r}(t) = 4\cos t\,\hat{i} + \sin t\,\hat{j}$

9. $\vec{r}(t) = e^{3t}\hat{i} + e^{-t}\hat{j}$

10. $\vec{r}(t) = (\cos^3 t)\hat{i} + (\sin^3 2t)\hat{j}\ ,\ 0 < + \pi/2$

11. $\vec{r}(t) = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + t\,\hat{k}$

12. $\vec{r}(t) = t\,\hat{i} + \frac{1}{2}t^2\,\hat{j} + \frac{1}{3}t^3\,\hat{k}$

13. $\vec{r}(t) = +\hat{i} + t\,\hat{j} + t^2\,\hat{k}$

14. $\vec{r}(t) = (\cosh t)\hat{i} - (\sinh t)\hat{j} + t\,\hat{k}$

15. $\vec{r}(t) = (\cos t + \sin t)\hat{i} + (\sin t - \cos t)\hat{j} + 3\hat{k}$

4. Tangential and Normal Components of Acceleration:

A particle is moving along a smooth curve, to find components of its acceleration along the tangent and the normal to the curve at any instant. This Question is answered by the following theorem.

Theorem 2: Tangential and Normal Components of Acceleration

An object moving along a smooth curve has velocity $\vec{v}$ and acceleration $\vec{A}$

$$\vec{v} = \left(\frac{ds}{dt}\right)\hat{T} \quad and$$

$$\vec{A} = \frac{d^2 s}{dt^2}\hat{T} + k\left(\frac{ds}{dt}\right)^2 \hat{N}$$

and s is the are length along the trajectory.

Proof: In order to derive the formula for $\vec{A}$

$$\frac{d\hat{T}}{ds} = \left|\frac{d\hat{T}}{ds}\right|\hat{N} = \kappa\,\hat{N}$$

Since the derivative of $\hat{T}(\text{s})$ is orthogonal to $\hat{T}$

We know that

$$\vec{A} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\hat{T}\frac{ds}{dt}\right)$$

By product Rule of Differentiation, we get

$$\vec{A} = \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\frac{d\hat{T}}{dt}$$

$$= \frac{d^2s}{dt^2}\hat{T} + \frac{ds}{dt}\left(\frac{d\hat{T}}{dt}\frac{ds}{dt}\right)$$

$$= \frac{d^2s}{dt^2}\hat{T} + \left(\frac{ds}{dt}\right)^2\frac{d\hat{T}}{ds}$$

$$= \frac{d^2s}{dt^2}\hat{T} + \left(\frac{ds}{dt}\right)^2\left(k\,\hat{N}\right) \quad \left(\text{since } k\,\hat{N} = \frac{d\hat{T}}{ds}\right)$$

Value Addition: Formula for Tangential and Normal components of Acceleration
The acceleration $\vec{A}$ of a moving object can be written as $\vec{A} = A_T\,\hat{T} + A_N\,\hat{N}$ Where, $A_T = \frac{d}{dt}\left\|\vec{v}\right\|$ is the tangential component of acceleration $A_N = k\left(\frac{ds}{dt}\right)^2 = \kappa\left\|\vec{v}\right\|^2$ is the normal component of acceleration.

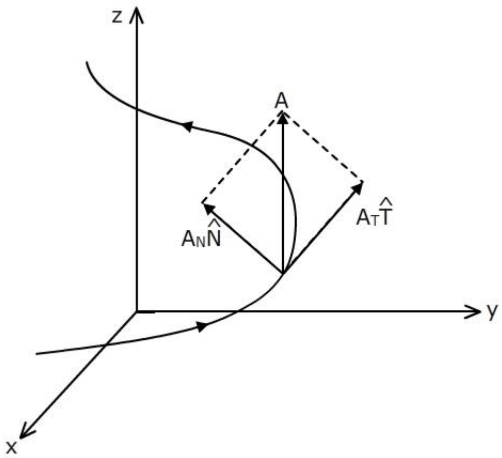


Figure 4: Components of acceleration $\vec{A} = A_T\,\hat{T} + A_N\,\hat{N}$

The following theorem provides alternative formulas for A_T and A_N in terms of derivatives $\vec{r}'$ and $\vec{r}''$ of the position vector $\vec{r}(t)$.

