Discipline Courses-I

Semester-I

Paper: Calculus-I

Lesson: Hyperbolic Functions and Successive Differentiation

Lesson Developer: J.K. Kohli

College/Department: Department of Mathematics, Hindu College, Delhi University

Hyperbolic Functions and Successive Differentiation

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Hyperbolic Functions and Successive Differentiation

1. Learning Outcomes:

The chapter deals with (1) Hyperbolic Functions. (2) Successive Differentiation and Higher order Derivatives.

Hyperbolic functions are defined in terms of e^x and e^{-x} and surprisingly have properties analogous to trigonometric functions.

After learning this chapter you will be at home with hyperbolic functions and inverse hyperbolic functions. You will be able to work with them, find their graphs, manipulate them, find their derivatives and integrals. You will also be able to use them in your scientific work and applications. Also using a graphic utility you will be able to generate their graphs by expressing these functions in terms of e^x and e^{-x}.

After learning successive differentiation and Leibnitz theorem you will be at ease to find higher order derivate and use them in your work.

2. Introduction:

Hyperbolic functions arise in many scientific and engineering applications, e.g., in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also arise when a cable is suspended between two poles as with a telephone line hanging between poles.

3. Hyperbolic Functions:

Hyperbolic functions are defined as certain combinations of e^x and e^{-x} and surprisingly have many properties similar to trigonometric functions.

3.1. Definitions of Hyperbolic functions:

There are defined some hyperbolic functions as follows:-

(i) Hyperbolic cosine $\cosh x = \dfrac{e^x + e^{-x}}{2}$

(ii) Hyperbolic sine $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(iii) Hyperbolic tangent $\tanh x = \dfrac{e^x - e^{-x}}{\left(e^x + e^{-x}\right)}$

(iv) Hyperbolic cotangent $\coth\, x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(v) Hyperbolic secant $\text{sech}\, x = \dfrac{2}{e^x + e^{-x}}$

(vi) Hyperbolic cosecant $(\text{csch}\, x) \equiv \text{cosech}\, x = \dfrac{2}{e^x - e^{-x}}$

3.1.1. Domain and Range of Hyperbolic functions:

The domain and range of the above defined hyperbolic functions is as follows:-

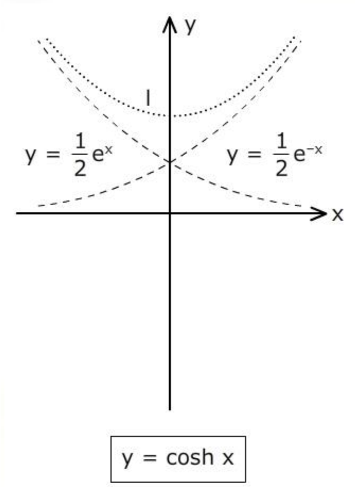
Hyperbolic functions	Domain	Range
(i) cosh x	$\mathbb{R} \equiv (-\infty, \infty)$	$[1, \infty)$
(ii) sinh x	$\mathbb{R}$	$\mathbb{R}$
(iii) tanh x	$\mathbb{R}$	$(-1, 1)$
(iv) coth x	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
(v) sech x	$\mathbb{R}$	$(0, 1]$
(vi) cosech x	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Value Addition: Do you know?

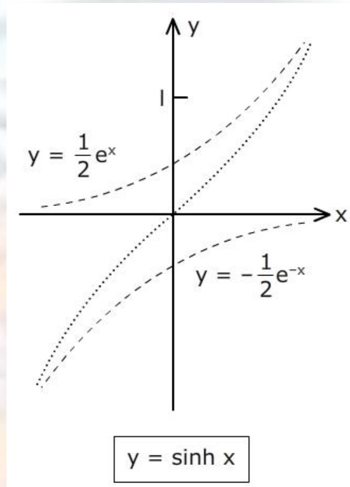
If a homogeneous, flexible cable is suspended between two poles as with a telephone line hanging between two poles, such a cable forms a curve, called catenary. In case a co-ordinate system is introduced such that the lowest point of the cable lies on the y–axis, then the cable (catenary) has an equation of the form $y = \cosh\left(x/a\right) + C$.

3.2. Graph of Hyperbolic Functions:

The graph of the above defined hyperbolic functions is as follows:-



(a)



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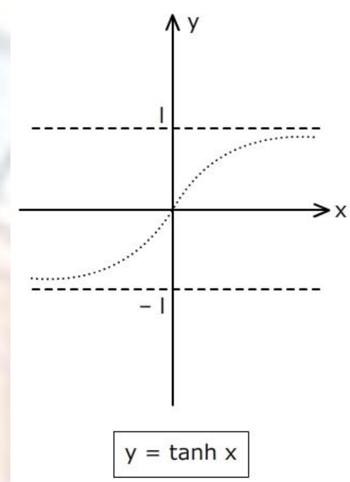

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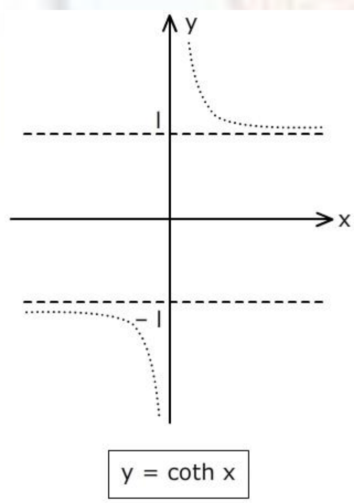

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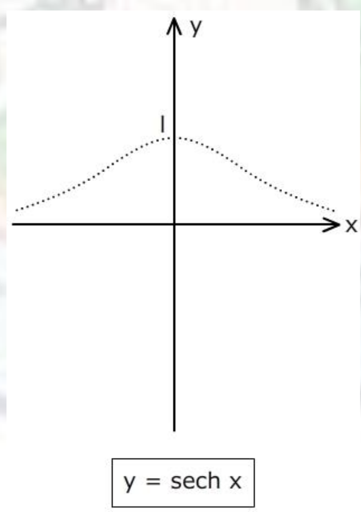

