Subject: Algebra-I
Semester-I
Lesson: Eigen Values and Eigen Vectors

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Eigen Values and Eigen Vectors

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1. Learning outcomes:

After studying this chapter you should be able to

- Understand the eigen values and eigenvectors of a matrix.
- find the characteristic equation of a matrix
- understand Cayley-Hamilton theorem
- define the algebraic multiplicity of an eigenvalue
- understand the similarity of matrices

2. Introduction:

The eigenvalue and eigenvectors are of considerable theoretical interest and wide-ranging application throughout the pure and applied mathematics. Eigen value and eigenvectors are used to solve the systems of differential equations, continuous dynamical systems, calculating powers of matrices (in order to define the exponential matrix) and analyzing the population growth models. They provide critical information in engineering design and arise naturally in fields such as physics, chemistry, statistics, biology, sociology and.

3. Characteristic Equation:

Let A be a square matrix of order n, such that

$$A = \begin{bmatrix} a_{11} & a_{12} & --- & a_n \\ a_{21} & a_{22} & --- & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}$$

Then we can form a matrix A–λI, where I is the unit matrix of order n and λ is any scalar.

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & --- & a_{1n} \\ a_{22} & a_{22} - \lambda & --- & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & --- & a_{mn} - \lambda \end{bmatrix}$$

Then, the determinant of this matrix equated to zero, i.e.

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & --- & a_{1n} \\ a_{22} & a_{22} - \lambda & --- & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & --- & a_{mn} - \lambda \end{vmatrix} = 0$$

is called the characteristic equation of the matrix A. On expanding the determinant, the characteristic equation can be written as a polynomial equation of degree n in λ and is of the form

$$(-1)^n \lambda^n + a^1 \lambda^{n-1} + a^2 \lambda^{n-2} + - - - + a_n = 0$$

where a_1, a_2, - - -, a_n are the constants.

The roots of this equation are called the characteristic roots of the matrix A.

4. Eigen Values and Eigenvectors:

We know that a linear transformation T: V $\rightarrow$ V can be expressed by a matrix representation as follows.

$$T(x) = A\,x \qquad (1)$$

where A is a matrix

In practice, we are more interested to find those vectors x which transforms into scalar multiples of themselves.

Let x be such a vector which transform into λ multiple of itself by the transformation T. Then

$$T(x) = \lambda x \qquad (2)$$

where λ is a scalar.

Then we have

$$Ax = \lambda x$$

$$\Rightarrow \quad (A - \lambda I)\,x = 0 \qquad (3)$$

This can be represented by the n–homogeneous linear equations as follows

$$(a_{11} - \lambda)x_1 + a_{12}x_2 + - - - + a_1n\, x_n = 0$$

$$a_{21}\, x_1 + (a_{22} - \lambda)\, x_2 + - - - + a_2n\, x_n = 0 \qquad (4)$$

$$a_{n1}\, x_1 + a_{n2}\, n_2 + - - - + (a_{nn} - \lambda)\, x_n = 0$$

We know that n–homogeneous equations in n–variables have a non–trivial solution if and only if the co–efficient matrix is singular, i.e. if

$$|A - \lambda I| = 0 \qquad (5)$$

We know that $|A - \lambda I| = 0$ is the characteristic equation of the matrix A, which has n–roots known as the eigenvalues of A.

Corresponding to each root or we can say to each eigen value of matrix A, the homogeneous system (3) has a non–zero solution

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

which is called an eigenvector or latent vector.

Thus the Eigenvalue and Eigenvector of a Matrix may be defined as follows:

Let A be a square matrix of order n. Then the non–trivial solution vector x of the equation

$Ax = \lambda x$

for any scalar λ. Then this solution vector x is called an eigenvector of matrix A corresponding to λ, which is called an eigenvalue of A.

Value Addition: Note
1. Consider an n x n non-zero matrix A. Then a nonzero vector x is called an eigenvector of the matrix A if there exists a nontrivial solution x of $Ax = \lambda x$ corresponding to λ. And the scalar λ is called an eigenvalue of matrix A.
2. The eigenvector corresponding to an eigenvalue may not be unique i.e. there may exists more than one eigenvectors corresponding to an eigenvalue.
3. Corresponding to a non–singular n x n square matrix A there exist eigenvalues. Which may be all real, some real and some complex or all complex.