Theorem 3: Formulas for the components of acceleration

Let $\vec{r}(t)$ be the position vector of an object moving along a smooth curve C. Then the tangential and normal components of the object's acceleration are given by

$$A_t = \frac{\vec{r}'.\vec{r}''}{\left|\vec{r}'\right|} \; and \; A_N \; \frac{\left|\vec{r}' \times \vec{r}''\right|}{\left|\vec{r}'\right|}$$

Proof : Let θ be the angel between $\vec{r}'$ and $\vec{r}''$. Since the acceleration is given by $\vec{A} = \vec{r}''$

Using formula, $\vec{r}'.\vec{r}'' = \left|\vec{r}'\right| \left|\vec{r}''\right| \cos\theta$

We have

$$A_T = \left|\vec{A}\right|\cos\theta = \frac{\left|\vec{r}'\right|\left|\vec{A}\right|\cos\theta}{\left|\vec{r}'\right|}$$

$$\Rightarrow \quad A_T = \frac{\vec{r}'.\vec{r}''}{\left|\vec{r}'\right|}$$

Using formula, $\sin\theta = \dfrac{\left|\vec{r}'\times\vec{r}''\right|}{\left|\vec{r}'\right|\left|\vec{r}''\right|}$, we have

$$A_N = \left|\vec{A}\right|\sin\theta = \frac{\left|\vec{r}'\right|\left|\vec{A}\right|\sin\theta}{\left|\vec{r}'\right|}$$

$$\Rightarrow \quad A_N = \frac{\left|\vec{r}'\times\vec{r}''\right|}{\left|\vec{r}'\right|}$$

Value Addition :
Alternative formula of calculating normal component of acceleration $A_N = \sqrt{\left|\vec{A}\right| - (A_T)^2}$

Example 5: Find the tangential and normal components of the acceleration of an object that moves with position vector $\vec{r}(t) = t^3\hat{i} + t^2\hat{j} + t\hat{k}$

Solution: Given $\vec{r}(t) = t^3\hat{i} + t^2\hat{j} + t\hat{k}$

Differentiating above w.r.t. t

$$\vec{r''}(t) = \vec{v} = 3t^2 \hat{i} + 2t^2 \hat{j} + \hat{k}$$

Again differentiate w.rt.t

$$\vec{r''}(t) = \vec{A} = 6 + \hat{i} + 2\hat{j}$$

Now

$$\vec{r'}(t).\vec{r''}(t) = (3t)(6t) + (2t)(2)$$

$$= 18t^3 + 4t$$

$$\vec{r'} \times \vec{r''} = \hat{v} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{t} & \hat{k} \\ 3t^2 & 2t & t \\ 6t & 2 & 0 \end{vmatrix}$$

$$= 2\hat{i} + 6t\hat{j} - 6t^2\hat{k}$$

Hence, the components of acceleration are

$$A_T = \frac{\vec{r'}.\vec{r''}}{\left|\vec{r'}\right|} = \frac{\vec{v}.\vec{A}}{\left|\vec{v}\right|} = \frac{18t^3 + 4t}{\sqrt{9t^4 + 4t^2 + 1}}$$

$$A_N = \frac{\left|\vec{r'} \times \vec{r''}\right|}{\left|\vec{r'}\right|} = \frac{\left|\vec{v} \times \vec{A}\right|}{\left|\vec{v}\right|} = \frac{\sqrt{(-2)^2 + (6t)^2 + (-6t^2)^2}}{\sqrt{9t^4 + 4t^2 + 1}}$$

$$= 2\sqrt{\frac{9t^4 + 9t^2 + 1}{9t^4 + 4t^2 + 1}}$$

Example 6: An object moves with position vector $\vec{r}(t) = (\cos t, \sin t, t)$. Find tangential and normal components of acceleration

