Lesson: Differentiation of Functions

Paper : Analysis-II

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Differentiation of Functions

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Chapter : Differentiation of Functions

1. Learning Outcomes

After you have read this chapter, you should be able to

- Define the derivative of a function.
- Find a relation between a continuous function and a differentiable function.
- Define and discuss algebra of derivatives of functions.
- Characterize differentiability in term of continuity of a function, via, Caratheodory's Theorem.
- State and Prove the Chain Rule of differentiable functions.

Differentiation of Functions

Differentiation of Functions

> "I turn away in fright and horror from this lamentable plague of functions that do not have derivatives".
>
> — C. Hermite, 1893

> "The method of Fluxions [the calculus] is the general key by help whereof the modern mathematicians unlock the secrets of Geometry, and consequently of Nature".
>
> — Bishop Berkeley

Introduction

Modern mathematics began with two great advances – analytic geometry and the calculus. Analytic geometry took definite form in the year 1637 while the calculus took the definite shape in 1966. Rene Descartes (1596-1650), the great French mathematician of the seventeenth century is generally credited as the founder of analytic geometry. On the other hand, both Newton (1642-1727) and Leibnitz (1646-1716), share the credit for inventing "Calculus" independently in the seventeenth century. The former used physical approach while the later used geometrical approach.

Further, Newton used the term "rate of change in his second law of motion. Thus, the calculus, sometimes, may be defined as mathematics of motion and change. It was inevitable after the work of Cavalieri (1598-1647), Fermat (1601-1665), Wallis (1616-1703), Barrow (1630-1677) and others that the calculus should presently get itself organized as an autonomous discipline. In fact, P. Laplace (1749-1827) considered Fermat the discoverer of differential calculus as Fermat developed a method for finding tangents and solving maximum and minimum problem using a difference quotient, identical to the one which we now use to define derivatives, although he did not have a theory of limit. Calculus grew with the stimulus of applied work that continued through the 18^{th} century, into analysis topics such as the calculus of variations, differential calculus etc. During this period, calculus techniques were applied to approximate discrete problems by continuous one.

Cauchy formulated calculus in term of geometric ideas and infinitesimals. Furthermore, he introduced the concept of the Cauchy sequence. In fact, Cauchy contributed to many areas including real and complex analysis, number theory, differential equation, mathematical physic and probability. He was one of the most important mathematicians in the first half of the nineteenth century. Karl Weierstrass in the 1870's developed (ε, δ)□ definition of limit approach. He brought the proper understanding to the idea of continuity. He did fundamental work on the foundation of arithmetic and analysis, on complex analysis, the calculus of variation, and algebraic geometry. He astonished the entire mathematical world by constructing a function which is continuous everywhere but nowhere differentiable. Undoubtedly, he is known as the father of Modern Analysis. In subsequent decades the subject developed further through the works of several mathematician, most notably Euler, Cauchy, Riemann and Cantor.

Today, calculus and its extension in real analysis which is a part of mathematical analysis are far reaching indeed. Now, every mathematician knew that analysis arose naturally in the nineteenth century out of the calculus, which involves the elementary concepts and techniques of analysis, of the previous two centuries.

Today not only the mathematics but many other subjects – such as Economic, Physics, Chemistry and Biological Sciences are enjoying the fruits of calculus.

Differentiation of Functions

The chapter illustrates the importance of the notion of derivative of a function as one of the most classified topics in calculus. Here we give a theoretical treatment of differentiation and related topics. We start the chapter with quotations taken from the literature. We now give the sectionwise summary of the chapter. The chapter is divided into four sections. In each section definition, examples, theorem, remarks etc. are furnished with substantial examples to stimulate the learning process and they are numbered consecutively in each section and we also indicate both the chapter and the section numbers. Frequently, there are some pertinent remarks before or after the statement of the definitions, theorems etc.

Section one (3.1) begin with the formal definition of a derivative of a function. Furthermore, we have discussed various basic concepts and interesting results related to differentiability. We, also establish a relation between differentiability and continuity of a function. In fact, we establish that differentiability of any function implies the continuity of that function, but the converse is not true.

Section two (3.2) deals with many results that are useful in finding derivatives of various combinations of functions. Moreover, these results will enlarge our collection of differentiable functions rather extensively.

In the third section (3.3) we discuss a theorem due to C. Caratheodory which makes it possible to reduce some of the theorems on derivatives to theorems on continuity. Indeed, this theorem, sometimes referred as the characterization of differentiability in terms of continuity. The section ends with the illustration of the theorem.

Lastly, in the fourth section (3.4), we study another fundamental property, namely, the chain rule of differentiable functions. Moreover the result forms the basis for calculating the derivatives of most elementary functions. In the end we conclude that the chain rule is probably the most important theorem about derivatives.