Example 1. Prove that 0 is a characteristic root of a matrix if and only if the matrix is singular.

Solution: The characteristic root of a matrix A is given by $|A - \lambda I| = 0$. If $\lambda = 0$, then it gives $|A| = 0$

$\Rightarrow$ A is singular.

Again if matrix A is singular, then

$|A - \lambda I| = 0$

$\Rightarrow \quad |A| - \lambda|I| = 0$

$\Rightarrow \quad 0 - \lambda.1 = 0$

$\Rightarrow \quad \lambda = 0.$

Example 2: The sum of the eigen value of a square matrix is equal to the sum of the elements of its principal diagonal.

Solution: Let $A = [a_{ij}]_{3x3}$ be a square matrix of order 3. Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + \lambda^2 (a_{11} + a_{22} + a_{33}) - \ldots = 0 \qquad \ldots(1)$$

But by the definition of characteristic equation, we have

$$|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= -\lambda^3 + \lambda^2 (\lambda_1 + \lambda_2 + \lambda_3) - \ldots \qquad \ldots(2)$$

Comparing equations (1) and (2), we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}.$$

Thus, the sum of the eigen value of a square matrix is equal to the sum of the elements of its principal diagonal.

5. Cayley–Hamilton Theorem:

Theorem 1: Every square matrix satisfies its own characteristic equation.

Proof: Let $A = [a_{ij}]_{n \times n}$ be a square matrix, then

$$A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & --- & a_{nn} \end{bmatrix}$$

Now

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & --- & a_{1n} \\ a_{21} & a_{22}-\lambda & --- & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & --- & a_{nn}-\lambda \end{bmatrix}$$

Therefore, the characteristic equation is

$|A - \lambda I| = 0$

$\Rightarrow \quad a_0\lambda^n + a_1\lambda^{n-1} + a_2A^{n-2} + --------- + a_nI = 0 \qquad --------- (1)$

We have to prove that the matrix A satisfies the equation (1), i.e.

$a_0A^n + a_1\lambda^{n-1} + a_2A^{n-2} + --------- + a_n = 0 \qquad --------- (2)$

Since, the elements of the matrix $A-\lambda I$ are of first degree in λ and the elements of Adj $(A - \lambda I)$ are at most of degree $(n - 1)$ in λ because some terms may cancelled.

Therefore, we can write

$\text{Adj } (A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + --------- + B_{n-2}\lambda + B_{n-1}$

where B_0, B_1, --------- B_{n-2} , B_{n-1} are matrices or order $n \times n$.

We know that

$(A - \lambda I) \text{ Adj.}(A - \lambda I) = |A - \lambda I|\, I$

$\Rightarrow \quad (A - \lambda I)\,(B_0\lambda^{n-1} + B_1\lambda^{n-2} + ------ + B_{n-2}\lambda + B_{n-2}\ )$

$= (a_0\lambda^{n-1} + a_1\lambda^{n-2} + ------ + a_n)\, I$

On equating the coefficients of like power of λ on both sides, we get

$-B_0 = a_0 I$

$AB_0 - B_1 = a_1 I$

$AB_1 - B_2 = a_2 I$

$AB_{n-1} = a_n I$

premultiplying above relations by A^n, A^{n-1}, A^{n-2}, --- A, and I respectively and adding we get

$$0 = a_0 A^n + a_1 A^{n-1} + \text{------} + a_{n-1} A + a_n$$

$$\Rightarrow \quad a_0 A^n + a_1 A^{n-1} + \text{-------} + a_{n-1} A + a_n = 0$$

Thus, every square matrix satisfies its own characteristic equation.

Value Addition: Inverse of a matrix can be found using Cayley–Hamilton Theorem.

We know that every square matrix satisfies its characteristic equation, i.e.,

$$a_0 A^n + a_1 A^{n-1} + - - - + a_{n-1} A + a_n I = 0 \quad \text{------ (1)}$$

on multiplying by A^{-1}, we have

$$a_0 A^{n-1} + a_1 A^{n-2} + - - - + a_{n-1} I + a_n A^{-1} = 0 \quad \text{------ (2)}$$

$$\Rightarrow \quad A^{-1} = -\frac{1}{a_n}[\, a_0 A^{n-1} + a_1 A^{n-2} + - - - + a_{n-1} I \,]$$

Thus, Cayley–Hamilton theorem gives another method for competing the inverse of a sequence matrix.