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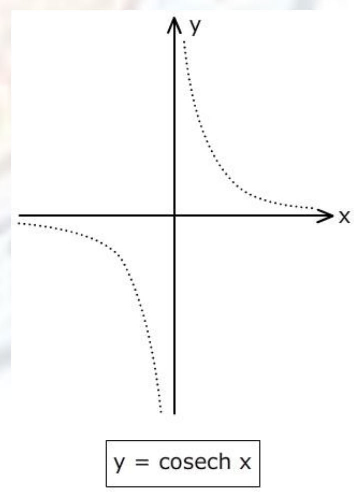


(f)

3.3. Hyperbolic Identities:

Properties of Hyperbolic Functions

(i) cosh x + sinh x = e^x

(ii) cosh x – sinh x = e^{-x}

(iii) $\cosh^2 x - \sinh^2 x = 1$

(iv) $\text{sech}^2 x + \tanh^2 x = 1$

(v) $\coth^2 x - \text{cosech}^2 x = 1$

(vi) cosh (–x) = cosh x

(vii) sinh (–x) = – sinh x

(Viii) cosh (x ± y) = cosh x cosh y ± sinh x sinh y

(ix) sinh (x ± y) = sinh x cosh y ± cosh x sinh y

(x) sinh 2x = 2 sinh x cosh x

(xi) cosh 2x = $\cosh^2 x + \sinh^2 x$

$= 2\sinh^2 x + 1$

$= 2\cosh^2 x + 1$

Proofs of (i) and (ii) are obvious from the definition of the hyperbolic functions.

Proof of (iv): $\sec h^2 x + \tanh^x x = \dfrac{4}{\left(e^x + e^{-x}\right)^2} + \dfrac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$

$$= \frac{4 + e^{2x} + e^{-2x} - 2}{\left(e^x + e^{-x}\right)^2} = \frac{\left(e^x + e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$$

$$= 1$$

Proofs of (iii) and (v) are similar to that of the proof of (iv) and are left as exercises for the reader.

Proof of (vi) and (vii) are easy

Proof of (viii): RHS = cosh x cosh y + sinh x sinh y

$$= \frac{\left(e^x + e^{-x}\right)}{2}\,\frac{\left(e^y + e^{-y}\right)}{2} + \frac{\left(e^x - e^{-x}\right)}{2}\,\frac{\left(e^y - e^{-y}\right)}{2}$$

$$= \frac{1}{4}\left[\left\{e^x e^y + e^x e^{-y} + e^{-x}e^y + e^{-x}e^{-y}\right\} + \left\{e^x e^y - e^x e^{-y} - e^{-x}e^y + e^{-x}e^{-y}\right\}\right]$$

$$= \frac{1}{4}\left[2e^x e^y + 2e^{-x}e^{-y}\right] = \frac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right) = \cosh(x+y)$$

$$= LHS$$

The proofs of remaining identities are similar and are left as exercises for the reader.

3.4. Derivative and integral formulas for hyperbolic functions:

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = sinh\,x$$

$$\frac{d}{dx}(sin\text{h}\,x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right)$$

$$= \frac{\text{sin}hx\,\frac{d}{dx}\cosh x - \cosh x\,\frac{d}{dx}\sinh x}{\sinh^2 x}$$

$$= \frac{\text{sin}h^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-\left(cosh^2 x - \sinh^2 x\right)}{\sinh^2 x} = \frac{-1}{\sinh^2 x}$$

$$= -\text{cosech}^2\,x$$

Similarly the derivatives of remaining hyperbolic function can be worked out. In the following we provide a complete list of derivative and integration formulas for the generalized functions

(i) $\frac{d}{dx}(\cosh w) = sinh\, w \frac{dw}{dx}$ (i) $\int sinh\, w\, dw = \cosh w + C$

(ii) $\frac{d}{dx}(\sinh w) = \cosh w \frac{dw}{dx}$ (ii) $\int \cosh w\, dw = \sinh w + C$

(iii) $\frac{d}{dx}(\coth w) = -\operatorname{cosech}^2 w. \frac{dw}{dx}$ (iii) $\int \operatorname{cosech}^2 w\, dw = -\coth w + C$

(iv) $\frac{d}{dx}(\tanh w) = \operatorname{sech}^2 w \frac{dw}{dx}$ (iv) $\int \operatorname{sech}^2 w\, dw = \tanh w + C$

(v) $\frac{d}{dx}(\operatorname{cosech} w) = -\operatorname{cosech} w \coth w \frac{dw}{dx}$ (v) $\int \operatorname{cosech} w \coth w\, dw = -\operatorname{cosech} w + C$

(vi) $\frac{d}{dx}(\operatorname{sech} w) = \operatorname{s}ech\, w \tanh w \frac{dw}{dx}$ (vi) $\int \operatorname{sech} w \tanh w\, d\, w = \operatorname{sech} w + C$

Example 1:

(i) $\frac{d}{dx}(\sinh x^4) = \cosh x^4 \frac{d}{dx}(x^4) = 4x^3 \cosh x^4$

(ii) $\frac{d}{dx}(\ln(\coth h\, x)) = \frac{1}{\cot h\, x} \frac{d}{dx}(\cot \mathrm{h}\, x)$

(iii) $= \frac{1}{\cot h\, x}(-\cos ec\, \mathrm{h}^2 x) = -\cos ech\, x \operatorname{sech} x$

Example 2:

(i) $\int \cos h^4 x \sin h\ x\, dx = \frac{1}{5}\cosh^5 x + C$

(ii) $\int \coth x\ dx = \int \frac{\cosh x}{\sinh\ x} dx$

$= \ln|\sin \mathrm{h}\, \mathrm{x}| + C$

Example 3: Find the length of the catenary

$$y = 20\cosh\left(\frac{x}{20}\right) \text{ from } x = -10 \text{ to } x = 10$$