Finally, the chapter ends with a list of exercises (with the answers / hints) and references for further reading.

3.1 Definition of Derivative

We start the chapter with the formal definition of a derivative of real- valued function and further, we discuss many basic concepts and interesting results related to differentiation. Throughout the chapter, the symbol *R* denotes the set of real numbers.

Definition 3.1.1. Let $A \subseteq R$, and let $a \in A$ be an element of *A* which is also *a* limit point of *A. A* function $f : A \to R$ is said to be differentiable at the point *a* of *A* if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. If the limit exists, it is called the derivative of *f* at the point *a* and the value of the limit is denoted by $f'(a)$ or $Df(a)$ or

$$\frac{df}{dx}(a) \ \ or \ \ \left.\frac{dy}{dx}\right|_{x=a} \qquad [\text{where } y = f(x)].$$

Therefore, for *a* function *f* which is differentiable at *a,* we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or, equivalently, we can write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \text{(Replace } x - a \text{ with } h \text{)}$$

Differentiation of Functions

From the definition of limit of a function, we can say that the real number $f'(a)$ is the derivative of *f* at a if given any ε > 0, there exists a δ > 0 (depending on ε).

such that

$$\left|\frac{f(x)-f(a)}{x-a}-f'(a)\right|<\varepsilon \text{ whenever } 0<|x-a|<\delta \text{ and } x\in A.$$

Definition 3.1.2: Let $A\subseteq R$, let *a* be a limit point of *A,* and let *f* : *A* → *R* be any function. Then *f* is said to be differentiable on *A* if *f* is differentiable at every point of *A.*

Value Addition : Remarks
(i) The function *f* itself is sometimes written as $f^{(0)}$.
(ii) The process by which *f′* is obtained from *f* is called differentiation
(iii) In case *f* is differentiable on *A,* we obtain a new function, *f′* whose domain is the set of points at which *f* is differentiable, thus dom (*f′*) ⊆ dom *f*. For example if $f(x)=x, x\in R$, then for any $a\in R$ we obtain $$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x-a}{x-a}=1$$ Thus $f'(a)$ = 1 for all *a* ∈ *R.* This implies that the function *f′* is defined for all of *R* and $f'(x)$ = 1 for all *x* ∈ *R*
(iv) If the $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ does not exist, or if *a* is not an element of *A* or *a* is not a limit point of *A,* then $f'(a)$ is not defined, and we say that *f* is not differentiable at the point *a* of *A.*

I.Q.1

Examples. 3.1.3 (i) Let *f* : *R* → *R* be any constant function. We take $f(x) = c$ for all *x* ∈ *R* and let *a* ∈ *A.* Now, we compute the limit.

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{c-c}{x-a} = 0$$

Thus, $f'(a) = 0,$ a ∈ *R* or $f'(x) = 0$, for all *x* ∈ *R.* This implies that the derivative of a constant function is the zero function.

(ii) Let *f* : *R* → *R* be the function $f(x) = x^2, x \in R$. Suppose that *a* ∈ *R.* In order to find $f'(a)$, we compute the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)-(x+a)}{x-a}$$

$$= \lim_{x\to a}(x+a) = 2a.$$

i.e. $f'(a) = 2a$, where *a* ∈ *R*.

Thus, in this case, the function $f(x)$ is differentiable at *a* and its derivative at *a* is 2*a*. In fact, we have $f'(x) = 2x$ for all *x* ∈ *R.*

It follows that *f* is differentiable on *R* and $f'(x) = 2x$, where *x* ∈ *R.*

(iii) Let *f* : (0, ∞) → *R* be defined by

$$f(x) = \frac{1}{x} \qquad (x > 0)$$

Let $a \in (0,\infty)$. We notice that

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a) = \lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \lim_{x\to a}\frac{-(x-a)}{a.x.(x-a)} = -\frac{1}{a^2}.$$

Thus, $f'(a) = -\frac{1}{x^2}.\ a \in R$.

Since $a \in (0,\infty)$ was taken arbitrarily, therefore it follows that

$$f'(x) = -\frac{1}{x^2} \text{ for } x > 0.$$

(iv) Let $f:[0,\infty) \to R$ be defined by

$$f(x) = \sqrt{x} \text{ for } x \geq 0.$$

For any *a* > 0, we observe that

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}$$

$$= \lim_{x\to a}\left(\frac{\sqrt{x}-\sqrt{a}}{x-a}\times\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right) = \lim_{x\to a}\frac{(x-a)}{(x-a)(\sqrt{x}+\sqrt{a})}$$

$$= \frac{1}{2\sqrt{a}} = \frac{1}{2}a^{-\frac{1}{2}}$$

Therefore, $f'(a) = \frac{1}{2}a^{-\frac{1}{2}}, \qquad a > 0$

In fact, $f'(a) = \frac{1}{2}x^{-\frac{1}{2}}$, for *x* > 0.