If we again multiply equation (2) by A^{-1}, we have

$$a_0 A^{n-2} + a_1 A^{n-3} + - - - + a_{n2} I + a_{n-1} A^{-1} + an\, A^{-2} = 0$$

$$\Rightarrow \quad A^{-2} = -\frac{1}{a_n}[\, a_0 A^{n-2} + a_1 A^{n-3} + - - - + a_{n-2} I + a_{n-1} A^{-1}]$$

Hence, by using Cayley–Hamilton theorem, we can also final A^{-2} and hence we can find A^{-3}, A^{-4}, - - - etc.

Theorem 2: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof: For the sack of simplicity, we have considered a 3×3, upper triangular matrix A such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Then,

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)x = 0$ has a nontrivial solution that is if and only if the equation has a free variable. Because of zero entries in $A - \lambda I$. It is easy to see that $(A - \lambda I)x = 0$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero. This happens if and only if λ equals of the entries a_{11}, a_{22}; a_{33} in A.

Theorem 3: The eigenvalues of a unitary matrix are of unit modulus.

Proof: Let A be a unitary matrix so that

$$A^* A = I = AA^* \qquad ...(1)$$

If λ is a characteristic root of the matrix A and x is its eigenvector, then we have

$$Ax = \lambda x \qquad ...(2)$$

Taking transpose conjugate of (2), we have

$$(Ax)^* = (\lambda x)^*$$

$$x^*A^* = \bar{\lambda} x^* \qquad ...(3) \quad [\text{Since } \lambda^* = \bar{\lambda}]$$

On multiplying (2) and (3), we get

$$(x^* A^*)(Ax) = (\bar{\lambda} x^*)(\lambda x)$$

$$\Rightarrow \quad x^* (A^*A)x = \lambda \bar{\lambda} (x^*x)$$

$$\Rightarrow \quad x^*x = \lambda \bar{\lambda} (x^*x) \qquad [\text{using (1)}]$$

$$\Rightarrow \quad (1-\lambda\bar{\lambda}) x^* x = 0 \qquad ...(4)$$

Since x is a characteristic vector, $x \neq 0$

Consequently, $x^*x \neq 0$

Hence equation (4) gives

$$1 - \lambda\bar{\lambda} = 0 \quad \Rightarrow \quad \lambda\bar{\lambda} = 1$$

$$\Rightarrow \quad |\lambda|^2 = 1 \quad \Rightarrow \quad |\lambda| = 1$$

Hence the characteristic roots of a unitary matrix are of unit modulus.

Value Addition: Unitary Matrix
A matrix A is said to be unitary matrix if and only if A* A = I = AA* Where A* is the transpose of A and I is the identity matrix.

Example 3. Show that for any square matrix A, A and A* have same set of eigen values.

Solution: Let A be a square matrix. Then the characteristic equation of A is

$$|A - \lambda I| = 0 \qquad \ldots(1)$$

Let A* be the transpose of A.

Then the characteristic equation of A* will be

$$|A^* - \lambda I| = 0 \qquad \ldots(2)$$

Since the interchange of rows and columns does not alter the value of the determinant we have.

$$|A^* - \lambda I| = |A - \lambda I|$$

$$|A - \lambda I| = |A - \lambda I|^* = |A^* - \lambda I^*| = |A^* - \lambda I| \text{ as } I^* = I$$

Hence the eigen values of matrix A and its transpose A* are same.

Example 4: Product of all eigen values of a square matrix A is equal to the determinant of matrix A.

Solution: Let $A = [a_{ij}]_{n \times n}$ be a given square matrix and $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots, \lambda_n$ be its eigen values. If $\phi(\lambda)$ be the characteristic polynomial then,

$$\phi(\lambda) = | A - \lambda I |$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n \{\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \ldots + p_n\}$$

$$= (-1)^n \{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \ldots (\lambda - \lambda_n)\}$$

Putting $\lambda = 0$, we get

$$\phi(0) = (-1)^n (-1)^n \lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

$$|A| = \lambda_1\lambda_2\lambda_3\ldots\lambda_n$$

Hence the product of all eigen values of A is equal to determinant (A).