Solution: Given $\vec{r}(t) = (\cos t, \sin t, t)$

On differentiating w.r.t.t. we get

$$\vec{v} = \vec{r}'(t) = (-\sin t, \cos t, 1)$$

Again differentiate w.r.t.t. we get

$$\vec{A} = \vec{r}''(t) = (-\cos t,\ -\sin t,\ 0)$$

$$\frac{ds}{dt} = \left|\vec{v}\right| = \sqrt{(\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

$$A_T = \frac{d^2 s}{dt^2} = 0$$

$$A_N = \sqrt{\left|\vec{v}\right|^2 - A_T^2} = \sqrt{\left(\sqrt{\cos^2 t + \sin^2 t}\right)^2 - (0)^2}$$

$$= \sqrt{1} = 1$$

$$\Rightarrow \qquad \vec{A} = A_T \hat{T} + A_N \hat{N} = o\hat{T} + 1\hat{N} = \hat{N}$$

Hence, the acceleration is normal to the curve.

4.1. Applications of Modeling:

In order to compute the tangential and normal components of acceleration A_T and A_N, we will illustrate some applications. According to Newton's second law of motion

$$\vec{F} = m\vec{A}$$

where $\vec{A}$ is the acceleration

since, $\vec{A} = A_T \hat{T} + A_N \hat{N}$

$$\Rightarrow \quad \vec{F} = m\vec{A} = mA_T \hat{T} + mA_N \hat{N}$$

$$= F_T \hat{T} + F_N \hat{N}$$

where, $F_T = m\dfrac{d^2s}{dt^2}$ *and* $F_N = mk\left(\dfrac{ds}{dt}\right)^2$

Example 7: Tendency of a vehicle to skid

A car weighing 2,700 lb makes a turn on a flat road while traveling at 56 ft/sec (about 38 ml/h). If the radius of the turn is 70 ft. How much frictional force is required to keep the car from skidding?

Solution : Given, w = 2,700 lb

g = 32 ft/s^2, ρ = 70ft

$\dfrac{ds}{dt}$ = 56 ft/s

We know that

$$m = \frac{W}{g} = \frac{2700}{32}$$

$$k = \frac{1}{\rho} = \frac{1}{70}$$

The frictional force required to keep the car from skidding is given by

$$F_N = mk\left(\frac{ds}{dt}\right)^2$$

$$= \left(\frac{2700}{32}\right)\left(\frac{1}{70}\right)(56)^2$$

$$= 3780\ \ell b$$

Theorem 4: Acceleration of an object with constant speed

The acceleration of an object moving with constant speed is always orthogonal to the direction of motion.

Proof: since an object has constant speed

$\Rightarrow \quad \left|\vec{r}'(t)\right|$ is constant

By theorem, $\vec{r}(t)$ is perpendicular to $\vec{r}''(t) = \vec{A}(t)$

But $\vec{r}'(t)$ is in the direction of the object's motion

$\Rightarrow \vec{A}(t)$ is orthogonal to the direction of motion

Example 8: Period of a satellite

An artificial satellite travels at constant speed in a stable circular orbit 20,000 km above the earth's surface. How long does it take for the satellite to make one complete circuit of the earth?

Solution : Since earth is a sphere of radius 6,440 k.m.

$$\therefore \quad R = \underbrace{6,440}_{Radious\ of\ earth} + \overbrace{20,000}^{height} = 26,440\,km$$

Let m and v be satellite's mass and speed

Thus for stability, we have

$$\frac{mv^2}{R} = \frac{GmM}{R^2}$$

Where M is earth's mass and G is gravitation constant

But GM = 398,600 km^3/s^2 and

By substituting R = 26,440, We get

$$v = \sqrt{\frac{GM}{R}} = \sqrt{\frac{398,600}{26,440}} = 3.88273653 \qquad (approximately)$$

Let T be the time required for the satellite to make one complete circuit of the earth

$$\therefore \qquad T = \frac{2\pi R}{v} = \frac{2\pi(26,440)}{3.88273653}$$

$$= 42,786.16852 \text{ seconds} \qquad (approximately)$$

= 11hr 53 min (approximately)

Example 9: Find the acceleration scalar components A_T, A_N without finding $\hat{T}$ and , write the acceleration of the motion

$\vec{r}(t) = (\sin t - t\cos t)\hat{i} + (\cos t + t\sin t)\hat{j},\ t > 0$ in the form $\vec{A} = A_T\,\hat{T} + A_N\,\hat{N}$.