Value Addition: Interpretations
(i) Geometrically, the derivative $f'(a)$ of a function *f* at a point *x* = *a* represents the slope of the tangent to the curve $y = f(x)$ at that point. (ii) Physically, the derivative of a function f at $t = t_0$ represent the instantaneous speed of the particle at the time $t = t_0$.

I.Q.2

Differentiation of Functions

Proposition 3.1.4: Let $f : A \to R$ be differentiable at $x = a$, and $g : A \to R$ be equal to f, i.e. $f(x) = g(x)$ for all $x \in A$, then g is also differentiable at $x = a$ and in fact, $g'(a) = f'(a)$, $a \in R$

Proof: By definition of differentiability of f at $x = a$, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a) \qquad (\, f(x) = g(x) \text{ for all } x \in A).$$

Thus, $f'(a) = g'(a)$ which we concludes that g is also differentiable at $x = a$.

Example 3.1.5. In the above result, if two functions f and g merely have the same value at x, i.e. $g(a) = f(a)$, this does not imply that $g'(a) = f'(a)$.

For example,

Let $f(x) = 1, \quad x \in R$ and $g(x) = x, \quad x \in R$.

Then, we conclude that $f'(x) = 0$ for all $x \in R$ and $g'(x) = 1$ for all $x \in R$.

Let us take $a = 1 \in R$. Then $f(1) = g(1)$. But $f'(1) \neq g'(1)$.

Value Addition: Note
Thus, there is a big difference between two function being equal on their whole domain and merely being equal at one point.

Now, we introduce the concepts of the right-hand derivative of f and left-hand derivative of f at a point a which are, respectively, determined by the expression

$$f'_{+}(a) = \lim_{\substack{x \to a \\ x > a}} \frac{f(x) - f(a)}{x - a} \text{ and } f'_{-}(a) = \lim_{\substack{x \to a \\ x < a}} \frac{f(x) - f(a)}{x - a}.$$

The right-hand and left-hand derivatives are also referred one-sided derivatives.

Value Addition: Remember
The necessary and sufficient condition for the derivative $f'(a)$ of *f* at *x* = *a* to exists is that the right-hand and left-hand derivatives of f at the point *x* = *a* both exist and are equal to each other. In this case, the value of each of them automatically coincides with $f'(a)$.

I.Q.3

Let us consider the function f defined by

$$f(x) = |x|, x \in [-1,1]$$

where,

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

We show that *f* is not differentiable at *x* = 0.

For this, we have

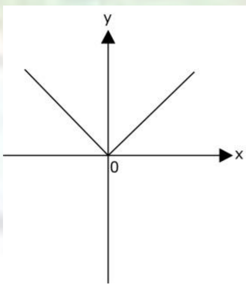

Figure 1

$$f'_{+}(0) = \lim_{x \to 0^{+}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^{+}} \frac{|x|}{x} = 1$$

and

$$f'_{-}(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Here, both left-hand and right-hand derivatives exist but they are not equal.

Thus, it follows that *f* is not differentiable at *x* = 0.

However, at all the remaining points the derivative of | *x* | exists and is expressed by the formula

$$|x|' = \operatorname{sgn} x = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Our next result establish that the differentiability of a function at a point implies continuity of the function at that point.

Theorem 3.1.6. Let $A \subseteq R$ and a be a limit point of *A.* If a function $f : A \to R$ is differentiable at *a,* then *f* is continuous at *a.*

Proof: We are given that *f* is differentiable at *a,* that is, $f'(a) = \lim_{x \to a} \frac{f(a) - f(a)}{x - a}$ exists and finite. We have to show that *f* is continuous at *a.* We notice that

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}.(x - a)$$

for $x \in A$ and $x \neq a$.

Since $\lim_{x \to a}(x - a) = 0$ and $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist and finite, therefore

$$\lim_{x \to a}[f(x) - f(a)] = 0.$$

Thus, $\lim_{x \to a} f(x) = f(a)$.

It follows from the definition of continuity that the function f is continuous at
$x = a$.

Value Addition: Remarks	
(i)	We conclude from the Theorems 3.1.6 that continuity of a function f at a point $x = a$ is a necessary condition for the existence of the derivative of f at $x = a$. However, it is not a sufficient condition.
(ii)	We also observe that from Theorem 3.1.6 that non-continuity of f implies non-differentiability of f.

Example 3.1.7: Let us consider a function $f : R \to R$ be defined by

$$f(x) = \begin{cases} x, & x \geq 1 \\ x^2, & x < 1 \end{cases}$$

Thus f is continuous at $x = 1$ but not differentiable at $x = 1$.