Solution: The length L of the catenary is given by

$$L = \int_{-10}^{10} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{-10}^{10} \sqrt{1+\sinh^2\left(\frac{x}{20}\right)}\, dx = \int_{-10}^{10} \cos h\ \frac{x}{20} dx$$

$$= 2\int_{0}^{10} \cosh \frac{x}{20}.dx = \left[2\left(\sinh\frac{x}{20}\right) 20\right]_0^{10} \qquad (\because\ \cosh t \ \textit{is an even function of } t)$$

$$= 40\left[\left(\sinh\frac{1}{2}\right)\right] = 40\left(\frac{e^{1/2} - e^{-1/2}}{2}\right) \qquad (\because\ \sinh 0 = 0)$$

Value Addition: Why the functions in this chapter are called Hyperbolic functions ?
The parametric equations $x = \cosh t,\ \ y = \sinh t \quad (-\infty < t < \infty)$ represent a portion of the hyperbola $x^2 - y^2 = 1$. Since $\cosh t \geq 1$ it represents the curve shown in the following Diagram I and is the right half of the larger curve called unit hyperbola. This is the reason that the functions dealt with in this chapter are called hyperbolic functions.

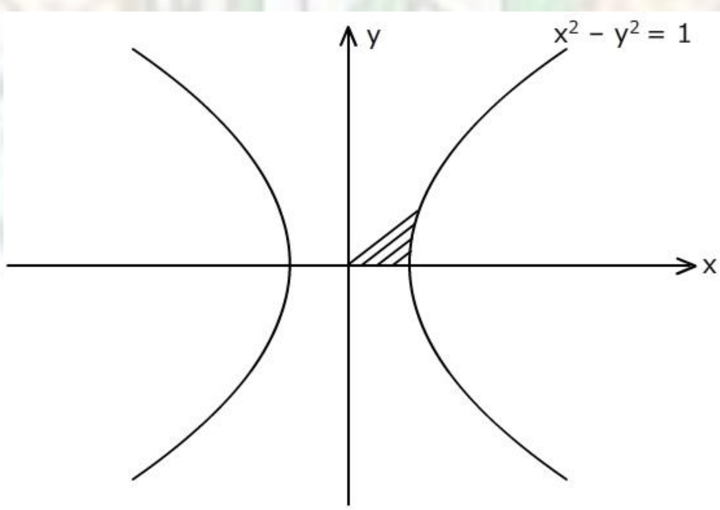


Diagram I

Value Addition: The following table summarizes the basic properties of the inverse hyperbolic functions

Function	Domain	Range	Basic Relationships
$\sinh^{-1} x$	[illegible]	[illegible]	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < \infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < \infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, \infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$	$\tan^{-1}(\tan x) = x$ if $-\infty < x < +\infty$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$
$\text{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\text{sech}^{-1} x(\text{sech}\, x) = x$ if $x \geq 0$ $\text{sech}(\text{sech}^{-1} x) = x$ if $0 < x < 1$
$\text{cosech}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, \infty)$	$\text{cosech}^{-1}(\text{cosech}\, x) = x$ if $x<0$ or $x>0$ $\text{cosech}(\text{cosech}^{-1} x) = x$ if $x<0$ or $x>0$

Value Addition: Do you know?

Horizontal line test : A function f has an inverse function if and only if its graph is cut at most once by any horizontal line.

3.5. Logrithonic forms of Inverse Hyperbolic Functions:

Since hyperbolic functions are expressible in terms of e^x, the inverse hyperbolic functions are expressible in terms of natural logrithmic. The following relationships hold for all x in the domain of the stated inverse hyperbolic functions:

(i) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

(ii) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

(iii) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

(iv) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$

(v) $\text{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$

(vi) $\text{cosech}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{1+\sqrt{1-x^2}}{|x|}\right)$

We shall prove parts (ii) and (iii) only.

Proof of (ii): Let $\sinh^{-1} x = y$. Then $\sinh y = x$

or $\dfrac{e^y - e^{-y}}{2} = x$

or $e^y - 2x - e^{-y} = 0$

or $e^{2y} - 2xe^{-y} - 1 = 0$

or $e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$

Since $e^y > 1$, the solution involving minus sign must be discarded. Thus

$e^y = x + \sqrt{x^2+1}$.

