Discipline Courses-I

Semester-I

Paper: Calculus-I

Lesson: Differentiation and Integration of Vector Functions

Lesson Developer: Chaman Singh & Kapil Kumar

College/Department: Department of Mathematics, Acharya Narendra Dev College , University of Delhi

Differentiation and Integration of Vector Functions

Table of Contents

1. Learning outcomes:

After studying this chapter you should be able to understand the

- Differentiation of vector functions.
- Differentiation of vector function in terms of its components
- Derivative of a Vector function of function
- Successive Differentiation
- Tangent Vector
- Geometrical significance of $\frac{dR}{dt}$ and Tangent to the curve

- Unit Tangent Vector
- Smooth Curve
- Orthogonality of a function of constant length and its derivative
- Application to curvilinear motion speed, velocity and acceleration
- Integration of Vector Functions.

2. Introduction:

Vector calculus is used to model mathematically a vast range of engineering phenomena including electromagnetic fields, electrostatics, heat flow in nuclear reactors and air flow around aircraft. It answers questions like how to define and measure the fluid velocity, variation of temperature, force, magnetic flux etc. In the 3 dimensional engineering world, one wants to know things like the stress and strain inside a structure, the induced electromagnetic field a round an aerial or the vorticity of the air flow over a wing. For such questions, we must know how to differentiate and integrate vector quantities with three components (in directions i, j and k) which depend on three co-ordinates x, y, z. Vector calculus provides the necessary mathematical notation and techniques for dealing with such issues. For the sack of simplicity, in this chapter, we have denoted the vector functions by the capital letters.

3. Differentiation of Vector Function:

Let F(t) be a vector function of the scalar variable t. Then the derivative of F(t) with respect to t is denoted by $\frac{dF}{dt}$ and defined as

$$\frac{dF}{dt} = \lim_{\delta t \to 0} \frac{F(t + \delta t) - F(t)}{\delta t}$$

The function F(t) is said to be differentiable if $\frac{dF}{dt}$ exists.

Value Addition: Note
The derivative of a vector function i.e., $\frac{dF}{dt}$ is also a vector fucntion.

3.1. Differentiation Formulae For The Vector Function:

Let F(t), G(t) and H(t) be differentiable vector function of a scalar variable t and ϕ(t) be a differentiable scalar function of t, then

(i) $\frac{d}{dt}(F \pm G) = \frac{dF}{dt} \pm \frac{dG}{dt}$

(ii) $\frac{d}{dt}(F.G) = F.\frac{dG}{dt} + \frac{dF}{dt}.G$

(iii) $\frac{d}{dt}(F \times G) = F \times \frac{dG}{dt} + \frac{dF}{dt} \times G$

(iv) $\frac{d}{dt}(\phi F) = \phi\frac{dF}{dt} + \frac{d\phi}{dt}F$

(v) $\frac{d}{dt}[F\ G\ H] = \left[\frac{dF}{dt}\ G\ H\right] + \left[F\ \frac{dG}{dt}\ H\right] + \left[F\ G\ \frac{dH}{dt}\right]$

(vi) $\frac{d}{dt}\{F \times (G \times H)\} = \frac{dF}{dt} \times (G \times H) + F \times \left[\frac{dG}{dt} \times H\right] + F \times \left[G \times \frac{dH}{dt}\right].$

Proof :

1. $$\frac{d}{dt}(F+G) = \lim_{\delta t \to 0} \frac{\{(F+\delta F) + (G+\delta G)\} - (F+G)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta F + \delta G}{\delta t} = \lim_{\delta t \to 0}\left(\frac{\delta F}{\delta t} + \frac{\delta G}{dt}\right)$$

$$= \lim_{\delta t \to 0} \frac{\delta F}{\delta t} + \lim_{\delta t \to 0} \frac{\delta G}{\delta t} = \frac{dF}{dt} + \frac{dG}{dt}$$

Thus the derivative of the sum of two vectors is equal to the sum of their derivatives, as it is also occurs in Scalar Calculus.

Similarly we can prove that $\frac{d}{dt}(F-G) = \frac{dF}{dt} - \frac{dG}{dt}$.

Value Addition: Note
If F_1, F_2,. . ., F_n are vector functions of a scalar *t*, then $\frac{d}{dt}(F_1 + F_2 + ... + F_n) = \frac{dF_1}{dt} + \frac{dF_2}{dt} + ... + \frac{dF_n}{dt}$.

2. $$\frac{d}{dt}\left(F \bullet G\right) = \lim_{\delta t \to 0} \frac{\left(F + \delta F\right) \bullet \left(G + \delta G\right) - F \bullet G}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{F \bullet G + F \bullet \delta G + \delta F \bullet G + \delta F \bullet \delta G - F \bullet G}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{F \bullet \delta G + \delta F \bullet G + \delta F \bullet \delta G}{\delta t}$$

$$= \lim_{\delta t \to 0} \left\{ F \bullet \frac{\delta G}{\delta t} + \frac{\delta F}{\delta t} \bullet G + \frac{\delta F}{\delta t} \bullet \delta G \right\}$$

$$= \lim_{\delta t \to 0} F \bullet \frac{bG}{\delta t} + \lim_{\delta t \to 0} \frac{\delta F}{\delta t} \bullet G + \lim_{\delta t \to 0} \frac{\delta F}{\delta t} \bullet \delta G$$

$$= F \bullet \frac{dG}{dt} + \frac{dF}{dt} \bullet G + \frac{dF}{dt} \bullet 0,$$ since $\delta G \to$ zero vector as $\delta t \to 0$

$$= F \bullet \frac{dG}{dt} + \frac{dF}{dt} \bullet G.$$

Value Addition: Note
We know that F • G = G • F. Therefore while evaluating $\frac{d}{dt}\left(F \bullet G\right)$, we should not bother about the order of the factors.

3. $$\frac{d}{dt}\left(F \times G\right) = \lim_{\delta t \to 0} \frac{\left(F + \delta F\right) \times \left(G + \delta G\right) - F \times G}{dt}$$

$$= \lim_{\delta t \to 0} \frac{F \times G + F \times \delta G + \delta F \times G + \delta F \times \delta G - F \times G}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{F \times \delta G + \delta F \times G + \delta F \times \delta G}{\delta t}$$

$$= \lim_{\delta t \to 0} \left\{ F \times \frac{\delta G}{\delta t} + \frac{\delta F}{\delta t} \times G + \frac{\delta F}{\delta t} \times \delta G \right\}$$

$$= \lim_{\delta t \to 0} F \times \frac{\delta G}{dt} + \lim_{\delta t \to 0} \frac{\delta F}{dt} \times G + \lim_{\delta t \to 0} \frac{\delta F}{dt} \times \delta G$$

$$= F \times \frac{dG}{dt} + \frac{dF}{dt} \times G + \frac{dF}{dt} \times 0$$ since $\delta G \to$ zero vector as $\delta t \to 0$

$$= F \times \frac{dG}{dt} + \frac{dF}{dt} \times G \text{ .}$$

Value Addition: Note
We know that cross product of two vectors is not commutative because F $\times$ G = –G $\times$ F. Therefore while evaluating $\frac{d}{dt}(F \times G)$, we must maintain the order of the factors a and b.