For this, we have

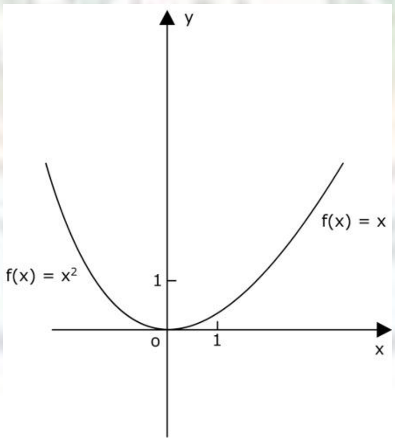

Figure 2

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f(1+h) - 1}{h} = 1$$

and

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{f(1+h)^2-1}{h}=2$$

The derivative of the above function *f* at the point *x* = 1 dos not exists since the right-hand and left-hand derivatives at this point are different. But at all the remaining points the derivative of *f* exist and is given by the formula

$$f'(x)=\begin{cases}1, & x>1\\ 2x, & x<1\end{cases}$$

The function *f* gives us an example of a function continuous at a point *x* = 1 and having no derivative at that point,

This example show that the converse of the Theorem 3.1.6 does not hold.

I.Q.4

3.2 Algebra of Derivatives

In this section we now prove some results that are useful in finding derivatives of various combination of functions without our having to indulge in the technical process of appealing to the definition of differentiable function. Further, these results will enlarge our collection of differentiable functions rather extensively.

Theorem 3.2.1. Let $A\subseteq R$, let *a* be a limit point of *A,* and let *f* : *A* → *R* and
g : *A* → *R* be any function. Further, let α and β be any real numbers.

If *f* and *g* are differentiable functions at *x* = *a,* then

(i) $\alpha f+\beta g$ is also differentiable at *x* = *a,* and

$$(\alpha f+\beta g)(a)=\alpha f'(a)+\beta g'(a)$$

(ii) fg is also differentiable at *x* = *a,* and

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

(iii) $\frac{1}{f}$ is also differentiable at *x* = *a* provided $f(x) \neq 0$ for all *x* ∈ *A,* and

$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{(f(a))^2}.$$

Proof: We are given that *f* and *g* are differentiable at the point *x* = *a* which implies that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \quad \text{and } g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

both exist and finite.

(i) Consider the limit

$$\lim_{x \to a} \frac{(\alpha f + \beta g)(x) - (\alpha f + \beta g)(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\alpha[f(x) - f(a)]}{x - a} + \lim_{x \to a} \frac{\beta[g(x) - g(a)]}{x - a}$$ (by using algebra of limits)

$$= \alpha \lim_{x \to a} \frac{f(x) + f(a)}{x - a} + \beta \lim_{x \to a} \frac{g(x) + g(a)}{x - a}$$

$$= \alpha . f'(a) + \beta g'(a).$$

Thus, $(\alpha f + \beta g)'(a) = \alpha . f'(a) + \beta g'(a), \; a \in A$.

Hence, we obtain that the function $\alpha f + \beta g$ is differentiable at *x* = *a* with derivative $\alpha . f'(a) + \beta g'(a)$.

(ii) We now, compute the limit

$$\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a}=\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}$$

$$=\lim_{x\to a}\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}$$

$$=\lim_{x\to a}\frac{f(x)(g(x)-g(a))+g(a)(f(x)-f(a))}{x-a}$$

$$=\lim_{x\to a}f(x).\lim_{x\to a}\frac{g(x)-g(a)}{x-a}+g(a)\lim_{x\to a}\frac{(f(x)-f(a))}{x-a}$$

(by using algebra of limits).

Since *f* is differentiable at *x* = *a,* therefore by using Theorem 3.1.6, we conclude that *f* is continuous at *x* = *a.* Thus, $\lim_{x\to a}f(x)=f(a)$.

Therefore, we deduce that

$$\lim_{x\to a}\frac{(fg)(x)-(fg)(a)}{x-a}=f(a)g'(a)+g(a)f'(a)$$

which implies that *fg* is differentiable at *x* = *a* with derivative $(fg)'(a)=f(a)g'(a)+g(a)f'(a)$.

(iii) Let us obtain the limit

$$\lim_{x\to a}\frac{\frac{1}{f(x)}-\frac{1}{f(a)}}{x-a}=\lim_{x\to a}\frac{f(a)-f(x)}{f(a)f(x)(x-a)}$$

$$=-\lim_{x\to a}\frac{f(x)-f(a)}{x-a}.\lim_{x\to a}\frac{1}{f(a)f(x)}$$

(by using algebra of limits).