Theorem 4: The eigenvalues of a Hermitian matrix are all real.

Proof: Let λ be the eigenvalue of a Hermitian matrix A. Then there exists a non-zero eigenvector x such that

$$Ax = \lambda x \qquad \ldots(1)$$

Pre-multiplying both sides of (1) by x*, we get

$$x^*Ax = x^*\lambda x \qquad \ldots(2)$$

Transpose conjugate of (2) gives

$$(x^*Ax)^* = (\lambda x^*x)^*$$

$\Rightarrow$ $x^*A^*(x^*) = x^*(x^*)^* \lambda^*$ [By reversal law]

$\Rightarrow$ $x^*A^*x = x^*x\bar{\lambda}$ [$\because \lambda^* = \bar{\lambda}$]

But A is a Hermitian matrix therefore, A* = A

Thus, we have

$$x^*Ax = \bar{\lambda}x^*x \quad \text{...(3)}$$

From (2) and (3), we have

$$x^*(\lambda x) = \bar{\lambda}x^*x$$

$$\Rightarrow (\lambda - \bar{\lambda})x^*x = 0 \quad \text{...(4)}$$

Since x is a non-zero eigenvector

$\therefore$ $x^*x \neq 0$

Hence from (4), we have

$$\lambda - \bar{\lambda} = 0 \quad \Rightarrow \quad \lambda = \bar{\lambda}$$

which is possible only when λ is real.

Hence the eigenvalues of a Hermitian matrix are all real.

Value Addition: Hermitian Matrix
A Matrix A is said to be Hermitian matrix if and only if A* = A.

Theorem 5: The eigenvalues of a skew-hermitian marix is either zero or purely an imaginary number.

Proof: Since A is a skew-Hermitian matrix

$\therefore$ *i*A is a Hermitian matrix.

Let λ be a characteristic root of A.

Then, $Ax = \lambda x \quad \Rightarrow (iA)x = (i\lambda)x$

$\Rightarrow \quad i\lambda$ is a characteristic root of matrix iA.

But $\quad i\lambda$ is a Hermitian matrix.

Therefore $\quad i\lambda$ should be real.

Hence λ is either zero or purely imaginary.

Value Addition: Skew-Hermitian Matrix
A Matrix A is said to be Skew-Hermitian matrix if and only if $A^* = - A$.

Theorem6: The characteristic roots of an idempotent matrix are either zero or unity.

Proof: Since A is an idempotent matrix.

$\therefore \quad A^2 = A.$

Let x be a eigenvector of the matrix A corresponding to the eigenvalue λ so that

$$Ax = \lambda x \qquad \ldots(1)$$

$\Rightarrow \quad (A - \lambda I)x = O \qquad$ such that $x \neq 0$

Pre-multiplying (1) by A

$$A(Ax) = A(\lambda x) = \lambda(Ax)$$

$\Rightarrow \quad (AA)x = \lambda(\lambda x) \qquad$ [by (1)]

$\Rightarrow \quad A^2x = \lambda^2 x \;\Rightarrow\; Ax = \lambda^2 x \qquad$ [Since $A^2 = A$]

$\Rightarrow \quad \lambda x = \lambda^2 x$ [by (1)]

$\Rightarrow \quad (\lambda^2 - \lambda)x = O \quad \Rightarrow \quad \lambda^2 - \lambda = 0$ [Since $x \neq 0$]

$\Rightarrow \quad \lambda(\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 0, 1.$

Value Addition: Idempotent Matrix
A Matrix A is said to be Idempotent matrix if and only if $A^2 = A$.

Theorem 7: The $v_1, v_2, . . ., v_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2,. . ., \lambda_n$ of an $n \times n$ matrix A, then the set $\{v_1, v_2, . . ., v_n\}$ is linearly independent.