Proof of : (iii) Let $\tanh^{-1} x = y$, *then*

$\tanh y = x$

or $\dfrac{e^y - e^{-y}}{e^y + e^{-y}} = x$

or $e^y - e^{-y} - x\left(e^y + e^{-y}\right) = 0$

or $(1-x)e^y - (1+x)e^{-y} = 0$

or $(1-x)e^{2y} = 1+x$

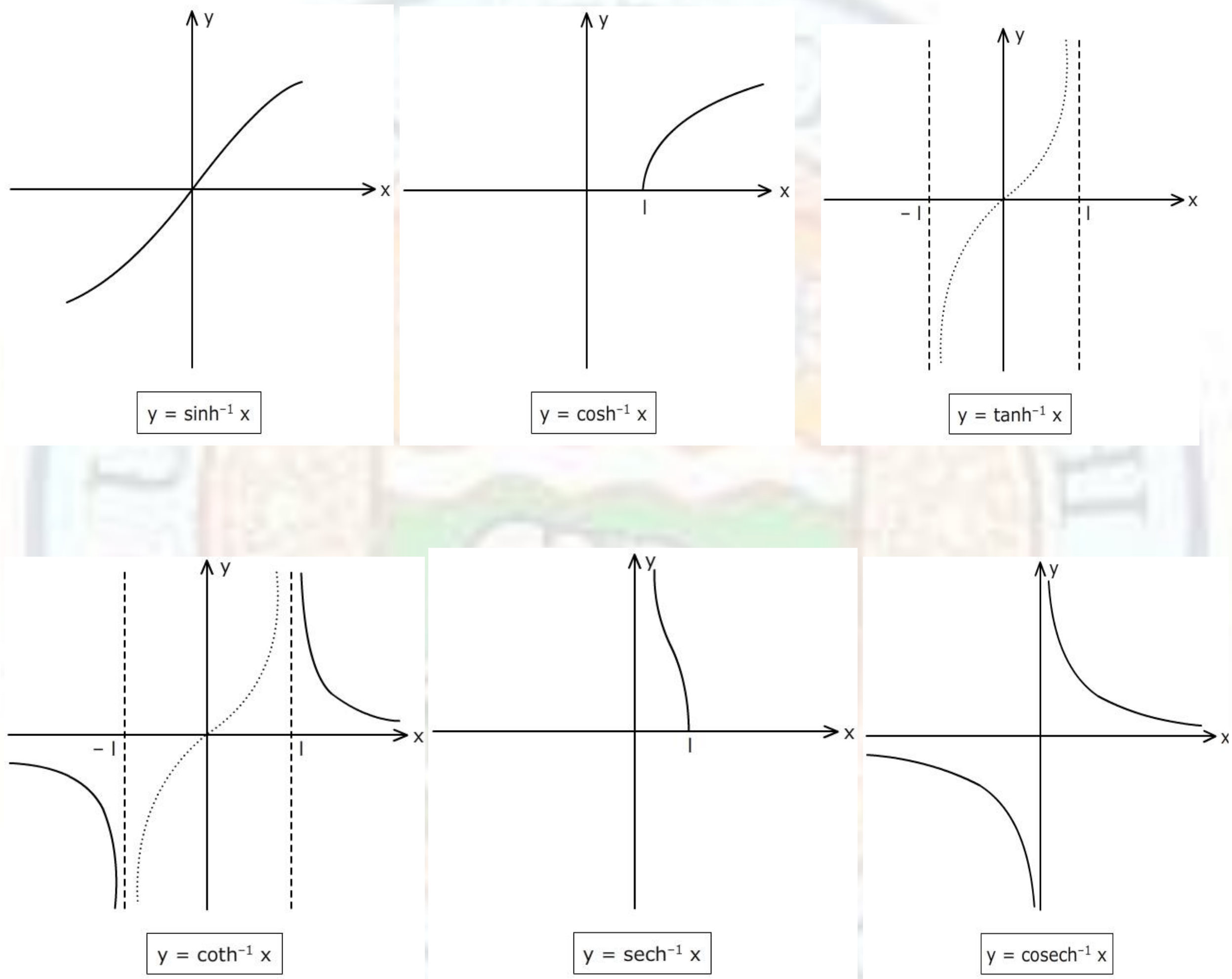
or $e^{2y} = \dfrac{1+x}{1-x}$

or $2y = \ln\left(\dfrac{1+x}{1-x}\right)$

or $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. *Thus* $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$.

Proof of remaining identities can be worked out similarly and are left as exercises for the reader.

3.6. Graphs of Inverse Hyperbolic functions:



Example 4:

$$\sinh^{-1}1 = \ln\left(1+\sqrt{1^2+1}\right) = \ln\left(1+\sqrt{2}\right) \approx 0.8814$$

$$\tanh^{-1}\tfrac{1}{2} = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln 3 \approx 0.5493$$

3.7. Derivatives and Integrals of Inverse Hyperbolic Functions:

$$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right] = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right)} = \frac{1}{\sqrt{x^2+1}}$$

Value Addition: Alternative Method for derivative of $\sinh^{-1} x$	
Let $y = \sinh^{-1} x$	
$\Rightarrow \sinh y = x$	$\cosh^2 x - \sinh^2 y = 1$
Differentiating	
$\Rightarrow \cosh y \frac{dy}{dx} = 1$	$\cosh^2 y = 1+\sinh^2 y$
$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$	$\cosh y = \sqrt{1+\sinh^2 y}$

In the following we list the generalized derivate formulas for the inverse hyperbolic functions

(i) $\frac{d}{dx}\left(\sinh^{-1} w\right) = \frac{1}{\sqrt{1+w^2}} \frac{dw}{dx}$ (ii) $\frac{d}{dx}\left(\coth^{-1} w\right) = \frac{1}{1-w^2} \frac{dw}{dx}$, $|w|>1$

(iii) $\frac{d}{dx}\left(\cosh^{-1} w\right) = \frac{1}{\sqrt{w^2-1}} \frac{dw}{dx}$ (iv) $\frac{d}{dx}\left(\text{sech}^{-1} w\right) = -\frac{1}{w\sqrt{1-w^2}} \frac{dw}{dx}$, $o < w < 1$

(v) $\frac{d}{dx}\left(\tanh^{-1} w\right) = \frac{1}{1-w^2} \frac{dw}{dx}$, $|w|<1$ (vi) $\frac{d}{dx}\left(\text{coseh}^{-1} w\right) = \frac{1}{|w|\sqrt{1+w^2}} \frac{dw}{dx}$, $w \neq 0$

In the following theorem we list the integration formulas for the inverse hyperbolic functions.

<u>Theorem:</u> If a > 0, then

(i) $\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right)+C \quad or \quad \ln\left(u+\sqrt{u^2+a^2}\right)+C$.

(ii) $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad or \quad \ln\left(u + \sqrt{u^2 - a^2}\right) + C$.

(iii) $\int \frac{du}{\sqrt{a^2 - u^2}} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C,\ |u| < a \quad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C,\ |u| \neq a \\ or \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, \end{cases}$.

(iv) $\int \frac{du}{u\sqrt{a^2 - u^2}} = \frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \quad or \quad -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 - u^2}}{|u|}\right) + C \ \ 0 < |u| < a$.

(v) $\int \frac{du}{u\sqrt{a^2 + u^2}} = \frac{1}{a}\operatorname{cosech}^{-1}\left|\frac{u}{a}\right| + C \quad or \quad -\frac{1}{a}\ln\left(\frac{a + \sqrt{a^2 + u^2}}{|u|}\right) + C,\ u \neq 0$.