Since *f* is differentiable at a point *x* = *a,* therefore by using Theorem 3.1.6, we conclude that *f* is continuous at *x* = *a.*

Thus, by definition of continuity, we have

$$\lim_{x \to a} f(x) = f(a)$$

Therefore, we conclude that

$$\lim_{x \to a} \frac{\frac{1}{f(x)} - \frac{1}{f(a)}}{x - a} = -\frac{f'(a)}{(f(a))^2}$$

It follows that the function $\frac{1}{f}$ is differentiable at $x = a$ and $\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{(f(a))^2}$.

Corollary 3.2.2. Let $f_1, f_2, \ldots, f_n$ be functions defined on a set $A \subseteq R$ and are differentiable at a point $a \in A$, then

(i) the function $f_1, f_2, \ldots, f_n$ is differentiable at a and

$$\left(f_1 + f_2 + \ldots + f_n\right)'(a) = f_1'(a) + f_2'(a) + \ldots + f_n'(a)$$

(ii) The function $f_1 f_2 \ldots\ldots f_n$ is differentiable at a and

$$(f_1 f_2 \ldots\ldots f_n)'(a) = f_1'(a) f_2(a) \ldots. f_n(a) + f_1(a) f_2'(a).$$

$$\ldots f_n(a) + \ldots. + f_1(a) f_2(a) \ldots f_n'(a)$$

Also, $(f^n)'(a) = n(f(a))^{n-1}.f'(a)$.

Proof: (i) We prove it by principle of mathematical induction.

By Theorem 3.2.1, the assertion is true for $n = 2$.Assume that the result is true for $n = k$, that is,

$$(f_1 + f_2 + \ldots. + f_k)'(a) = f_1'(a) + f_2'(a) + \ldots + f_k'(a).$$

Now, we consider

$$(f_1 + f_2 + ... + f_k + f_{k+1})'(a) = (f_1 + f_2 + ... + f_k)'(a) + f'_{k+1}(a)$$

$$= f'_1(a) + f'_2(a) + ... + f'_k(a) + f'_{k+1}(a).$$

This implies that the result is true for $n = k + 1$. Thus the result hold true by induction.

(ii) Similar arguments as given in the part (i).

Let us take $f_1 = f_2 = ... = f_n = f$ in (ii), we obtain

$$(f^n)'(a) = n(f(a))^{n-1}.f'(a).$$

and this completes the proof.

As a special case, if we take $f(x) = x$, then

$$(f^n)'(x) = (x^n)' = nx^{n-1}.1 = nx^{n-1},\ x \in R,\ n \in N$$

Corollary 3.2.3. Let $A \subseteq R$, let $x = a$ be a limit point of A. If $f : A \to R$ and

$g : A \to R$ are differentiable at $x = a$ and $g(x) \neq 0$ for all $x \in A$, then $f \mid g$ is also differentiable at $x = a$ and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a).g(a) - g'(a)f(a)}{(g(a))^2}$$

Proof: Then, by using Theorem 3.2.1 (ii), we obtain

$$\left(\frac{f}{g}\right)'(a) = \left(f.\frac{1}{g}\right)'(a)$$

$$= f(a)\left(\frac{1}{g}\right)'(a) + f'(a).\frac{1}{g(a)}$$

$$= f(a)\left(\frac{-g'(a)}{(g(a))^2}\right) + \frac{f'(a)}{g(a)}$$ (by using Theorem 3.2.1(iii))

$$= \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$$

which established that the function *f* | *g* is differentiable at *x* = *a* with the derivative

$$\frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$$

I.Q.5.

Value Addition : Remarks	
(i)	In order to prove the Corollary 3.2.3, we have used parts (ii) and (iii) of Theorem 3.2.1. However, a direct, proof may be given.
(ii)	The converse of the above Corollary 3.2.3 is not true.
(iii)	The converse of the above theorem 3.2.1 is not true.
(iv)	The part (ii) of Theorem 3.2.1 is famously known as the product rule or the Leibnitz's rule after Gdtfried Leibnitz (1646-1716) who was also the founder of differential and integral calculus besides the great scientist and mathematician Sir Isaac Newton (1642-1727)
(v)	Corollary 3.2.3 is famously known as the quotient rule.

For part (iii),i.e., Theorem 3.2.1(i) of the above remarks, let us take

$f(x) = |x|$, $x \in R$, $g(x) = |x|$, $x \in R$, $\alpha = 1$, $\beta = 1$.

Then $(\alpha f + \beta g)(x) = 0$ which is differentiable at $x = 0$ but *f* and *g* are not differentiable at 0.

Again, for part (iii), that is, Theorem 3.2.1(ii) of the above remark, we consider

$$f(x) = |x|,\ x \in R,\ g(x) = |x|,\ x \in R.$$

Then *f* and *g* are not differentiable at $x = 0$.