Proof: Suppose $\{v_1, v_2, . . ., v_n\}$ is linearly dependent. Since v_1 is nonzero, Let *p* be the least index such that v_{p+1} is a linear combination of the preceding (linearly independent) vectors. Then there exist scalars a_1, $a_1, . . ., a_p$ such that

$$a_1v_1 + a_2v_2 + . . . + a_p v_p = v_{p+1} \quad ... (1)$$

Multiplying both sides of (1) by A and using the fact that $Av_k = \lambda_k v_k$ for each k, we obtain

$$a_1Av_1 + a_2 Av_2 + . . . + a_p Av_p = Av_{p+1}$$

$$a_1 \lambda v_1 + a_2 \lambda v_2 + . . . + c_p\lambda_p v_p = \lambda_{p+1} v_{p+1} \quad ... (2)$$

Multiplying both sides of (1) by λ_{p+1} and subtracting the result from (2), we have

$$a_1 (\lambda_1 - \lambda_{p+1})v_1 + a_2 (\lambda_2 - \lambda_{p+1})v_2 + . . .+c_p(\lambda_p - \lambda_{p+1})v_p = 0 \quad ... (3)$$

Since $\{v_1, v_2, . . ., v_p\}$ is linearly independent, the weights in (3) are all zero. But none of the factors $(\lambda_1 - \lambda_{p+1})$ are zero, because the eigenvalues are distinct. hence $a_i = 0$ for i = 1, ..., *p*. But then (1) says that $v_{p+1} = 0$, which is impossible. Hence $\{v_1, v_2, . . ., v_n\}$ cannot be linearly dependent and therefore must be linearly independent.

Value Addition: Note
A set of vectors $\{v_1, v_2, . . ., v_p\}$ is called linearly dependent if any one of them can be written as a linear combination of other vectors.

Example 5: Find the eigen values and eigen vector of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}.$$

Solution: The characteristic equation of the given matrix is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} I-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$\lambda^2 - 5\lambda - 6 = 0$$

$\Rightarrow$ $$\lambda = 6, - 1.$$

Thus, the eigen values of A are 6, - 1.

Corresponding to $\lambda = 6$, the eigen vectors are given by

$$(A - 6I)X_1 = 0$$

or $$\begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

or $$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

We get only one independent equation $- 5x_1 - 2x_2 = 0$

$\Rightarrow$ $$\frac{x_1}{2} = \frac{x_2}{-5} = k_1 \text{ (say)}$$

$$x_1 = 2k_1, \ x_2 = 5k_1$$

∴ eigen vectors are $X_1 = k_1\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Corresponding to λ = -1, the eigenvectors are given by

$(A + I) X_2 = O$

⇒ $\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

⇒ $x_1 - x_2 = 0$

⇒ $\frac{x_1}{1} = \frac{x_2}{1} = k_2$ (say)

∴ The eigen vectors are $X_2 = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Example 6. Find the eigen values and the corresponding eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Solution: The characteristic equation of the given matrix is $|A - \lambda I| = 0$

or $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

or $(-2 -\lambda) [-\lambda(1 - \lambda) - 12] - 2 [-2\lambda -6] - 3 [-4 + 1(1-\lambda)] = 0$

or $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

⇒ $(\lambda + 3)(\lambda^2 - 2\lambda - 15) = 0$

⇒ $(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0 \Rightarrow \lambda = -3, -3, 5$

Thus, the eigen values of A are -3, -3, 5.

Corresponding to λ = -3, the eigen vectors are given by

$(A + 3I)\, X_1 = O$

or $$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O$$

We get only one independent equation $x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = k_1$ and $x_2 = k_2$ then $x_1 = 3k_1 - 2k_2$

$\therefore$ The eigen vectors are given by

$$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Corresponding to $\lambda = 5$, the eigen vectors are given by $(A - 5I)\, X_2 = O$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{bmatrix} \quad R_1 \to (-1)R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 16 & 32 & 0 \end{bmatrix} \quad \begin{cases} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{cases}$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & -8 & -16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 8R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{-1}{8} R_2$$

$x_1 + 2x_2 + 5x_3 = 0$

$x_2 + 2x_3 = 0$

$\therefore \quad x_3 = -k,\ x_2 = 2k,\ x_1 = k_3,$

Hence the eigen vectors are given by

$$X_2 = k\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Example 7: Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$, and hence find its inverse A^{-1} using Cayley-Hamilton theorem.