3.8. Exercise:

1. Prove the following identities

(i) $\sinh 2\theta = \frac{2\tanh\theta}{1 - \tanh^2\theta}$

(ii) $\cosh 2\theta = \frac{1 + \tanh^2\theta}{1 - \tanh^2\theta}$

(iii) $\tanh 2\theta = \frac{2\tanh^2\theta}{1 + \tanh^2\theta}$

(iv) $\sinh 3\theta = 4\sinh^3\theta + 3\sinh\theta$

(v) $\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta$

2. Prove the following

(i) $\frac{1 + \tanh x}{1 - \tanh x} = \cosh 2x + \sinh 2x$

(ii) $\left(\frac{1 + \tanh x}{1 - \tanh x}\right)^4 = \cosh 8x + \sinh 8x$

(iii) $(\cosh\theta \pm \sinh\theta)^6 = \cosh 6x \pm \sinh 6x$

(iv) $\sinh^{-1} x = \cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\dfrac{x}{\sqrt{1+x^2}}$

3. In each part rewrite the expression as a ratio of polynomials

(i) sinh (lnx) (ii) cosh (lnx)

(iii) tanh (2lnx) (iv) cosh (–lnx)

4. In each part value for one hyperbolic function is given at x_0. Find the values of the remaining five hyperbolic functions at x_0.

(i) $\sinh x_0 = 2$ (ii) $\cosh x_0 = 5/4$ (iii) $\tanh x_0 = \dfrac{4}{5}$

5. Find $\dfrac{dy}{dx}$ for the following functions

(i) $y = \coth(\ln x)$ (ii) $y = \sqrt{4x + \cosh^2(5x)}$

(iii) $y = x^3 \tanh^2\left(\sqrt{x}\right)$ (iv) $y = \dfrac{1}{\tanh^{-1} x}$

(v) $y = e^x \operatorname{sech}^{-1}\sqrt{x}$

6. Evaluate the following integrals

(i) $\int \sqrt{\tanh x}\ \operatorname{sech}^2 x\, dx$ (ii) $\int \tanh x\, dx$

(iii) $\int \tanh x \operatorname{sech}^3 x\, dx$ (iv) $\int \dfrac{dx}{\sqrt{1+9x^2}}$

(v) $\int \dfrac{dx}{x\sqrt{1+4x^2}}$ (vi) $\int_0^{1/2} \dfrac{dx}{\sqrt{1-x^2}}$

7. Find the arc length of catenary y = cosh x between x = 0 and x = ln2

8. Prove that

(i) $\text{sech}^{-1} x = \cosh^{-1} \left(\frac{1}{x}\right)$, $0 < x \leq 1$

(ii) $\coth^{-1} x = \tanh^{-1} \left(\frac{1}{x}\right)$, $|x| > 1$

(iii) $\text{cosech}^{-1} x = \sinh^{-1} \left(\frac{1}{x}\right)$, $x \neq 0$

9. Approximate the following expressions to 4 decimal places.

(a) sinh3 (b) tanh (ln4) (c) $\cosh^{-1}3$

(d) coth1 (e) $\text{sech}^{-1} \left(\frac{1}{2}\right)$ (f) $\text{cosech}^{-1}(-1)$

4. Successive Differentiation: Higher order Derivatives:

The derivative f' of a differentiable function is itself a function. If it is differentiable we denote its derivative by f''. Similarly if it is also differentiable we denote the third derivative by f''' and so on.

If we write y = f(x), the symbols

$$y_1, y_2, y_3, \text{————————} , y_n \text{————————}$$

$$\text{or} \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \text{————} , \frac{d^ny}{dx^n} \text{——————}$$

are used to denote the successive derivative of y.

4.1. Calculation of the n^{th} Derivative, some standard results:

1. $y = (ax + b)^m$

$y_1 = ma\,(ax + b)^{m-1}$

$y_2 = m(m - 1)\,a^2\,(ax + b)^{m-2}$

$y_n = m(m - 1)$ ------ $(m - n + 1)a^n\,(ax + b)^{m-n}$.

In case n is a positive integer, then

$$y_n = \frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}$$

for m = –1, we get

$$y_n = (-1)(-2)(-3)-----------------(-n)a^n(ax+b)^{-1-n}$$

$$y_n = (-1)^n \frac{n!a^n}{(ax+b)^{n+1}}$$

2. $y = \ln(ax+b)$

$$y_1 = \frac{a}{(ax+b)}$$

$$y_n = \frac{a^n(-1)^{n-1}(\mathrm{n}-1)!}{(ax+b)^n}$$

3. $y = a^{mx}$

$$y_1 = ma^{mx}.\ln \mathrm{a}$$

$$y_n = m^n a^{mx}(\ln \mathrm{a})^n$$

4. $y = e^{mx}$

$$y_1 = me^{mx}$$

$$y_n = m^{ns}e^{mx}$$

5. $y = \sin(ax+b)$

$$y_1 = \cos(ax+b)$$

$$= a\sin(\mathrm{ax}+\mathrm{b}+\frac{\pi}{2})$$

$$y_2 = a^2\cos(\mathrm{ax}+\mathrm{b}+\frac{\pi}{2})$$

$$= a^2\sin(\mathrm{ax}+\mathrm{b}+2\frac{\pi}{2})$$

$$y_n = a^n \sin(\text{ax} + \text{b} + \frac{n\pi}{2})$$

Similarly

6. $y_1 = \cos(ax+b)$

$$y_1 = a\sin(ax+b)$$

$$= a\cos(\text{ax} + \text{b} + \frac{\pi}{2})$$

$$y_n = a^n \cos(\text{ax} + \text{b} + \frac{n\pi}{2})$$

7. $y = e^{ax}\sin(bx+c)$

$$y_1 = ae^{ax}\sin(bx+c) + be^{ax}\cos(bx+c)$$

Put $a = r\cos\phi,\ b = r\sin\phi$. Then $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(b/a\right)$

$$y_1 = r\,e^{ax}\sin(bx+c+\phi)$$

$$y_2 = r^2\,e^{ax}\sin(bx+c+2\phi)$$

$$y_n = r^n\,e^{ax}\sin(bx+c+n\phi) = r^n\,e^{ax}\sin\left(bx+c+\tan^{-1}\left(b/a\right)\right)$$

Similarly

8. $\frac{d^n}{dx^n}\left(e^{ax}\cos(bx+c)\right) = r^n\,e^{ax}\cos(bx+c+\phi)$, where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(b/a\right)$

4.2. Determination of nth Derivative of Rational Functions:

To determine the nth derivative of a rational function we have to decompose it into partial fractions.