But $(f.g)(x) = f(x)g(x) = |x|^2 = x^2$ which is differentiable at the point $x = 0$.

Likewise, the other parts of the theorem 3.2.1 can be discussed in the similar manner.

Example 3.2.4. Let *n* be a natural number, and let $f : R \to R$ be the function
$f(x) = x^n$, $x \in R$ Then, for any *x*, we have

$$f'(x) = nx^{x-1}$$

We prove it by using Principle of Mathematical Induction.

If $n = 1$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h} = 1.$$

It follows that the result is obvious for $n = 1$.

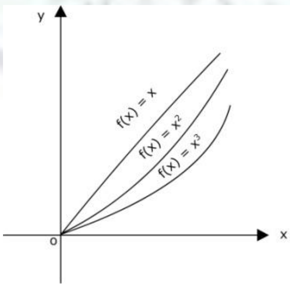

Figure 3

Now let us assume that the result is true for *n* = *k*.

Thus $f'(x) = kx^{k-1}$, $x \in R$.

By using Theorem (3.2.1)(ii), we obtain

$$f'(x) = (x^{k-1})' = (x^k.x)' = x^k.1 + x.kx^{k-1}$$

$$= (k+1)x^k = (k+1)x^{(k+1)-1}$$

and the result follows by induction.

I.Q.6

Example 3.2.5. Let *n* be *a* positive integer, and let $r(x) = x^{-n}$ where $x \neq 0$. Let us take $f(x) = 1$ and $g(x) = x^n$ for all $x \in R$. Suppose that $a \in R$ and $a \neq 0$. By using quotient rule (Corollary 3.2.3), we have

$$r'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}$$

$$= \frac{a^n.0 - 1.na^{n-1}}{(a^n)^2}$$

$$= \frac{-n}{a^{n+1}} = -na^{-n-1}$$

We observe that the derivative of x^n is nx^n for both positive as well as negative integers.

I.Q.7

3.3 Caratheodory's Theorem

In this section we study a theorem which makes it possible to reduce some of the theorems on derivatives to theorem on continuity and is also due to Constantin Caratheodory (1873-1950).

Theorem 3.3.1 (Caratheodory). Let $A \subset R$ and let a be a limit point of A. Then a function $f : A \to R$ is differentiable at a if and only if there exists a function $f^* : A \to R$ such that

$$f(x) - f(a) = f^*(x)(x-a)$$

for all $x \in A$, f^* is continuous at a. Moreover, if these conditions hold, then $f'(a) = f^*(a)$.

Proof: Let us first assume that f is differentiable at $x = a$. Now we can defined $f^* : A \to R$ by

$$f^*(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a}, & \text{if } x \in A,\ x \neq a \\ f'(a), & \text{if } x = a \end{cases}$$

The f^* is continuous at a, since $\lim_{x\to a} f^*(x) = f'(a) = f^*(a)$ and

$$f(x) - f(a) = f^*(x)(x-a) \quad \text{for } x \in A .$$

Conversely, if there exists a continuous function $f^* : A \to R$ such that

$$f(x) - f(a) = f^*(x)(x-a) \quad \text{for } x \in A.$$

We need to show that f is differentiable at $x = a$. We have

$$f(x) - f(a) = f^*(x)(x-a) \quad \text{for } x \in A$$

Then by dividing the above expression by $x - a$ and letting $x \to a$, we find that $f'(a)$ exists and equals $f^*(a)$ which established the theorem.

Value Addition: Do you Know
(i) The above theorem is sometimes referred as the characterization of differentiability in terms of continuity.
(ii) We observe that the above theorem states that every

differentiable function is approximated by a linear function whose slope is the derivative.

This approximation was observed by the great scientist and mathematician Isaac Newton (1642-1727), one of the founders of differential and integral calculus.

To give an illustration of Caratheodory's theorem 3.3.1, we discuss the following examples:

Example 3.3.2. Let us consider the function $f : R \to R$ defined by $f(x) = x^4$ for all $x \in R$.

Then for any $a \in R$, we observe that

$$x^4 - a^4 = (x-a)(x+a)(x^2+a^2)$$

Here, we take $f^*(x) = (x+a)(x^2+a^2)$. Then f^* satisfies the conditions of Theorem 3.3.1.

Thus, we deduce that f is differentiable at $x = a$ and

$$f'(a) = f^*(x) = 2a.2a^2 = 4a^3.$$

Now we consider the chain rule.

I.Q. 8

3.4 The Chain Rule

This section deals with another fundamental property, namely, the chain rule of differentiable function which together with Theorem 3.2.1 forms the basis for calculating the derivatives of most elementary functions. The chain rule establish a formula for finding the derivative of a composite function $g \circ f$ in terms of the derivatives of g and f and is probably the most important theorem about derivatives.