Solution: The characteristic equation of the given matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 2-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 2-\lambda \\ 1 & -2 \end{vmatrix} = 0$$

$$(2-\lambda)\ \{(2-\lambda)^2 - 2\} - (1-\lambda) + \lambda = 0\,.$$

thus the required characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

Using Cayley Hamilton Theorem, we have

$$A^{-1} = \frac{1}{3}\left(A^2 - 6A + 8I\right)$$

$$A^{-1} = \frac{1}{3}\left(\begin{bmatrix} 6 & -6 & 5 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{3}\left(\begin{bmatrix} 6 & -6 & 5 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 12 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right)$$

Therefore inverse of matrix A is $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix}$.

Example 8. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence, compute A^{-1}. Also find the matrix represented by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

Solution: The characteristic equation of A is

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0$$

or $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem, $A^3 - 5A^2 + 7A - 3I = O$ …(1)

Pre-multiplying (1) by A^{-1}, we get

$$A^{-1} A^3 - 5A^{-1}A^2 + 7A^{-1} A - 3A^{-1}I = A^{-1}O$$

$\Rightarrow$ $A^2 - 5A + 7I - 3A^{-1} = O$

$\Rightarrow$ $3A^{-1} = A^2 - 5A + 7I$

$\Rightarrow$ $A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$ …(2)

Now, $A^2 = A\ .\ A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

From (2), $3A^{-1} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

Now, $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + A\left(A^3 - 5A^2 + 7A - 3I\right) + \left(A^2 + A + I\right)$$

$$= A^2 + A + I \qquad \text{[Using (1)]}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Example 9. Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ find Adj. A by using Cayley-Hamilton theorem.

Solution: The characteristic equation of the given matrix A is

$| A - \lambda I | = 0$

$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (1 - \lambda)\ [(1 - \lambda)^2 - 1] - 2\ [3] - 1\ [-3(1 - \lambda)] = 0$

$\Rightarrow \quad (1 - \lambda)(1 + \lambda^2 - 2\lambda - 1) - 6 + 3 - 3\lambda = 0$

$\Rightarrow \quad \lambda^2 - \lambda^3 - 2\lambda + 2\lambda^2 - 3 - 3\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 3\lambda^2 - 5\lambda - 3 = 0$

$\Rightarrow \quad \lambda^3 - 3\lambda^2 + 5\lambda + 3 = 0$

By Cayley-Hamilton theorem, matrix A should satisfy the equation

$$A^3 - 3A^2 + 5A + 3I = O$$

Pre-multiplying by A^{-1}, we get

$$A^2 - 3A + 5I + 3A^{-1} = O$$

$\Rightarrow \quad A^{-1} = -\frac{1}{3}\left(A^2 - 3A + 5I\right) \qquad \ldots(1)$

Now, $A^2 = A \,.\, A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{bmatrix}$

$$3A = \begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 9 & -3 & 3 \end{bmatrix}$$

$\therefore$ From (1), $A^{-1} = -\frac{1}{3}\left\{\begin{bmatrix} -2 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 9 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right\} = -\frac{1}{3}\begin{bmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{bmatrix}$

We know that, $A^{-1} = \frac{Adj.\,A}{|A|}$

$\therefore \quad$ Adj. A = A^{-1} | A |

Now, $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -3$

$\therefore$ Adj. A = (-3) $\left(-\frac{1}{3}\right)\begin{bmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -3 & 4 & 1 \\ -3 & 7 & 1 \end{bmatrix}$

Example 10: Let $A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$, $u = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ *and* $v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Are u and v eigenvectors of A.

Solution : Given $A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$

$$Au = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 - 21 + 9 \\ -16 + 15 + 1 \\ 8 - 12 + 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

Thus, $u = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of matrix A correspondence to an eigenvalue (0).

Now,

$$Av = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 - 14 + 9 \\ -4 + 10 + 1 \\ 2 - 8 + 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Thus, $v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is not an eigenvector of Matrix A.

Example 11: Show that λ = 4, is an eigenvalue of the matrix find the corresponding eigenvectors.