Solution : Given $\vec{r}(t) = (\sin t - t\cos t)\hat{i} + (\cos t + t\sin t)\hat{j}$

$$\Rightarrow \qquad \vec{v} = \frac{d\vec{r}}{dt}(\cos t - \cos t + t\sin t)\hat{i} + (-\sin t + \sin t + t\cos t)\hat{j}$$

$$= t\sin t\,\hat{i} + t\cos t\,\hat{j}$$

$$\Rightarrow \quad \left|\vec{v}\right| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = t$$

We know that

$$A_T = \frac{d\left|\vec{v}\right|}{dt} = 1$$

$$\Rightarrow \quad \left|\vec{A}\right| = \frac{d\vec{v}}{dt}\left(\sin t + t\cos t\right)\hat{i} + \left(\cos t - t\sin t\right)\hat{j}$$

$$\Rightarrow \quad \left|\vec{A}\right|^2 = t^2 + 1$$

We know that

$$A_N = \sqrt{\left(\vec{A}\right)^2 - \left(A_T\right)^2} = \sqrt{t^2 + 1 - 1} = \sqrt{t^2} = t$$

$$\vec{A} = A_T\,\hat{T} + A_N\,\hat{N} = \hat{T} + t\,\hat{N}$$

4.2. Exercise-II:

Find the tangential and normal components of the object's acceleration for the following curves.

1. $\vec{r}(t) = \frac{5}{13}\cos t\,\hat{i} + \frac{12}{13}(1-\cos t)\,\hat{j} + \sin t\,\hat{k}$

2. $\vec{r}(t) = \sin t\,\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k}$

3. $\vec{r}(t) = (a\sin t)\,\hat{i} + (a\cos t)\,\hat{j} + at\,\hat{k}$

4. $\vec{r}(t) = (1+3t)\hat{i} + (t-2)\hat{j} - 3t\hat{k}$

5. $\vec{r}(t) = (3\cos t, 2\sin t)$

6. $\vec{r}(t) = (t\sin t, t\cos t)$

7. $\vec{r}(t) = \left(t\hat{i} + e^{+}\hat{j}\right)$

8. $\vec{r}(t) = \left(t\hat{i} + t^2\hat{j}\right)$

Write $\vec{A}$ in the form $\vec{A} = A_T\hat{T} + A_N\hat{N}$ for the following curves at the green value of t without finding $\vec{T}$ and $\vec{N}$

9. $\vec{r}(t) = (t+1)\hat{i} + 2t\hat{j} + t^2\hat{k}\ ,\ t = 1$

10. $\vec{r}(t) = (t\cos t)\hat{i} + (t\sin t)\hat{j} + t^2\hat{k}\ ,\ t = 0$

11. $\vec{r}(t) = t^2\hat{i} + \left(t + \frac{1}{3}t^3\right)\hat{j} + \left(t - \frac{1}{3}t^3\right)\hat{k}\ ,\ t = 0$

12. $\vec{r}(t) = (e^t\cos t)\hat{i} + (e^t\sin t)\hat{j} + \sqrt{2}e^t\hat{k}\ ,\ t = 0$

For the following problem the speed $\left|\vec{v}\right|$ of a moving object is given. Find A_T, and A_N the tangential and normal component of acceleration at the indicated time

13. $\left|\vec{v}\right| = \sqrt{5t^2+3}\ ;\ t = 1$

14. $\left|\vec{v}\right| = \sqrt{t^2 + t + 1}\,;\;\; t = 3$

15. $\left|\vec{v}\right| = \sqrt{\sin^2 t + \cos 2t}\;\;;\;\; t = 0$

16. $\left|\vec{v}\right| = \sqrt{e^{-t} + t^2}\;\;;\;\; t = 0$

Summary:

In this lesson, we have emphasize on the followings

- Definition of curvature,
- how to find the radius of curvature,
- interpretation of curvature in two dimensional and three dimensional space ,
- to find the tangential and normal components of acceleration.

References for further readings:

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