Example 5: Find the nth differential coefficient of $\frac{x^3}{(x-1)(x-2)}$.

Solution: We have

$$\frac{x^3}{(x-1)(x-2)} = x + 3 + \frac{7x-6}{(x-1)(x-2)}$$

$$= x + 3 - \frac{1}{x-1} + \frac{8}{x-2}$$

Therefore for n > 1,

$$\frac{d^n}{dx^n}\left(\frac{x^3}{(x-1)(x-2)}\right) = n!\left[\frac{(-1)^{n+1}}{(x-1)^{n+1}} + \frac{8(-1)^n}{(x-2)^{n+1}}\right]$$

$$= (-1)^{n+1}\left[\frac{1}{(x-1)^{n+1}} - \frac{8}{(x-2)^{n+1}}\right]$$

Example 6: Find nth differential coefficient of $y = \tan^{-1}\left(x/a\right)$.

Solution: On differentiating, we have

$$y_1 = \frac{a}{x^2+a^2}$$

$$= \frac{a}{(x-ia)(x+ia)} = \frac{1}{2i}\left(\frac{1}{(x-ia)} - \frac{1}{(x+ia)}\right]$$

$$\therefore \quad y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\left\{\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right]$$

Putting x = r cosθ, a = r sinθ, and using Demoivre's theorem, we obtain

$$y_n = \frac{1}{2i}(-1)^{n+1}(n-1)!\, r^{-n}\, 2i \sin n\theta$$

$$= (-1)^{n-1}(n-1)!\; r^{-n} \sin n\theta$$

Since $a = r\sin\theta, \quad r^{-n} = a^{-n}\sin^n\theta,$

$\therefore \quad y_n = (-1)^{n-1}(n-1)!\ a^{-n}\sin^n\theta\sin n\theta\,, \qquad where\ \theta = \tan^{-1}\left(a/x\right).$

Example 7: Find y_n for the function $y = \cos^4 x$.

Solution: Given that

$$y = \left(\cos^2 x\right)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$$

$$= \frac{1}{4} + \frac{\cos^2 2x}{4} + \frac{\cos 2x}{2}$$

$$= \frac{1}{4} + \frac{1}{8}\left(1+\cos 4x\right) + \frac{\cos 2x}{2}$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}$$

$$\therefore \quad y_n = \frac{4^n}{8}\cos\left(4x+\frac{n\pi}{2}\right) + \frac{2^n}{2}\cos\left(2x+\frac{n\pi}{2}\right)$$

$$= 2^{n-1}\left(\cos 2x + \frac{n\pi}{2}\right) + \frac{1}{2}\,4^{n-1}\cos\left(4x+\frac{n\pi}{2}\right)$$

Example 8: If $y = e^{ax}\cos^2 x\sin x$, find y_n.

Solution: We have

$$\cos^2 x\sin x = \frac{1}{2}\left(1+\cos 2x\right)\sin x$$

$$= \frac{1}{2}\sin x + 2\sin x\cos 2x$$

$$= \frac{1}{2}\sin x + \frac{1}{4}\left(\sin 3x - \sin x\right)$$

$$= \frac{1}{4}\sin x + \frac{1}{4}\sin 3x$$

So $\quad y = \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$$\therefore \qquad y_n = \frac{1}{4}\left(a^2+1\right)^{n/2} e^{ax}\sin\left(x+n\tan^{-1} 1/a\right)+\frac{1}{4}\left(a^2+9\right)^{n/2} e^{ax}\sin\left(x+n\tan^{-1} 3/a\right)$$

4.3. Leibnitz Theorem:

Let u and v be functions of x whose n-th order derivatives exist. Then the n-th derivative of the product function $u\,v$ is given by

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + ---- + {}^nC_r u_{n-r} v_r + ----+ {}^nC_{n_1} u v_n \quad ---------- (1)$$

Proof: The theorem is proved by mathematical induction. Clearly the result is true for n = 1. Assume that result is true for a particular value of n. Then differentiating (1) with respect to x, we get

$$(uv)_{n+1} = \left(u_{n+1}v + u_n v_1\right) + {}^nC_1\left(u_{n+1}v + u_{n-2}v_2\right) + {}^nC_2\left(u_{n-1}v_2 + u_{n-2}v_3\right) + -------- +$$

$${}^nC_r\left(u_{n-r+1}v_r + u_{n-r}v_{r+1}\right) + ----- \; {}^nC_n\left(u_1 v_n + uv_{n+1}\right)$$

$$= u_{n+1}v\left({}^nC_1 + 1\right)u_1 v_n + \left({}^nC_1 + {}^nC_2\right)u_{n-1}v_2 + ------$$

$$+\left({}^nC_r + {}^nC_{r+1}\right)u_{n-r}v_{r+1} + ----- + {}^nC_n\left(u_1 v_n + uv_{n+1}\right)$$

We know that

$${}^nC_1 + 1 = n + 1 = {}^{n+1}C_1$$

$${}^nC_1 + {}^nC_2 = {}^{n+1}C_1$$

$${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$$

Hence

$$(uv)_{n+1} = u_{n+1}v + {}^{n+1}C_1 u_n v_1 + {}^{n+1}C_2 u_{n-1} v_2 + ------ + {}^{n+1}C_r u_{n-r} v_{r+1} + ---- + {}^{n+1}C_{n+1} uv_{n+1}$$

Thus we conclude that theorem is true for all positive integers n

Example 9: If $y = x^2 e^x \cos x$, find y_n.