4. $$\frac{d}{dt}(\phi F) = \lim_{\delta t \to 0} \frac{(\phi + \delta\phi)(F + \delta F) - \phi F}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\phi F + \phi \delta F + \delta\phi F + \delta\phi\delta F - \phi F}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\phi \delta F + \delta\phi F + \delta\phi\delta F}{\delta t}$$

$$= \lim_{\delta t \to 0} \left\{ \phi \frac{\delta F}{\delta t} + \frac{\delta\phi}{\delta t} F + \frac{\delta\phi}{\delta t} \phi F \right\}$$

$$= \lim_{\delta t \to 0} \phi \frac{\delta F}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\phi}{\delta t} F + \lim_{\delta t \to 0} \frac{\delta\phi}{\delta t} \phi F$$

$$= \phi \frac{dF}{dt} + \frac{d\phi}{dt} F + \frac{d\phi}{dt} 0$$ since δ**F** $\to$ zero vector as $\delta t \to 0$

$$= \phi \frac{dF}{dt} + \frac{d\phi}{dt} F \text{ .}$$

Value Addition: Note
ϕF is the multiplication of a vector by a scalar. In the case of such multiplication we usually write the scalar in the first position and the vector in the second position.

5. $$\frac{d}{dt}[F \ G \ H] = \frac{d}{dt}\{F \cdot (G \times H)\}$$

$$= F \cdot \frac{d}{dt}(G \times H) + \frac{dF}{dt} \cdot (G \times H) \qquad [\text{by rule } (2)]$$

$$= F \cdot \left(\frac{dG}{dt} \times H + G \times \frac{dH}{dt} \right) + \frac{dF}{dt} \cdot (G \times H) \qquad [\text{by rule } (3)]$$

$$= F \cdot \left(G \times \frac{dH}{dt} \right) + F \cdot \left(\frac{dG}{dt} \times H \right) + \frac{dF}{dt} \cdot (G \times H)$$

$$= \left[F \;\; G \;\; \frac{dH}{dt} \right] + \left[F \;\; \frac{dG}{dt} \;\; H \right] + \left[\frac{dF}{dt} \;\; G \;\; H \right]$$

$$= \left[\frac{dF}{dt} \;\; G \;\; H \right] + \left[F \;\; \frac{dG}{dt} \;\; H \right] + \left[F \;\; G \;\; \frac{dH}{dt} \right].$$

Value Addition: Note
Here [F G H] is the scalar triple product of three vectors F, G and H. Therefore while evaluating $\frac{d}{dt}[F\ G\ H]$ we must maintain the cyclic order of each factor.

6. $$\frac{d}{dt}\{F \times (G \times H)\} = F \times \frac{d}{dt}(G \times H) + \frac{dF}{dt} \times (G \times H) \qquad [by\ rule\ (3)]$$

$$= F \times \left(\frac{dG}{dt} \times H + G \times \frac{dH}{dt} \right) + \frac{dF}{dt} \times (G \times H)$$

$$= F \times \left(\frac{dG}{dt} \times H \right) + F \times \left(G \times \frac{dH}{dt} \right) + \frac{dF}{dt} \times (G \times H)$$

$$= \frac{dF}{dt} \times (G \times H) + F \times \left(\frac{dG}{dt} \times H \right) + F \times \left(G \times \frac{dH}{dt} \right)$$

3.2. Derivative of vector function in terms of its components:

Let F(t) be a vector function of a scalar variable t such that

$$F(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

where the components $f_1(t)$, $f_2(t)$ and $f_3(t)$ are scalar functions of the variable t and i, j and k are the fixed unit vector.

Thus, we have

$$F + \delta F = (f_1 + \delta f_1)\hat{i} + (f_2 + \delta f_2)\hat{j} + (f_3 + \delta f_3)\hat{k}$$

$$\Rightarrow \qquad \delta F = (F + \delta F) - F = \delta f_1 \hat{i} + \delta f_2 \hat{j} + \delta f_3 \hat{k}$$

$$\Rightarrow \quad \frac{\delta F}{\delta t} = \frac{\delta f_1}{\delta t}\hat{i} + \frac{\delta f_2}{\delta t}\hat{j} + \frac{\delta f_3}{\delta t}\hat{k}$$

$$\Rightarrow \quad \lim_{t \to 0} \frac{\delta F}{\delta t} = \lim_{t \to 0} \left\{ \frac{\delta f_1}{\delta t}\hat{i} + \frac{\delta f_2}{\delta t}\hat{j} + \frac{\delta f_3}{\delta t}\hat{k} \right\}$$

$$\Rightarrow \quad \frac{dF}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Value Addition: Note
A vector function $F(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ is differentiable if and only if all its components $f_1(t)$, $f_2(t)$ and $f_3(t)$ are differentiable.

3.3. Derivative of a Constant Vector:

A vector is said to be constant vector if and only if its magnitude as well as direction are fixed. If either of these changes then the vector will change and thus it will not be constant.

Let F(t) be a constant vector, then

$F(t) = C$ (C is a constant vector) -----------(1)

$F + \delta F = C$ -----------(2)

Subtract (1) from (2), we get

$$\delta F = 0$$

Dividing by δt and taking the limit as δt → 0,

we get

$$\lim_{\delta t \to 0} \frac{\delta F}{\delta t} = 0 \quad or \quad \frac{df}{dt} = 0$$

Hence the derivative of a constant vector is a zero vector.

Value Addition: Alternative Method
Let $F(t) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is constant function f_1, f_2 and f_3 all are constant functions. Now,

$$\frac{dF}{dt} = \frac{d}{dt}\left(f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \right)$$

$$\Rightarrow \quad \frac{dF}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

$$\frac{dF}{dt} = \left(0\hat{i} + 0\hat{j} + 0\hat{k} \right)$$ Since f_1 , f_2 and f_3 are all constant

$$= 0$$

Thus, derivative of a constant function is null vector.

3.4. Derivative of a Vector function of function:

Let F is a differentiable vector function of a scalar variable g and g is a differentiable scalar function of another scalar variable t. Then F is a function of t.

An increment δt in t produces an increment δF in F and an increment δg in g. When

$\delta t \to 0$, $\delta F \to 0$, and $\delta g \to 0$,

We have

$$\frac{dF}{dt} = \lim_{\delta t \to 0} \frac{\delta F}{\delta t} = \lim_{\delta t \to 0} \left(\frac{\delta g}{\delta t} \cdot \frac{\delta F}{\delta g} \right)$$

$$= \left(\lim_{\delta t \to 0} \frac{\delta g}{\delta t} \right)\left(\lim_{\delta t \to 0} \frac{\delta F}{\delta g} \right)$$

$$= \frac{dg}{dt} \cdot \frac{dF}{dg}$$

Value Addition: Remember

That $\frac{dF}{dt}$ is a vector quantity and $\frac{dg}{dt}$ is a scalar quantity. Thus, $\frac{dg}{dt} . \frac{dF}{dg}$ is nothing but the multiplication of the vector $\frac{dF}{dg}$ by the scalar $\frac{dg}{dt}$.

3.5. Successive Differentiation:

If $\frac{dF}{dt}$ exists then the vector function F(t) is called differentiable. If we again differentiate $\frac{dF}{dt}$ with respect to t, then it is denoted by $\frac{d^2F}{dt}$ and is called second derivative of F(t) w.r.t. t and so on we continue differentiating successively up to n time and get

$$\frac{dF}{dt}, \frac{d^2F}{dt^2}, \frac{d^3F}{dt^3}, ---, \frac{d^nF}{dt^n}$$

where $\frac{d^nF}{dt^n}$ is called the n[th] derivative of f w.r.t. t.

Value Addition: Note
The derivatives of F(t) w.r.t. t i.e., $\frac{dF}{dt}, \frac{d^2F}{dt^2}, \frac{d^3F}{dt^3}, ---$ may also be denoted by $\dot{F}, \ddot{F}, \dddot{F}, ---$ respectively.

Example 1: If $F(t) = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$, find $\frac{dF}{dt}$ *and* $\frac{d^2F}{dt^2}$.