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$$

Solution : We know that, scalar 4 is an eigenvalue of A if the equation

Ax = 4x

has a non–trivial solution, Thus

$\Rightarrow \quad (A - 4I)\, x = 0$

Now we have,

$$A - 4I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$$

Now

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \qquad R_1 \to (-1)R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \qquad \begin{matrix} R_1 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 + 4R_2$$

since the rank $(A - \lambda I) = 2 <$ No. of variable, therefore the equation $Ax = 4x$ has a non–trivial solution.

Hence, 4 is an eigenvalue of A.

To find the corresponding eigenvector, we have

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we have

$x_1 + x_3 = 0$

$x_2 + x_3 = 0$

Let $x_3 = -k \Rightarrow x_2 = k$ and $x_1 = k$

Thus general solution has the form $\begin{bmatrix} k \\ k \\ -k \end{bmatrix}$.

Thus, Each vector of the form $k\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, k \neq 0$, is an eigenvector corresponding to the eigenvalue 4.

or in particular $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, is an eigenvector corresponding to the eigenvalue 4.

Value Addition: Cautions
1. We have used the row reduction method in this example to find the eigenvector, but this method cannot be used to find the eigenvalues. 2. To find the eigenvector corresponding to the eigenvalue λ, we find a non–zero solution of the equation $(A - \lambda I)x = 0$. 3. The set of all solutions of the equation $(A - \lambda I)x = 0$ is called the eigenspace of matrix A corresponding to the eigenvalue λ and this set of solutions is just the null space of the matrix $(A - \lambda I)$, therefore it is also a subspace of R^n. 4. The eigenspace of matrix A consists of the zero vector and all the eigenvectors corresponding to any eigenvalue λ.

Example 12: Find a basis for the matrix $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ for the eigenspace corresponding to the eigenvalue $\lambda = 4$.

Solution: We have

$$A - 4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$$

Now to find the solution of the equation

$(A - 4I)x = 0$.

We have

$$\begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \qquad R_1 \to \frac{1}{3} R_1$$

$$\sim \begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 - 2R_1$$

Thus, we have $2x_1 - 3x_2 = 0$

Let $x_2 = k$

$\Rightarrow \quad x_1 = \frac{3}{2}k$

Thus the general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}k \\ k \end{bmatrix} = k\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Thus, the required basis is $\left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \right\}$.

Example 13: Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

Solution: We have

$$A - 2I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

To find the solution of the equation (A - 2I) x = 0, we have

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

Thus, we have

$2x_1 + x_3 = 0$

$-x_2 + x_3 = 0$

Hence, x_3 is a tree variable here, let x_3 = k

$\Rightarrow$ $x_2 = k$ and $x_1 = \frac{1}{2}k$

Thus, the general solution is

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k}{2} \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the required basis is $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$.

Value Addition: Basis for the Eigenspace
The basis for the eigenspace is set of linearly independent vectors which spans the eigenspace.

Example 14: If λ be an eigen value of a non-singular matrix A, show that

(i) λ^{-1} is an eigen value of A^{-1}. (ii) $\frac{|A|}{\lambda}$ is an eigen value of adj. A.

Solution: (i) λ is an eigen value of A.

$\Rightarrow$ There exists a non-zero vector x such that $Ax = \lambda x$

$\Rightarrow$ $x = A^{-1}(\lambda x)$

$\Rightarrow$ $x = \lambda(A^{-1}x)$

$\Rightarrow$ $\frac{1}{\lambda}x = A^{-1}x$

$\Rightarrow$ $A^{-1}x = \lambda^{-1}x$

$\Rightarrow$ λ^{-1} is an eigen value of A^{-1}.

(ii) λ is an eigen value of A.

$\Rightarrow$ There exists a non-zero vector x such that $Ax = \lambda x$

$\Rightarrow$ $(\text{adj. } A)(Ax) = (\text{adj. } A)(\lambda x)$

$\Rightarrow$ $((\text{adj. } A)A)\, x = \lambda\, (\text{adj. } A)\, x$

$\Rightarrow$ $|A|\, Ix = \lambda\, (\text{adj. } A)\, x$ $\quad [\because \; (\text{adj. } A)\, A = |A|\, I]$

$\Rightarrow$ $|A|\, x = \lambda\, (\text{adj. } A)\, x$

$\Rightarrow$ $\frac{|A|}{\lambda}\, x = (\text{adj. } A)\, x$

$\Rightarrow$ $(\text{adj. } A)\, x = \frac{|A|}{\lambda} x$

$\Rightarrow$ $\frac{|A|}{\lambda}$ is an eigen value of adj. A.