Solution: Using the Leibnitz theorem, we have

$$\left(x^2 e^x \cos x\right) = \left(e^x \cos x\right)_n \; x^2 \; + \; {}^nC_1\left(e^x \cos x\right)_{n-1}(2x) + \; {}^nC_2\left(e^x \cos x\right)_{n-2} \; 2$$

$$y_n = 2^{n/2} e^x \cos\left(x + n\tan^{-1}1\right) + 2nx.2^{\frac{n-1}{2}}.e^x \cos\left\{x + (n-1)\tan^{-1}1\right\}$$

$$+ \, n(n-1)e^x \cos\left\{x + (n-2)\tan^{-1}1\right\}2^{\frac{n-2}{2}}$$

$$= \; 2^{\frac{n-2}{2}} e^x \left[cox\left(x + \frac{n\pi}{4}\right) + 2^{3/2} nx\cos\left\{x + (n-1)\frac{\pi}{4}\right\} + n(n-1)\cos\left\{x + (n-2)\frac{\pi}{4}\right\}\right]$$

$$= \; 2^{\frac{n-2}{2}} e^x \left[cox\left(x + \frac{n\pi}{4}\right) + 2^{3/2} nx\cos\left\{x + (n-1)\frac{\pi}{4}\right\} + n(n-1)\cos\left\{x + (n-2)\frac{\pi}{4}\right\}\right]$$

Example 10: If $y^{1/m} + y^{-1/m} = 2x$, then prove that

$$\left(x^2 - 1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2 - m^2\right)y_n = 0$$

Solution: Given

$$y^{1/m} + y^{-1/m} = 2x$$

$$\Rightarrow \qquad y^{2/m} - 2xy^{1/m} + 1 = 0$$

$$\Rightarrow \qquad y^{1/m} = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

$$\Rightarrow \qquad y = \left(x \pm \sqrt{x^2 - 1}\right)^m$$

$$\Rightarrow \qquad y_1 = \; m\left(x \pm \sqrt{x^2 - 1}\right)^{m-1}\left(1 \pm \frac{x}{\sqrt{x^2 - 1}}\right)$$

Squaring both sides and simplifying, we get

$$\left(x^2 - 1\right)y_1^2 = m^2 y^2$$

Differentiating w.r. to x, we obtain

$$2\left(x^2 - 1\right)y_1 y_2 + \; 2xy_1^2 = 2m^2 y\, y_1$$

$\Rightarrow \quad \left(x^2-1\right)y_2 + xy_1 = m^2 y$

Applying Leibnitz Theorem, we get

$$\left(x^2-1\right)y_{n+2} + {}^nC_1 y_{n+1}\left(2x\right) + {}^nC_2 y_n\left(2\right) + xy_{n+1} + {}^nC_1 y_n = m^2 y_n$$

$\Rightarrow \quad \left(x^2-1\right)y_{n+2} + 2nxy_{n+1} + n\left(n-1\right)y_n + xy_{n+1} + ny_n = m^2 y_n$

$\Rightarrow \quad \left(x^2-1\right)y_{n+2} + \left(2n+1\right)xy_{n+1} + \left(n^2-n+n\right)y_n = m^2 y_n$

$\Rightarrow \quad \left(x^2-1\right)y_{n+2} + \left(2n+1\right)xy_{n+1} + \left(n^2-m^2\right)y_n = 0$

Example 11: If $y = \ln\left(x+\sqrt{1+x^2}\right)$, then find $y_n\left(0\right)$

Solution: Given

$$y = \ln\left(x+\sqrt{1+x^2}\right) \quad ---------------------- \quad (1)$$

$$y_1 = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$y_1 = \frac{1}{\sqrt{1+x^2}} \quad ---------------------------- \quad (2)$$

So $\quad y\left(0\right) = 0$

$y_1\left(0\right) = 1$

$\Rightarrow \quad (1+x^2)\,y_1^2 = 1$

Differentiating w.r.to x, wet obtain

$$2y_1 y_2\,(1+x^2) + 2xy_1^2 = 0$$

$\Rightarrow \quad (1+x^2)\,y_2 + xy_1 = 0 \quad ------------------------------- \quad (3)$

Differentiating n times and using Leibnitz theorem

$$(1+x^2)y_{n+2} + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n 2 + xy_{n+1} + {}^nC_1 y_n = 0$$

$$(1+x^2)y_{n+2} + 2nx y_{n+1} + n(n-1)y_n + x y_{n+1} + ny_n = 0$$

$$(1+x^2)y_{n+2} + (2n+1)x y_{n+1} + n^2 y_n = 0$$

Putting x = 0, we get

$$y_{n+2}(0) + n^2 y_n(0) = 0 \text{ ---------------------------------- } (4)$$

$$y_{n+2}(0) = -n^2 y_n(0)$$

Putting n = 1, 2, 3, 4, ---------- in (4), we get

$$y_3(0) = -y_1(0) = -1^2 \qquad\qquad y(0)=0$$

$$y_4(0) = -2^2 y_2(0) = 0 \qquad\qquad y_1(0) = 1$$

$$y_5(0) = -3^2 y_3(0) = (-1)^2(-3^2) \qquad\qquad y_2(0) = 0$$

In general

$$y_n(0) = \begin{cases} o,\ if\ n\ is\ even \\ (-1)^{\frac{(n-1)}{2}}\ 1^2.\,3^2.\,5^2.\text{----------}(n-2)^2,\ if\ n\ is\ odd \end{cases}$$

Example 12: If $y = \cos(m\sin^{-1}x)$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0 \text{ and find } y_n = 0$$