Solution : Given that

$$F = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$$

$$\therefore \quad \frac{dF}{dt} = \frac{d}{dt}(t+1)\hat{i} + \frac{d}{dt}(t^2+t+1)\hat{j} + \frac{d}{dt}(t^3+t^2+t+1)\hat{k}$$

$$= \hat{i} + (2t+1)\hat{j} + (3t^2+2t+1)\hat{k}.$$

Again differentiating we have,

$$\frac{d^2F}{dt^2} = \frac{d}{dt}\left(\frac{dF}{dt}\right) = \frac{d}{dt}(1)\hat{i} + \frac{d}{dt}(2t+1)\hat{j} + \frac{d}{dt}(3t^2+2t+1)\hat{k}$$

$$= 0\hat{i} + 2\hat{j} + (6t+2)\hat{k} = 2\hat{j} + (6t+2)\hat{k}$$

Example 2: If $F = \sin t\,\hat{i} + \cos t\,\hat{j} + t\,\hat{k}$, find

(i) $\dfrac{dF}{dt}$, (ii) $\dfrac{d^2F}{dt^2}$,

(iii) $\left|\dfrac{dF}{dt}\right|$, (iv) $\left|\dfrac{d^2F}{dt^2}\right|$.

Solution : Given that $F = \sin t\,\hat{i} + \cos t\,\hat{j} + t\,\hat{k}$

Therefore

(i) $$\frac{dF}{dt} = \frac{d}{dt}(\sin t)\hat{i} + \frac{d}{dt}(\cos t)\,\hat{j} + \frac{d}{dt}(t)\hat{k} = \cos t\,\hat{i} - \sin t\,\hat{j} + \hat{k}.$$

(ii) $$\frac{d^2F}{dt^2} = \frac{d}{dt}\left(\frac{dF}{dt}\right) = \frac{d}{dt}\,(\cos t)\hat{i} - \frac{d}{dt}(\sin t)\,\hat{j} + \frac{d}{dt}(1)\hat{k}$$

$$= -\sin t\, i - \cos t\, j + 0 = -\sin t\, i - \cos t\, j.$$

(iii) $$\left|\frac{dF}{dt}\right| = \sqrt{\left[(\cos t)^2 + (-\sin t)^2 + (1)^2\right]} = \sqrt{2}.$$

(iv) $$\left|\frac{d^2F}{dt^2}\right| = \sqrt{\left[(-\sin t)^2 + (-\cos t)^2\right]} = 1.$$

Theorem 1: If a vector function F is differentiable vector function of t, then

$$\frac{d}{dt}\left(F \times \frac{dF}{dt}\right) = F \times \frac{d^2F}{dt^2}$$

Proof : Since, we have that

$$\frac{d}{dt}(F \times G) = F \times \frac{dG}{dt} + \frac{dF}{dt} \times G$$

$$\therefore \frac{d}{dt}\left(F \times \frac{dF}{dt}\right) = F \times \frac{d}{dt}\left(\frac{dF}{dt}\right) + \frac{dF}{dt} \times \frac{dF}{dt}$$

$$= F \times \frac{d^2F}{dt^2} + 0$$ Since the cross product of same vector is zero

$$= F \times \frac{d^2F}{dt^2} \ .$$

4. Curves in three dimensional space:

Let F(x, y, z) and G(x, y, z) represents the equations of two surfaces, then a curve in a three dimensional Euclidean space may be obtained by the intersection of two surfaces represented by two equations of the form

F(x, y, z) = 0 and G(x, y, z) = 0.

Therefore, if can be easily seen that the parametric equation of the form

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

also represent a curve in three dimensional space where x, y, z are scalar functions of the scalar variable t and t lies between a and b i.e. a ≤ t ≤ b.

let (x, y, z) are coordinates of a point on the curve and let

$$R = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{and} \quad F(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

Therefore, from the equation $R = F(t)$, we have

$$x\hat{i} + y\hat{j} + x\hat{k} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

Thus, this equation represents a curve in three dimensional space and this equation can be written in parametric equation al follows

$$x = f_1(t), \quad y = f_2(t), \quad \text{and} \quad z = f_3(t)$$

Value Addition: Note
(1) The vector equation $R = at^2\hat{i} + 2at\hat{j} + 0\hat{k}$ represents the vector equation of a parabola for different values of t.
(2) Similarly the vector equation $R = a\cos\hat{i} + b\sin t\hat{j} + 0\hat{k}$ represents an ellipse for different values of t.
(3) The vector equation $R = (a\sin t)\hat{i} + (b\tan t)\hat{j} + 0\hat{k}$ represents the equation of the hyperbola for different values of t.

4.1. Tangent Vector:

Let $F(t)$ is differentiable at t_0 and that $F^1(t_0) \neq \bar{0}$. Then $F^1(t_0)$ is called a tangent vector to the graph of $F(t)$ at the point where $t = t_0$ and points in the direction of increasing t.

Example 3: Find the tangent vector at the point t = 0, t = 1 on the graph of the vector function $F(t) = t^2 \hat{i} + 2t \hat{j} + (t^3 - t^2)\hat{k}$, also find the parametric equation for the tangent line to the graph of the given vector function at the corresponding point.

Solution : Given that $F(t) = t^2 \hat{i} + 2t \hat{j} + (t^3 - t^2)\hat{k}$

The derivative of $F(t)$ is

$$F^1(t) = 2t\hat{i} + 2\hat{j} + (3t^3 - 2t)\hat{k}$$

(i) Tangent vector at the point where t = 0 is

$$F^1(0) = 0.\hat{i} + 2\hat{j} + 0.\hat{k} = 2\hat{j}$$

Thus, the tangent line to the graph of $F(t)$ at the point P_0 is the line that passes through P_0 and is parallel to the vector $F(0)$.

Since

$$F(0) = 0.\hat{i} + 0.\hat{j} + 0.\hat{k} = \bar{0}$$

the point of tangency is (0, 0, 0).

Paramedic equation of the tangent line is

x = 0 + 0.t = 0

y = 0 + 2t = 2t

z = 0 + 0.t =0

Thus, the paramedic equation of tangent line is

x = 0,   y = 2t,   z = 0

(ii) Tangent vector at the point t = 1

$$F^1(1) = 2\hat{i} + 2\hat{j} + \hat{k}$$

and, $F(1) = \hat{i} + 2\hat{j}$

Thus, point of tangency is (2, 2, 0)

Thus, the parametric equation of the tangent line is

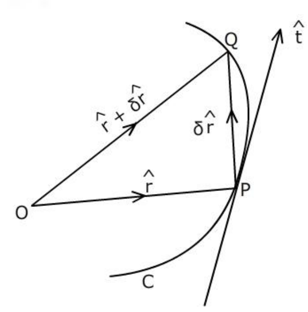
x = 1 + 2t, y= 2 + 2t, z = 0 + t

⇒ x = 1 + 2t, y = 2 + 2t, z = t

Value Addition: Note
Let $F(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, for the equation of tangent at the point t = t_0, we have $F^1(t_0) = f_1^1(t_0)\hat{i} + f_2^1(t_0)\hat{j} + f_3^1(t_0)\hat{k}$ provided $f_1^1(t_0)$, $f_2^1(t_0)$ and $f_3^1(t_0)$ all are not zero. and the point of tangency for the function $F(t_0) = f_1(t_0)\hat{i} + f_2(t_0)\hat{j} + f_3(t_0)\hat{k}$ at the point t = t_0 is $(f_1(t_0), f_2(t_0), f_3(t_0))$ Then, the parametric equation of tangent line is. $x = f_1(t_0) + t.f_1^1(t_0)$ $y = f_2(t_0) + t.f_2^1(t_0)$ $y = f_3(t_0) + t.f_3^1(t_0)$

4.2. Geometrical significance of $\frac{dR}{dt}$ and Tangent to the curve:

Let $R = F(t)$ be a vector equation of a curve in three dimensional space and let P and Q be two neighboring points on this curve whose position vectors are R and $R + \delta R$ respectively as shown in figure



Thus, we have

$$\vec{OP} = R = F(t)$$

and $\vec{OQ} = R + \delta R = F(t + \delta t)$

$$\vec{PQ} = \vec{OQ} - \vec{PQ} = (R + \delta R) - R = \delta R$$

Thus $\frac{\delta R}{\delta t}$ is a vector parallel to the chord $\vec{PQ}$ tends to a line at P to the curve. This line is known as tangent

$$\lim_{\delta t \to 0} \frac{\delta R}{\delta t} = \frac{dR}{dt}$$

is a vector parallel to the line (tangent) at P to the curve $R = F(t)$.