Let us assume that the composition $(g \circ f)(x)$ is defined in a neighbourhood of *x* = *a* and $f'(a)$ and $g'(f(a))$ both exist.

To evaluate $(g \circ f)'(a)$, we must have written

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a}$$

$$= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a}$$

Letting *x* → *a*, then we must argue that the second term in the last product above converges to $f'(a)$ and the first term converges to $g'(f(a))$ as *x* → *a*.

But for values of *x* near *a*, the quantity $f(x) - f(a)$ equals zero which makes the first term in the above expression undefined and we face a problem, which can be avoided by the use of Caratheodory's theorem.

I.Q.9

Theorem 3.4.1 (Chain Rule)

Let *A, B* be subsets of *R,* let *a*∈ *A* be a limit point of *A,* an let *b* ∈ *B* be *a* limit point of *B.* Let *f* : *A* → *B* be a function such that f(a) = *b,* and such that *f* is differentiable at *a.* Suppose that *g* : *B* → *R* is a function which is differentiable at *b.* Then the function $g \circ f$: *A* → *R* is differentiable at *a,* and $(g \circ f)'(a) = g'(f(a)),\ f'(a)$.

Proof: Since *f* is differentiable at *x* = *a,* therefore by using Caratheodory's Theorem 3.3.1, we can write

$f(x) - f(a) = (x - a) f^*(x)$ for all *x* ∈ *A,* where f^* is continuous at $x = a$ and $f^*(a) = f'(a)$. Again, since $g'(b)$ exists, by using Caratheodory's Theorem 3.3.1, we obtain

$$g(y) - g(b) = (y - b)g^*(y), \text{ for all } y \in b.$$

Here g^* is continuous at *b*, that is, $f(a)$ and $g^*(b) = g'(b)$.

By putting $y = f(x)$ and $b = f(a)$, we deduce

$$g(f(x)) - g(f(a)) = g^*(f(x))(f(x) - f(a))$$

$$= g^*(f(x)).f^*(x)(x-a)$$

$$= [(g^* \circ f(x)).f^*(x)](x-a)$$

for all $x \in A$ such that $f(x) \in B$.

By the continuity theorem for composite function,

$$g^*(f(x)) \to g^*(f(a)) = g'(f(a)) \text{ as } x \to a$$

Therefore, $\dfrac{g(f(x)) - g(f(a))}{x-a} = (g^* \circ f(x)).f^*(x)$.

Letting $x \to a$ in the above expression, we obtain

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x-a} = g'(f(a)).f'(a).$$

It follows that

$$(g \circ f)'(a) = g'(f(a)).f'(a).$$

Example 3.4.2. (i) If $f : R \setminus \{0\} \to R$ is the function $f(x) = \dfrac{x-1}{x}$ and $g : R \to R$ is the function $g(y) = y^2$, then $(g \circ f)(x) = \left(\dfrac{x-1}{x}\right)^2$ and the chain rule gives

$$(g \circ f)'(a) = 2\left(\frac{a-1}{a}\right)\frac{1}{a^2}, \text{ where } a \in R \setminus \{0\}$$

(ii) Let $h(x) = \sin(x^3 + 5x),\ x \in R$

We set $f(x) = x^3 + 5x, \quad x \in R$ so that $h = g \circ f$.

Then $f'(x) = 3x^2 + 5$ and $g'(x)$ = cos y.

The Chain rule provides

$$(g \circ f)'(x) = h(x) = g'(f(x)).f'(x)$$

$$= \cos(x^3 + 5x).(3x^2 + 5) .$$

(iii) Let $g : R \to R$ be any function defined by

$$g(x) = x^n,\ x \in R$$

Then, $g'(x) = nx^{n-1},\ x \in R$.

Let us consider a function $f : A \to R$ which is differentiable on A.

Now we compute $(g \circ f)$.

By applying Chain rule 3.4.1, we obtain

$$(g \circ f)'(x) = g'(f(x)).f'(x) .$$

Thus, we have $(f^n)'(x) = n(f(x))^{n-1}.f'(x)$ for all $x \in R$.

(iv) For any rational number r, we have

$(x^r)' = rx^{r-1}$, provided $x > 0$.

We write $x = \frac{m}{n}$ where m is an integer and n is a natural number.

Then

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$$(x^r)' = ((x^m)^{1/n})' = \frac{1}{n}(x^m)^{1/n-1}.mx^{m-1}.1.$$

$$= \frac{m}{n}.x^{m/n-m+m-1} = \frac{m}{n}x^{\frac{m}{n}-1}.$$

We notice that if $r > 1$, then the derivative also exists at $x = 0$.