Example 15: Show that if $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the characteristic roots of the matrix A, then A^3 has the characteristic roots $\lambda_1^3, \lambda_2^3, \ldots\ldots, \lambda_n^3$.

Solution: Let λ be a characteristic root of the matrix A. Then there exists a non-zero vector x such that

$Ax = \lambda x$ ...(1)

$\Rightarrow$ $A^2(Ax) = A^2(\lambda x)$

$\Rightarrow$ $A^3x = \lambda(A^2x)$

But $A^2x = A(Ax) = A(\lambda x)$ [using (1)]

$= \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$

$\therefore$ $A^3x = \lambda(\lambda^2 x) = \lambda^3 X$

$\Rightarrow$ λ^3 is a characteristic root of A^3.

Hence, if λ_1, λ_2, ……, λ_n are the characteristic roots of A, then $\lambda_1^3, \lambda_2^3, \ldots\ldots, \lambda_n^3$ are the characteristic roots of A^3.

6. Algebraic Multiplicity of an Eigengalue:

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Example 16: Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, find the eigenvalues of A and their multiplicities.

Solution: We have

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

Thus, characteristic equation is given by

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (5-\lambda)\,(3-\lambda)\,(5-\lambda)\,(1-\lambda) = 0$$

$$\Rightarrow \quad (5-\lambda)^2\,(3-\lambda)\,(1-\lambda) = 0$$

Thus 5, 3 and 1 are the eigenvalues of A. The eigenvalue 5 have multiplicity of order 2 because $(5 - \lambda)$ occurs two times as a factor of the characteristic polynomial. And the multiplicity of eigenvalues 3 and 1 is one.

7. Similarity of Matrices:

If A and B are two n $\times$ n matrix. Then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$ or Equivalently

$$A = PBP^{-1}$$

Changing A into $P^{-1}AP$ is called similarity transformation.

Theorem 8: If n $\times$ n matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Proof: If $B = P^{-1}AP$, then

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

$$\therefore \quad |B - \lambda I| = |P^{-1}(A - \lambda I)P| = |P^{-1}|.|A - \lambda I|.|P|$$

$$= |A - \lambda I|.|P^{-1}|.|P| = |A - \lambda I|.|P^{-1}P|$$

$$= |A - \lambda I|.|I| = |A - \lambda I| \qquad [\text{Since } |I| = 1]$$

Hence matrices A and $P^{-1}AP$ have the same characteristic roots.

Value Addition: Cautions
1. Similarity is not the same as row equivalence. (If A is row equivalent to B, then B = EA for some invertible matrix E.) 2. Row operation on a matrix usually change its eigenvalues.

Exercises:

1. Is $\lambda = 2$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Why or why not ?

2. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$? If so, find the corresponding eigenvalue.

3. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the corresponding eigenvalue.

4. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find the corresponding eigenvector.

5. Is $\lambda = 3$ an eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find the corresponding eigenvector.

6. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

7. Find all the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ -4 & 8 & 1 \\ -1 & -2 & 0 \end{bmatrix}$.

8. *Find the eigen values and eigen vectors of the following matrix*

(i) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

For the following questions find a basis for the eigenspace corresponding to each listed eigenvalue.

9. $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}, \lambda = 4$

10. $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}, \lambda = 1, 5$

11. $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \lambda = 1, 2, 3$

12. $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}, \lambda = 3$

13. Verify the Cayley–Hamilton theorem for the following theorems and also find the inverse of using Cayley-Hamilton theorem

(i) $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

14. Find the inverse of the matrix A using Cayley-Hamilton theorem, given the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

15. Show that the matrix, $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence obtain A^{-2}.

16. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then find the eigen values of the matrix $(A - \lambda I)^2$.

17. Show that for a square matrix, there are infinitely many eigen vectors corresponding to a single eigen value.

Summary:

In this lesson, we have emphasized on the following

- eigen values and eigenvectors of a matrix.
- characteristic equation of a matrix
- Cayley-Hamilton theorem
- algebraic multiplicity of an eigenvalue
- similarity of matrices

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