4.3. Unit Tangent Vector:

Let C be any fixed point on the curve and let in place of the scalar parameter t we take the parameter as s where s denotes the arc length measured along the curve from the fixed C on the curve.

Then,

arc CP = s

and arc CQ = s + δs

Thus, $\frac{\delta R}{\delta t}$ will be a vector along the tangent a P to the curve $R = F(s)$ in the direction of s increasing.

$$\therefore \quad \frac{dR}{ds} = \lim_{\delta t \to 0} \frac{\delta R}{\delta s}$$

$$= \lim_{\delta t \to 0} \frac{\delta R}{arc\, PQ} \qquad \begin{pmatrix} \therefore \ as\ \delta s \to 0, \quad Q \to P\ and \\ \delta s = arc\ \ length\ \ PQ \end{pmatrix}$$

$$or \quad \left|\frac{dR}{ds}\right| = \lim_{\delta t \to 0} \left|\frac{\delta R}{\delta s}\right|$$

$$= \lim_{Q \to P} \frac{|\delta R|}{arc\,PQ} = \lim_{Q \to P} \frac{Chord\,PQ}{arc\,PQ} = 1 \qquad \left(\therefore \;\; as\; \delta R = \vec{PQ}0 \right)$$

Thus, $\frac{dR}{ds}$ is a unit vector along the tangent at P to the curve $R = F(s)$

This unit vector is called the unit tangent vector and is denoted by $\hat{t}$

$$\therefore \;\; \hat{t} = \frac{dR}{ds}$$

5. Smooth Curve:

The graph of a vector function $F(t)$ is said to be smooth on any interval of t where $F^1(t)$ is continuous and $F^1(t) \neq 0$ for all t in the interval.

5.1. Piecewise Smooth Curve:

A curve is said to be piecewise smooth curve on an interval that can be subdivided into a finite number of subintervals on which $F(t)$ is smooth.

Example 4: Check whether the graph of the vector function $F(t) = (2\cos t)\hat{i} + t^2\hat{j} + (\sin t)\hat{k}$ is smooth or not.

Solution : Given vector function is

$$F(t) = (2\cos t)\hat{i} + t^2\hat{j} + (\sin t - t)\hat{k}$$

The derivative of the vector function $F(t)$ is

$$F'(t) = (-2\sin t)\hat{i} + 2t\hat{j} + (\cos t - 1)\hat{k}$$

The vector function $F'(t)$ is continuous for all t since all the components of $F^1(t)$ are continuous for all t. But we have

$$F(0) = 0.\hat{i} + 0.\hat{j} + (1-1).\hat{k}$$

$$= 0.\hat{i} + 0.\hat{j} + 0.\hat{k} = \bar{0}$$

Thus, the graph of the vector function is not smooth for all t, but it is smooth an any interval not containing t = 0. Therefore it is piecewise smooth curve.

Value Addition: Note
A graph cannot be smooth an any interval containing a point where there is an abrupt change in the direction.

6. Orthogonality of a function of constant length and its derivative:

Theorem 2: Let $F(t)$ be a non–zero vector which is differentiable and has constant length, then $F(t)$ is orthogonal to its derivative $F^1(t)$.

Proof : Given $F(t)$ is a non–zero vector of constant length and let the length of the vector $F(t)$ is r, then

$$|F(t)| = r$$

on squaring both sides, we have

$$|F(t)|^2 = r^2$$

$$\Rightarrow \quad F(t) . F(t) = r^2 \quad \text{for all t} \qquad \left[|F|^2 = F . F\right]$$

on differentiating both sides, we have

$$F(t) . F^1(t) + F^1(t) . F(t) = 0 \qquad [\text{since r is constant}]$$

$$\Rightarrow \quad 2\left[F(t) . F^1(t)\right] = 0$$

$$\Rightarrow \quad F(t) . F^1(t) = 0$$

Thus, $F(t)$ and $F^1(t)$ are orthogonal vectors.

7. Application to curvilinear motion speed, velocity and acceleration:

Suppose a particle moves in 2 dimensional space or 3 dimensional space in such a way that its position at time t relative to some coordinate system is given by the position vector $R = F(t)$

Let the particle is at a point P on the curve at any instant t whose position vector is R . In some time internal let δt the particle reached to the point Q whose position vector is $R + \delta R$ where δR is the displacement of the particle in the time interval δt.

Thus the vector $\frac{\delta R}{\delta s}$ gives the average velocity of the particle during the interval δt. If the velocity at the point P is represented by the vector V then

$$V = \lim_{\delta t \to 0} \frac{\delta R}{\delta t} = \frac{dR}{dt}$$

since $\frac{dR}{dt}$ is a vector along the tangent at P to the curve in which the particle is moving therefore the direction of the velocity vector V is along the tangent. If δV be the change in the velocity vector V during the time δt, then $\frac{\delta V}{\delta t}$ represents the average acceleration of the particle. Let the acceleration vector of the particle at time t is represented by A, then

$$A = \lim_{\delta t \to 0} \frac{\delta V}{\delta t} = \frac{dV}{dt}$$

since, $V = \frac{dR}{dt}$

$$\Rightarrow \quad A = \frac{dV}{dt} = \frac{d}{\delta t}\left(\frac{dR}{dt}\right) = \frac{d^2R}{dt^2}$$

Value Addition: Note

Let $R = x\hat{i} + y\hat{j} + z\hat{k}$ represents the position vector of a moving particle at any time t such that x, y, and z are the function of t. Then

$$V = \frac{dR}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

represents the velocity vector

and $|V| = \left|\frac{dR}{dt}\right|$ represents the speed of the particle. Now

$$A = \frac{dV}{dt} = \frac{d^2R}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

represents the acceleration vector of the moving particle.

Example 5: A particle's position at time t is determined by the vector

$$R(t) = e^{t}\hat{i} + e^{-t}\hat{j} + e^{2t}\hat{k}$$

find the particle's velocity, speed, acceleration, and direction of motion at time t = ln2.

Solution : Given $R(t) = e^{t}\hat{i} + e^{-t}\hat{j} + e^{2t}\hat{k}$

Then velocity vector, $V = \frac{dR}{dt} = \frac{d}{dt}\left(e^{t}\hat{i} + e^{-t}\hat{j} + e^{2t}\hat{k}\right)$

Thus, $V = e^{t}\hat{i} - e^{-t}\hat{j} + 2e^{2t}\hat{k}$

and acceleration vector is

$$A = \frac{dV}{dt} = \frac{d}{dt}\left(e^{t}\hat{i} - e^{-t}\hat{j} + 2e^{2t}\hat{k}\right)$$

$$A = \left(e^{t}\hat{i} + e^{-t}\hat{j} + 4e^{2t}\hat{k}\right).$$

The particle's velocity at time t = ln2

$$V(\ln 2) = e^{\ln 2}\hat{i} - e^{-\ln 2}\hat{j} + 2e^{2\ln 2}\hat{k} = 2\hat{i} - \frac{1}{2}\hat{j} + 2.4\hat{k} = 2\hat{i} - \frac{1}{2}\hat{j} + 8\hat{k}$$

Particle's speed at time t = ln2 is

$$\left|v(\ln 2)\right| = \left|2\hat{i} - \frac{1}{2}\hat{j} + 8\hat{k}\right| = \sqrt{(2)^2 + \left(-\frac{1}{2}\right)^2 + (8)^2}$$

$$= \sqrt{4 + \frac{1}{4} + 64} = \sqrt{\frac{16 + 1 + 256}{4}} = \frac{\sqrt{273}}{2}$$