Exercise:

1. Let $f(x) = \begin{cases} x^{4/3}\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

   Then show that

   (i) f is differentiable on R.

   (ii) If f' is bounded?

2. Determine the set of points at which the following function are not differentiable.

   (i) $f(x) = |x^3 - 8|$

   (ii) $f(x) = |x^2 - 2|$

   (iii) $f(x) = |\sin x|$

3. Show that the function f defined as

   $f(x) = |x-1| + |x+1|$ for all $x \in R$.

   is not derivable at the points $x = -1$ and x 1 and is derivable at every other point.

4. Show that the function f defined on R as follows:

   $$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

   is differentiable at 0 and $f'(0) = 0$. Further show that f is not differentiable for all $x \neq 0$.

5. Construct an example of *a* function which is discontinuous for all $x \neq 0$, but differentiable at $x = 0$.

6. Prove or disprove : every continuous function is also differentiable function.

7. Find the derivative of the following function using Chain rule:

   (i) $f(x) = \sqrt{1 - x^2}$

   (ii) $f(x) = (\sin x^k)^m$, for $m, k \in N$

   (iii) $f(x) = \left(1 + x^{100}\right)^{200}$

8. Show that $f(x) = x^{1/3}, x \in R$ is not differentiable at $x = 0$.

9. Let $h(x) = (x^5 + 24x^2)^{99}$. Find $h'(x)$ by using Chain rule.

10. Let $h(x) = (\cos x + e^{4x})^{50}$. Find $h'(x)$ by using chain rule.

11. Differentiate the function $\cos(e^{x^4 - 5x})$.

12. Prove Leibnitz's rule of finding nth derivative of product of the function:

$$(fg)^n(x) = \sum_{k=0}^{n} {}^nC_k f^{(k)}(x) g^{(n-k)}(x)$$

13. Let $f(x) = \sqrt{x}, x \geq 0$. Prove that the derivative of f is $f'(x) = \frac{1}{2} x^{-1/2}$, for $x > 0$.

14. Prove or disprove : The function $f(x) = x|x|, x \in R$ is differentiable on *R.*

15. Find two functions *f* and *g* which are not differentiable at a point but their quotient $\left(\frac{f}{g}\right)$, is differentiable at that point.

Solutions:

1. (i) Use definition of a derivative.

   (ii) f' is not bounded, on any interval containing 0.

2 (i) {2}

(ii) $\{\sqrt{2}, -\sqrt{2}\}$

(iii) $\{n : n \in Z\}$

2. Hint: Here

$$\begin{aligned} f(x) &= -(x-1)-(x+1) = -2x, && \text{if } x < -1 \\ &= (x+1)-(x-1) = 2, && \text{if } -1 \le x < 1 \\ &= (x-1)+x+1 = 2x, && \text{if } x \ge 1 \end{aligned}$$

and then compute left hand and right hand derivatives

4. Hint: We observe that

$$\frac{f(x)-f(0)}{x-0} = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Thus, $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0$, that is, $f'(0) = 0$

5. Hint: Consider Q.4.

6. Every continuum function may not be differentiable. For example let $f(x) = |x|,\ x \in [-1,1]$.

Then, f is continuum at $x = 0$ but not differentiable at $x = 0$

7. (i) $-f'(x) = -x \,|\, \sqrt{1-x^2}, \qquad x \in R$

(ii) $f'(x) = m(\sin x^k)^{m-1} \cos x^k . kx^{k-1}$.

(iii) $f'(x) = 200(1+x^{100})^{199}.100x^{99}$

8. Let us compute

$$f'(a) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{x^{1/3}}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

which concludes that *f* is not differentiable at *x* = 0.

9. $h'(x) = 99(x^5 + 24x^2)^{98}.(5x^4.48x)$

10. $h'(x) = 50(\cos x + e^{4x})^{49}.(-\sin x)e^{4x}4)$

11. $f'(x) = -\sin(e^{x^4-5x}).(e^{x^4-5x}).(4x^3-5)$

12. Use induction

13. Apply definition of a derivative

14. $f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x|x|-0}{x} = 0$

and

$f'_-(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x|x|}{x} = 0$

15. Take $f(x) = |x|+1,\ x \in [-1,1]$

$g(x) = |x|+1,\ x \in [-1,1]$

Then $\frac{f}{g}(x) = 1$ which is differentiable at x=0 but f and g are not differentiable at *x* = 0.

Summary:

In this chapter, we have emphasized on the followings:

- Definition and examples of a derivative of a functions.
- Relation between a continuous function and a differentiable function.
- Statement and proofs of theorems based on the algebra of derivatives.
- Characterization of differentiable function in term of continuous function which was established by Caratheodory.

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- Statement and proof of the Chain Rule of differentiable functions.

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