Solution: Given

$$y = \cos(\text{m}\sin^{-1}x) \text{ ---------------------------------- } (1)$$

$$y_1 = -\sin(\text{m}\sin^{-1}x)\frac{m}{\sqrt{1-x^2}} \text{ -------------------------- } (2)$$

$$(1-x^2)y_1^2 = m^2\sin^2(\text{m}\sin^{-1}x)$$

$$= m^2\left(1-\cos^2(\text{m}\sin^{-1}x)\right)$$

$$= m^2\left(1-y^2\right)$$

Differentiating w. r. to x, we get

$$2\left(1-x^2\right)y_1y_2 - 2xy_1^2 = -2m^2\, y\, y_1$$

$$\Rightarrow \quad \left(1-x^2\right)y_2 - xy_1 + m^2 y = 0 \quad \text{----------------------} \quad (3)$$

Differentiating n times by Leibnitz theorem, we obtain

$$\left(1-x^2\right)y_{n+2} + {}^nC_1\, y_{n+1}\left(-2x\right) + {}^nC_2\, y_n\left(-2\right) - xy_{n+1} + {}^nC_1\, y_n\left(-1\right) + m^2 y_n = 0$$

$$\left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - xy_{n+1} - n\left(n-1\right)y_n - ny_n + m^2 y_n = 0$$

$$\left(1-x^2\right)y_{n+2} - \left(2n+1\right)xy_{n+1} + \left(m^2-n^2\right)y_n = 0 \quad \text{-----------} \quad (4)$$

Putting x = 0 in (4), we get

$$y_{n+2}\left(0\right) + \left(m^2-n^2\right)y_n\left(0\right) = 0 \qquad \text{or} \qquad y_{n+2}\left(0\right) = \left(n^2-m^2\right)y_n\left(0\right)$$

From (1), (2) and (3), we obtain

$$y\left(0\right) = 1,\ y_1\left(0\right) = 0,\ y_2\left(0\right) = -m^2 y\left(0\right) = -m^2$$

Putting n = 1, 2, 3, 4, -------------- in relation (4), we get

$$y_3\left(0\right) = \left(1^2-m^2\right)y_1\left(0\right)$$

$$y_4\left(0\right) = \left(2^2-m^2\right)y_2\left(0\right) = -m^2\left(2^2-m^2\right)$$

$$y_5\left(0\right) = \left(3^2-m^2\right)y_3\left(0\right) = 0$$

$$y_6\left(0\right) = \left(4^2-m^2\right)y_4\left(0\right) = -\left(4-m^2\right)m^2\left(2^2-m^2\right)$$

$$= -m^2\left(2^2-m^2\right)\left(4^2-m^2\right)$$

In general,

$$y_n\left(0\right) = \begin{cases} o, \text{ if } n \text{ is odd} \\ -m^2\left(2^2-m^2\right)\left(4^2-m^2\right)\text{-------}\left\{\left(n-2\right)^2-m^2\right\}, \text{ if } n \text{ is even} \end{cases}$$

Example 13:

If y = $(\sin^{-1}x)^2$, then prove that

$(1 - x^2)\, y_2 - xy_1 - 2 = 0$ -------------------------------- (1)

and hence show that

$(1 - x^2)\, y_{n+2} - (2n + 1)xy_{n+1} - n^2y_n = 0$----------------- (2)

Now,

$$y = \left(\sin^{-1} x\right)^2$$

$$y_1 = 2\sin^{-1} x \, \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x)^2\, y_1^2 = 4\left(\sin^{-1} x\right)^2 = 4y$$

Differentiating w. r. to x, we get

$$(1-x)^2\, 2y_1y_2 - 2xy_1^2 - 4y_1 = 0$$

$$\Rightarrow \quad (1-x)^2\, y_2 - xy_1 - 2 = 0$$

Applying Leibniz Leibnitz theorem, we obtain

$$(1-x)^2\, y_{n+2} + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n\,(-2) - xy_{n+1} - {}^nC_1\, y_n = 0$$

$$\Rightarrow \quad (1-x)^2\, y_{n+2} - 2nxy_{n+1} - xy_{n+1} - \left(n^2 - n\right)y_n - ny_n = 0$$

$$\Rightarrow \quad (1-x)^2\, y_{n+2} - (2n+1)\, xy_{n+1} + n^2y_n = 0$$

This was to be proved.

4.4. Exercise:

1. Using Leibnitz theorem find the 3rd differential coefficients of

   (i) x^2e^{2x} (ii) $x^2\sin 3x$ (iii) $xe^{ax}\sin bx$

2. Use Leibnitz theorem to find n the differential coefficients of

(i) $e^{2x}(ax+b)^3$, (ii) $x^3\cos x$ (iii) $x^2 \ln x$

3. Differentiate the following equation n–times w.r. to x

   (i) $(1-x^2)y_2 - xy_1 + a^2y = 0$ (ii) $x^2y_2 - xy_1 + (a^2-m^2)y = 0$

4. If $y = \sin(m\sin^{-1}x)$, then prove that

   $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$ and find $y_n(0)$

5. If $y = \left(x+\sqrt{1+x^2}\right)^m$, then find $y_n(0)$

6. If $y = (\sinh^{-1}x)^2$, then prove that

   $(1-x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

7. If $y = e^{a\sin^{-1}x}$, then prove that

   $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

8. If $y = \cos^{-1}\left(\frac{y}{b}\right) = \ln\left(\frac{x}{n}\right)^n$, then prove that

   $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

9. If $y = e^{m\sin^{-1}x}$, then find $y_n = (0)$

10. If $y = \ln\left(x+\sqrt{1+x^2}\right)$, then find $y_n(0)$.

Summary: This chapter includes the following:-

1. Definitions of hyperbolic function. Elaborates on domains, range and graphs of hyperbolic functions.

2. Properties and interconnections among different hyperbolic functions, Hyperbolic identities.

3. Inverse hyperbolic functions. Their domain's ranges and graph.

4. Derivative formulas and integration formulas for the hyperbolic functions and inverse hyperbolic functions.

5. Higher order derivatives.

6. Leibnitz theorem and its applications.

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