Particle's acceleration at t = ln2 is

$$A(\ln 2) = e^{\ln 2}\hat{i} + e^{-\ln 2}\hat{j} + 4e^{2\ln 2}\hat{k} = 2\hat{i} + \frac{1}{2}\hat{j} + 4.4\hat{k} = 2\hat{i} + \frac{1}{2}\hat{j} + 16\hat{k}$$

Particle's direction of motion at t = ln2 is

$$\frac{V(\ln 2)}{|V(\ln 2)|} = \frac{2\hat{i} - \frac{1}{2}\hat{j} + 8\hat{k}}{\frac{\sqrt{273}}{2}} = \frac{2}{\sqrt{273}}\left(2\hat{i} - \frac{1}{2}\hat{j} + 8\hat{k}\right)$$

$$= \frac{1}{\sqrt{273}}\left(4\hat{i} - \hat{j} + 16\hat{k}\right)$$

Example: 3. If $R = (\cos nt)i + (\sin nt)j$, where n is a constant and t varies, show that $R \times \frac{dR}{dt} = nk$.

Solution: We have

$$\frac{dR}{dt} = \frac{d}{dt}(\cos nt)i + \frac{d}{dt}(\sin nt)j = n\sin nt\, i + n\cos nt\, j.$$

$$\therefore \quad R\times\frac{dR}{dt} = (\cos nt\, i + \sin nt\, j) \times (-\sin nt\, i + \cos nt\, j)$$

$$= -n\cos nt\sin nt\, i \times i + n\cos^2 nt \times j - n\sin^2 nt\, j\times i + n\cos nt\sin nt\, j\times j$$

$$= n\cos^2 nt\, k + n\sin^2 nt\, k \qquad [\because\ i\times i = 0,\ j\times j = 0,\ i\times j = k,\ \ j\times i = -k,]$$

$$= n(\cos^2 nt + \sin^2 nt)k = nk.$$

Example 4: If ω is a constant, and R is a vector function of the scalar variable t given by $R = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$, show that

(i) $\frac{d^2R}{dt^2} + \omega_2 R = 0$ and (ii) $R \times \frac{dR}{dt} = \omega a \times b$.

Solution: We have

(i) $\frac{dR}{dt} = \frac{d}{dt}(\cos\omega t)\hat{i} + \frac{d}{dt}(\sin\omega t)\hat{j} = \omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}.$

$$\therefore \quad \frac{d^2R}{dt^2} = -\omega^2\cos\omega t\,\hat{i} - \omega^2\sin\omega t\,\hat{j}$$

$$= -\omega^2(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) = -\omega^2 R.$$

$$\therefore \quad \frac{d^2R}{dt^2} + \omega^2 R = 0.$$

(ii) $$R \times \frac{dR}{dt^2} = \left(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right) \times \left(-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}\right)$$

$$= \omega\cos^2\omega t\,\hat{i} \times \hat{j} - \omega\sin^2\omega t\,\hat{j} \times \hat{i} \qquad \left[\because\ \hat{i}\times\hat{i}=0,\ \hat{j}\times\hat{j}=0\right]$$

$$= \omega\cos^2\omega t\,\hat{i} \times \hat{j} + \omega\sin^2\omega t\,\hat{i} \times \hat{j}$$

$$= \omega\left(\omega\cos^2\omega t + \sin^2\omega t\right)\hat{i} \times \hat{j} = \omega\hat{i} \times \hat{j}.$$

Example 6: If $R = a\cos t\,\hat{i} + a\sin t\,\hat{j} + at\tan\alpha\,\hat{k}$, find $\left|\frac{dR}{dt} \times \frac{d^2R}{dt^2}\right|$ and $\left|\frac{dR}{dt}, \frac{d^2R}{dt^2}, \frac{d^3R}{dt^3}\right|$.

Solution: we have $\frac{dR}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + a\tan\alpha\,\hat{k}$

$$\frac{d^2R}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j}, \qquad \left[\because \frac{dk}{dt} = 0\right]$$

$$\frac{d^3R}{dt^3} = a\sin t\,\hat{i} - a\cos t\,\hat{j}.$$

$$\therefore \quad \frac{dR}{dt} \times \frac{d^2R}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = -a\sin^2 t\tan\alpha\,\hat{i} - a^2\cos t\tan\alpha\,\hat{j} + a^2\,\hat{k}.$$

$$\therefore \quad \left|\frac{dR}{dt} \times \frac{d^2R}{dt^2}\right| = \sqrt{\left(a^2\sin^2 t\tan^2\alpha + a^2\cos^2 t\tan^2\alpha + a^4\right)} = a^2\sec\alpha.$$

also $$\left[\frac{dR}{dt}, \frac{d^2R}{dt^2}, \frac{d^3R}{dt^3}\right] = \left(\frac{dR}{dt} \times \frac{d^2R}{dt^2}\right) \cdot \frac{d^3R}{dt^3}$$

$$= \left(a\sin^2 t\tan\alpha\,\hat{i} - a^2\cos t\tan\alpha\,\hat{j} + a^2\,\hat{k}\right) \cdot \left(a\sin t\,\hat{i} - a\cos t\,\hat{j}\right)$$

$$= a^3\sin^2 t\tan\alpha\,\hat{i}\cdot\hat{i} + a^2\cos^2 t\tan\alpha\,\hat{j}\cdot\hat{j} \qquad [\because\ i \cdot j = 0\ etc.]$$

$$= a^3\tan\alpha\left(\sin^2 t + \cos^2 t\right) \qquad [\because\ i \cdot i = 1 = j \cdot j\,0]$$

$= a^3 \tan\alpha$

Example 7: A particle moves along the curve $x = 4\cos t,\ y = 4\sin t,\ z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t.

Solution: Let R be the position vector of the particle at time t. Then

$$R = x\hat{i} + y\hat{j} + z\hat{k} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j},\ z = 6t\,\hat{k}.$$

If V is the velocity of the particle at time t and A is the acceleration at that time then

$$V = \frac{dR}{dt} = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$$

$$A = \frac{d^2R}{dt^2} = -4\cos t\,\hat{i} - 4\sin t\,\hat{j}$$

Magnitude of the velocity at time t $= |V|$

$$= \sqrt{\left(16\sin^2 t + 16\cos^2 t + 36\right)} = \sqrt{(52)} = 2\sqrt{(13)}.$$

Magnitude of the acceleration at time t $= |A|$

$$= \sqrt{\left(16\cos^2 t + 16\sin^2 t\right)} = 4.$$

at time $t = \frac{1}{2}\pi,$ the velocity and accelration vectors are

$$V = -4i + 6k\ ,\ A = -4j$$

Example 8: If $A = \sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}$, $B = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} + 3\hat{k}$, and $c = 2\hat{i} + 3\hat{j} - 3\hat{k}$ find $\frac{d}{d\theta}\{A \times (B \times C)\}$ at $\theta = \pi/2$.

Solution: We know that

$$A \times (B \times C) = (A \cdot C)\,B - (A \cdot B)C. \qquad \text{............... (1)}$$

$$\therefore \qquad A \cdot C = \left(\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}\right) \cdot \left(2\hat{i} + 3\hat{j} - 3\hat{k}\right)$$

$$= \left(2\sin\theta + 3\cos\theta - 3\theta\right) \qquad \left(\because \ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,\ \hat{i}\cdot\hat{j} = 0 \ \ etc.\right)$$

and $\quad A \cdot B = \left(\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}\right) \cdot \left(\cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\hat{k}\right)$

$$= \sin\theta\cos\theta - \cos\theta\sin\theta - 3\theta = -3\theta.$$

From (1), we have

$$A \times \left(B \times C\right) = \left(2\sin\theta + 3\cos\theta - 3\theta\right)\left(\cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\hat{k}\right) + 3\theta\left(2\hat{i} + 3\hat{j} - 3\hat{k}\right)$$

$$= \left(2\sin\theta\cos\theta + 3\cos^2\theta - 3\theta\cos\theta + 6\theta\right)\hat{i} + \left(-2\sin^2\theta - 3\cos\theta\sin\theta + 3\theta\sin\theta + 9\theta\right)\hat{j}$$

$$+ \left(-6\sin\theta - 9\cos\theta\right)\hat{k}$$

$$\therefore \qquad \frac{d}{d\theta}\left\{A \times \left(B \times C\right)\right\} = \left(2\cos 2\theta - 6\cos^2\theta\sin\theta - 3\theta\cos\theta + 3\sin\theta + 6\right)\hat{i}$$

$$+ \left(-4\sin\theta\cos\theta - 3\cos 2\theta\sin\theta + 3\theta\cos\theta + 3\sin\theta + 9\right)\hat{j}$$

$$+ \left(-6\cos\theta + 9\sin\theta\right)\hat{k}$$

At $\theta = \pi/2$, we have

$$\frac{d}{d\theta}\left\{A \times \left(B \times C\right)\right\} = \left(-2 + \frac{3\pi}{2} + 6\right)\hat{i} + \left(3 + 3 + 9\right)\hat{j} + 9\hat{k}$$

$$= \left(4 + \frac{3\pi}{2}\right)\hat{i} + 15\hat{j} + 9\hat{k}\ .$$

Example 9: A particle moves along the curve $x = t^3 + 1,\ y = t^2,\ z = 2t + 5,$ where t is the time. Find the components of its velocity and acceleration at t = 1 in the direction $\hat{i} + \hat{j} + 3\hat{k}$.

Solution: Let r be the position vector of a moving particle at any time t at the point (x, y, z) on the curve, then

$$R = x\hat{i} + y\hat{j} + z\hat{k}$$

$$R = (t^3 + 1)\hat{i} + t^2\hat{j} + (2t + 5)\hat{k} \qquad \text{..........} \quad (1)$$

Where $\hat{i}, \hat{j}, \hat{k}$ are constant vector.

$$\therefore \qquad \frac{dR}{dt} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k} \qquad \text{..........} \quad (2)$$

At t = 1

$$\frac{dR}{dt} = 3\hat{i} + 2\hat{j} + 2\hat{k}.$$

it is a velocity vector whose components are 3, 2 and 2.

Again from (2)

$$\frac{d^2R}{dt^2} = 6t\hat{i} + 2\hat{j}.$$

At t = 1

$$\frac{d^2R}{dt^2} = 6\hat{i} + 2\hat{j}$$

That is the acceleration vector and whose components arc 6, 2 and 0. Now we have to find the components in the directions $\hat{i} + \hat{j} + 3\hat{k}$.

$\therefore$ Unit vector along the direction of $\hat{i} + \hat{j} + 3\hat{k} = \dfrac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}$

Thus the component of velocity in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ is

$$= 3\hat{i} + 2\hat{j} + 2\hat{k} \cdot \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}}$$

$$= \frac{3+2+2}{\sqrt{11}} = \frac{11}{\sqrt{11}} \sqrt{11} \text{ units.}$$

and the component of acceleration in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ is

$$= \left(6\hat{i} + 2\hat{j}\right) \cdot \frac{\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{11}} = \frac{6+2}{\sqrt{11}} = \frac{8}{\sqrt{11}} \text{ unit .}$$

8. Integration of Vector Function:

It is well known fact that the integration is the reverse process of the differentiation. Let $F(t)$ and $G(t)$ be two vector functions of the scalar t such that

$$\frac{d}{dt}G(t) = F(t) \qquad \text{--------- (1)}$$

Then the vector function $G(t)$ is called the indefinite integral of the vector function $F(t)$ with respect to t. Symbolically, we may write it as follows.

$$\int F(t)dt = G(t) \qquad \text{---------- (2)}$$

The vector function $F(t)$ which is being integrated is called the integrand.

Let C be a constant vector which is independent of t. Then the equation (1) can be written as follows.

$$\frac{d}{dt}\left[G(t) + C\right] = F(t) \qquad \text{------------------- (3)}$$

$$\int F(t)dt = G(t) + C \qquad \text{------------------- (4)}$$

The constant vector C is called the constant of integration. If $\frac{d}{dt}G(t) = F(t)$ is defined over the closed interval [a, b], then the definite integral between the limits t = a and t = b is defined as

$$\int_a^b F(t)dt = \left[G(t) + C\right]_a^b = G(b) - G(a)$$

Where a and b are called the limits of integration.

Theorem 3: If $F(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$, then $\int F(t)dt=\int f_1(t)dt\,\hat{i}+\int f_2(t)dt\,\hat{j}\int f_3(t)dt\,\hat{k}$.

Proof : Let $\frac{d}{dt}G(t)=F(t)$ ....(1)

Then $\int F(t)dt\ =G(t)$ ....(2)

Let $G(t)=g_1(t)i+g_2(t)j+g_3(t)k$.

Then from (1), we have

$$\frac{d}{dt}\{g_1(t)i+g_2(t)j+g_3(t)k\}\ =F(t)$$

or $$\left\{\frac{d}{dt}g_1(t)\right\}\hat{i}\ +\ \left\{\frac{d}{dt}g_2(t)\right\}\hat{j}\ +\ \left\{\frac{d}{dt}g_3(t)\right\}\hat{k}\ =\ f_1(t)\hat{i}\ +\ f_2(t)\hat{j}\ +\ f_3(t)\hat{k}.$$

Equating the coefficients of i, j and k we have

$$\frac{d}{dt}g_1(t)\ =f_1(t),\quad \frac{d}{dt}g_2(t)\ =\ f_2(t),\quad \frac{d}{dt}g_3(t)=f_3(t).$$

$$\therefore\quad g_1(t)=\int f_1(t)dt,\quad g_2(t)=\int f_2(t)\,\mathrm{dt},\quad g_3(t)=\int f_3(t)dt.$$

$$\therefore\quad G(t)=\left\{\int f_1(t)dt\right\}\hat{i}+\left\{\int f_2(t)dt\right\}\hat{j}+\left\{\int f_3(t)dt\right\}\hat{k}\ .$$

So from (2), we get

$$\int F(t)dt\ =\int f_1(t)dt\ \hat{i}+\int f_2(t)dt\ \hat{j}+\int f_3(t)dt\ \hat{k}.$$

Value Addition: Note
In order to integrate a vector function we should integrate its components.

Example 10: If $F(t)=\left(t-t^2\right)\hat{i}+2t^3\hat{j}-3\hat{k}$, find $\int F(t)dt$.

Solution : We have

$$\int F(t)dt\ =\int\left\{\left(t-t^2\right)\hat{i}\ +\ 2t^3\hat{j}\ -\ 3\hat{k}\right\}dt$$

$$=\int\left(t-t^2\right)dt\ \hat{i}+\int 2t^3\,dt\ \hat{j}+\int -3\,dt\,\hat{k}$$

$$= \left(\frac{t^2}{2} - \frac{t^3}{3}\right)\hat{i} + \left(\frac{2t^4}{4}\right)\hat{j} + (-3t)\hat{k} + C,$$

where C is an arbitrary constant vector

Thus,

$$\int F(t)\,dt = \left(\frac{t^2}{2} + \frac{t^3}{3}\right)\hat{i} + \frac{t^4}{2}\hat{j} - 3t\hat{k} + C.$$

Example 11: If $F(t) = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$, find $\int_0^1 F(t)\,dt$.

Solution : Since $F(t) = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$, then

$$\int_0^1 F(t)\,dt = \int_0^1 \left[(t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}\right]dt$$

$$= \int_0^1 (t+1)\,dt\;\hat{i} + \int_0^1 (t^2+t+1)\,dt\;\hat{j} + \int_0^1 (t^3+t^2+t+1)\,dt\;\hat{k}$$

$$= \left(\frac{t^2}{2} + t\right)_0^1 \hat{i} + \left(\frac{t^3}{3} + \frac{t^2}{2} + t\right)_0^1 \hat{j} + \left(\frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} + t\right)_0^1 \hat{k}$$

$$= \frac{3}{2}\hat{i} + \frac{11}{6}\hat{j}\;\frac{25}{12}\hat{k}.$$

Example 12: If $R(t) = 5t^2\,\hat{i} + t\hat{j} - t^3\,\hat{k}$, prove that

$$\int_1^2 \left(R \times \frac{d^2R}{dt^2}\right)dt = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Solution : We know that

$$\int \left(R \times \frac{d^2R}{dt^2}\right)dt = \left(R \times \frac{dR}{dt}\right) + c.$$

Thus, $\int_1^2 \left(R \times \frac{d^2R}{dt^2}\right)dt = \left[R \times \frac{d^2R}{dt^2}\right]_1^2.$

We have $\frac{dR}{dt} = \frac{d}{dt}\left(5t^2\,\hat{i} + t\,\hat{j} - t^3\,\hat{k}\right) = 10t\,\hat{i} + \hat{j} - 3t^2\hat{k}$.

Now, $R \times \frac{dR}{dt} = \left(5t^2\,\hat{i} + t\,\hat{j} - t^3\hat{k}\right)\times\left(10t\,\hat{i} + \hat{j} - 3t^2\hat{k}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -2t^3\,\hat{i} + 5t^4\,\hat{j} - 5t^2\hat{k}.$$

$$\therefore \quad \int_1^2 \left(R \times \frac{d^2R}{dt^2}\right)dt = \left[-2t^3\,\hat{i} + 5t^4\,\hat{j} - 5t^2\hat{k}\right]_1^2$$

$$= \left[-2t^3\right]_1^2 i + \left[5t^4\right]_1^2 - \left[5t^2\right]_1^2 k = -14i + 75j - 15k.$$

Example 13: The velocity of a particle moving in space is

$$V(t) = t^2\,\hat{i} - e^{2t}\,\hat{j} + \sqrt{t}\,\hat{k}$$

Find the particle's position as a function of t if the particle's position at time t = 0 is $R(0) = \hat{i} + 4\hat{j} - \hat{k}$.

Solution: Given that particle's velocity at time t is

$$V(t) = t^2\,\hat{i} - e^{2t}\,\hat{j} + \sqrt{t}\,\hat{k}$$

We know that

$$V(t) = \frac{dR}{dt} = t^2\,\hat{i} - e^{2t}\,\hat{j} + \sqrt{t}\,\hat{k}$$

On integrating both sides we have

$$\int dR = \int t^2 dt\,\hat{i} - \int e^{2t} dt\,\hat{j} + \int \sqrt{t}\,dt\,\hat{k}$$

$$\Rightarrow \quad R(t) = \frac{t^3}{3}\hat{i} - \frac{e^{2t}}{2}\hat{j} + \frac{2}{3}t^{3/2}\hat{k} + C \qquad \text{.............} (1)$$

At time t = 0

$$R(0) = 0.\hat{i} - \frac{1}{2}\hat{j} + 0\hat{k} + C$$

$$\Rightarrow \quad \hat{i} + 4\hat{j} - \hat{k} = -\frac{1}{2}\hat{j} + C$$

$$\Rightarrow \quad C = \hat{i} + 4\hat{j} - \hat{k} + \frac{1}{2}\hat{j} = \hat{i} + \frac{9}{2}\hat{j} - \hat{k}$$

Thus, from equation (1), we have

Thus, the particle's position at any time t is

$$R(t) = \frac{t^3}{3}\hat{i} - \frac{e^{2t}}{2}\hat{j} + \frac{2}{3}t^{3/2}\hat{k} + \left(\hat{i} + \frac{9}{2}\hat{j} - \hat{k}\right)$$

$$= \left(\frac{t^3}{3} + 1\right)\hat{i} - \frac{1}{2}\left(e^{2t} - 9\right)\hat{j} + \left(\frac{2}{3}t^{3/2} - 1\right)\hat{k}$$

Example 14: The acceleration of a particle at any time $t \geq 0$ is given by

$$A = 12\cos 2t\ \hat{i} - 8\sin 2t\ \hat{j} + 16t\,\hat{k}\ .$$

If the velocity V and displacement R are zero at t = 0, find V and R at any time.

Solution: Given

$$A = 12\cos 2t\ \hat{i} - 8\sin 2t\ \hat{j} + 16t\,\hat{k}$$

We know that

$$\frac{dV}{dt} = A = 12\cos 2t\ \hat{i} - 8\sin 2t\ \hat{j} + 16t\,\hat{k}$$

On integrating, we have

$$V = \int 12\cos 2t\,dt\ \hat{i} + \int -8\sin 2t\,dt\ \hat{j} + \int 16t\,dt\ \hat{k}$$

or $\quad V = 6\sin 2t\,\hat{i} + 4\cos 2t\,\hat{j} + 8t^2\,\hat{k} + C$

When, $t = 0$, $V = 0$

$$\therefore \qquad 0 = 0\hat{i} + 4\hat{j} + 0\hat{k} + C \quad or \quad C = -4\hat{j}.$$

Thus, $V = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}$.

We know that

$$\frac{dR}{dt} = V = 6\sin 2t\,\hat{i} + (4\cos 2t - 4)\hat{j} + 8t^2\hat{k}$$

On integrating, we get

$$R = \int 6\sin 2t\,dt\,\hat{i} + \int (4\cos 2t - 4)\,dt\,\hat{j} + \int 8t^2\,dt\,\hat{k}$$

$= -3\cos 2t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k} + C$, Where C is constant.

When, $t = 0$, $R = 0$.

$$\therefore \qquad 0 = -3\hat{i} + 0\hat{j} + 0\hat{k} + C \quad \Rightarrow \quad C = 3\hat{i}.$$

Thus,

$$R = -3\cos 2t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + \frac{8}{3}t^3\hat{k} + 3\hat{i}$$

$$= 3(1 - \cos 2t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + \frac{8}{3}t^3\hat{k}$$

Value Addition: Remember

Let F and G are two vector functions then

1. $\frac{d}{dt}(F\cdot G) = \frac{dF}{dt}\cdot G + G\cdot\frac{dF}{dt}$, then $\int\left(\frac{dF}{dt}\cdot G + F\cdot\frac{dG}{dt}\right)dt = F\cdot G + C$,

Where C is the constant of integration.

2. $\frac{d}{dt}(F)^2 = 2F\cdot\frac{dF}{dt}$, then $\int\left(2F\cdot\frac{dF}{dt}\right)dt = F^2 + C$.

Where C is the constant of integration.

3. $\frac{d}{dt}\left(\frac{dR}{dt}\right)^2 = 2\frac{dR}{dt}\cdot\frac{d^2R}{dt^2}$, then we have $\int\left(2\frac{dR}{dt}\cdot\frac{d^2R}{dt^2}\right)dt = \left(\frac{dR}{dt}\right)^2 + c$.

Where C is the constant of integration.

4. We have $\frac{d}{dt}\left(R\times\frac{dR}{dt}\right) = \frac{dR}{dt}\times\frac{dR}{dt} + R\times\frac{d^2R}{dt^2} = R\times\frac{d^2R}{dt^2}$

$$\therefore \qquad \int\left(R\times\frac{d^2R}{dt^2}\right)dt = R\times\frac{dR}{dt} + C.$$

Here the constant of integration C is a vector quantity since the integrant $R \times \frac{d^2R}{dt^2}$ is also a vector quantity.

5. If A is a constant vector, then $\frac{d}{dt}(A \times R) = \frac{dA}{dt} \times R + A \times \frac{dR}{dt} = A \times \frac{dR}{dt}$.

Therefore $\int \left(A \times \frac{dR}{dt} \right) dt = A \times R + C$

Here the constant of integration C is a vector quantity.

Exercise:

1. $R = (t+1)\hat{i} + (t^2+t+1)\hat{j} + (t^3+t^2+t+1)\hat{k}$, find $\frac{dR}{dt}$ *and* $\frac{d^2R}{dt^2}$.

2. (i) if $R = \sin t\,\hat{i} + \cos t\,\hat{j} + t\hat{k}$, find $\left| \frac{dR}{dt} \times \frac{d^2R}{dt^2} \right|$.

(ii) $R = a\cos t\,\hat{i} + a\sin t\,\hat{j} + at\tan\alpha\,\hat{k}$, find $\left| \frac{dR}{dt} \times \frac{d^2R}{dt^2} \right|$ and $\frac{dR}{dt} \cdot \left(\frac{dR}{dt} \times \frac{d^2R}{dt^2} \right)$.

3. If $R = t^3\hat{i} + \left(2t^3 - \frac{1}{5t^2} \right)\hat{j}$, show that $R \times \frac{dR}{dt} = \hat{k}$.

4. If $A = 5t\hat{i} + \mathrm{t}\hat{j} - t^3\hat{k}$ and $B = \sin t\,\hat{i} - \cos t\,\hat{j}$, then find

(i) $\frac{d}{dt}(A \cdot B)$ (ii) $\frac{d}{dt}(A \times B)$ (iii) $\frac{d}{dt}(A \cdot A)$.

5. If $R = e^{nt}\hat{i} + e^{-nt}\hat{j}$, then show that $\frac{d^2R}{dt^2} - n^2R = 0$.

6. If $R = \sin\omega t\,\hat{i} + \frac{ct}{\omega^2}\sin\omega t\,\hat{j}$, then prove that $\frac{d^2R}{dt^2} + \omega^2 R = \frac{2c}{\omega}\cos\omega t$.

7. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at t = 0.

8. A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{\pi}{2}$.

9. Find the unit tangent vector to any point on the curve

$$x = a\cos t,\ \ y = a\sin t,\ \ z = bt\,.$$

10. If R is the position vector for a particle in space at time t. Find the particle's velocity and acceleration vectors and then find the speed and direction of motion for the given value of t.

i. $R(t) = (1-2t)\hat{i} - t^2\hat{j} + e^t\hat{k}\quad at\ \ t = 0$

ii. $R(t) = (2\cos t)\hat{i} + t^2\hat{j} + (2\sin t)\hat{k}\quad at\ \ t = \dfrac{\pi}{2}$

iii. $R(t) = e^t\hat{i} + e^{-t}\hat{j} + e^{2t}\hat{k}\quad at\ \ t = \ln 2$

iv. $R(t) = (\ln t)\hat{i} + \dfrac{1}{2}t^3\hat{j} - t\hat{k}\quad at\ \ t = 1$

11. Find the tangent vector to the graph of the given vector function F at the points indicated in

i. $F(t) = \dfrac{t\hat{i} + t^2\hat{j} + t^3\hat{k}}{1+2t};\ t = 0,\ t = 2$

ii. $F(t) = t^2\hat{i} + (\cos t)\hat{j} + (t^2\cos t)\hat{k};\ \ t = 0,\ \ t = \dfrac{\pi}{2}$

iii. $F(t) = (e^t\sin \pi t)\hat{i} + (e^t\cos \pi t)\hat{j} + (\sin \pi t + \cos \pi t)\hat{k};\ \ t = 0,\ t = 1,\ t = 2$

12. Find parametric equation for the tangent line to the graph of the given vector function F(t) at the point P_0 corresponding to t_0.

i. $F(t) = \langle t^{-3}, t^{-2}, t^{-1}\rangle;\ t_0 = -1$

ii. $F(t) = \langle \sin t, \cos t, 3t\rangle;\ t_0 = 0$

13. Evaluate $\displaystyle\int_0^1 \left(e^t\,\hat{i} + e^{-2t}\hat{j} + t\,\hat{k}\right)dt\,.$

14. If $F(t) = (t - t^2)\hat{i} + 2t^3\hat{i} - 3\hat{k}$, find

(i) $\int F(t)\,dt$ (ii) $\int_1^2 F(t)\,dt$.

15. If $F(t) = t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 - 3t^2)\hat{k}$, find $\int_0^1 F(t)\,dt$.

16. Find $\int_2^3 \left(e^t\,\hat{i} + e^{-2t}\,\hat{j} + t\,\hat{k}\right)dt$.

17. $A = t\,\hat{i} - 3\,\hat{j} + 2t\,\hat{k},\ B = \hat{i} - 2\,\hat{j} + 2\,\hat{k},\ C = 3\hat{i} + t\,\hat{j} - \hat{k}$, then evaluate

$$\int_0^1 A\,(B \times C)\,dt.$$

18. If $R = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $S = 2t^2\hat{i} + 6t\,\hat{k}$, evaluate

(i) $\int_0^2 R \cdot S\,dt$, (ii) $\int_0^2 (R \times S)\,dt$.

19. Find the position vector R(t), given the velocity V(t) and the initial position R(0) for the following problems.

i. $V(t) = t^2 i - e^{2t} j + \sqrt{t}\,k;\ R(0) = i + 4j - k$

ii. $V(t) = ti - \sqrt[3]{t}\ j + e^t\,\mathrm{k};\ R(0) = i - 2j + k$

iii. $V(t) = \sqrt[2]{t}\ i + (\cos t)\,j;\ R(0) = i + j$

iv. $V(t) = -3t\ i + (\sin^2 t)\,j + (\cos^2 t)\,k;\ R(0) = j$

20. Find the velocity and the position vector of a moving particle whose acceleration at any time t is given by $A = 6t\,\hat{i} - 24t^2\,\hat{j} + 4\sin t\,\hat{k}$, given that $R = 2\hat{i} + \hat{j}$ and $V = -\hat{i} - 3\hat{k}$ at $t = 0$.

21. The acceleration of a particle at any time t is given by $A = e^t\,\hat{i} + e^{2t}\,\hat{j} + \hat{k}$, find the velocity at any time t given that $V = \hat{i} + \hat{j}\ \ at\ t = 0$.

22. Find the position vector R(t) and the velocity vector V(t), given the acceleration A(t) and initial position and velocity vector R(0) and V(0), respectively, for $A(t)=(\cos t)\hat{i}\ -(t\sin t)\hat{k};\ \ R(0)=\hat{i}-2\hat{j}+\hat{k};\ \ V(0)=2\hat{\mathrm{i}}+3\hat{\mathrm{k}}$.

Summary:

In this lesson we have emphasize on the followings

- Differentiation of vector functions.
- Differentiation of vector function in terms of its components
- Derivative of a Vector function of function
- Successive Differentiation
- Tangent Vector
- Geometrical significance of $\frac{dR}{dt}$ and Tangent to the curve
- Unit Tangent Vector
- Smooth Curve
- Orthogonality of a function of constant length and its derivative
- Application to curvilinear motion speed, velocity and acceleration
- Integration of Vector Functions.

References:

1. M. J. Strass, G.L. Bradley and K. J. Smith, Calculus (3rd Edition), Dorling Kindersley (India) Pvt. Ltd. (Pearson Education), Delhi, 2007.
2. H. Anton, I. Bivens and S. Davis, Calculus (7th Edition), John Wiley and sons(Asia), Pt Ltd., Singapore, 2002.