Discipline Courses-I
Semester-I
Paper: Calculus-I
Lesson: Indeterminate Forms and Application of Derivatives

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1. Learning out comes:

After the study of this chapter, reader can appreciate, understand and apply the concepts of limits and derivatives to use in different areas. In fact, the students are encouraged to formulate their own problems in business, economics, science & biology and solve them using the tools of this chapter.

2. Introduction:

In this chapter we shall discuss various indeterminate forms which occur while evaluating limits of functions. For example, if $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$. Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ to said to be a indeterminate form $\left(\frac{0}{0}\right)$. Similarly, other indeterminate forms are of the type $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0 and $\infty - \infty$. Further, we consider the applications of the derivatives to solve the problems in business, life sciences and physical sciences.

3. L' Hôpital's Rules:

In case $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ and if $B \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

However, if $A \neq 0$, and $B \neq 0$, then the limit is infinite.

In the case A = 0 and B = 0, the limit of the quotient f/g is said to be "indeterminate". The limit in this case may be any real number or many not exist, depending on the particular functions f and g. The symbol % is used to refer to this situation. For example if α is any real number and we take $f(x) = \alpha . x$ and $g(x) = x$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\alpha x}{x} = \lim_{x \to 0} \alpha = \alpha$$

This the indeterminate form % can lead to any real number α as limit.

Other indeterminate forms are represented by the symbols ∞/∞, $0 \cdot \infty$, 0^0, 1^∞, ∞^0 and $\infty - \infty$. These symbols correspond to the limiting behavior of the functions f and g. We shall focus on the indeterminate forms $0/0$ and ∞/∞. The other indeterminate forms are usually reducible to the from $0/0$ or ∞/∞ by algebraic manipulations or by taking logarithms or exponentials,

3.1. L' Hôpital's Rules for form $\frac{0}{0}$:

Theorem 1: Suppose that f and g are differentiable functions with $g(x) \neq 0$ on an open interval containing $x = a$, except (possibly at a itself) and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$.

If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ a finite limit $or +\infty\ or -\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L \ .$$

Furthermore, the result is also valid case for $x \to a+, x \to a-, x \to +\infty\ or\ x \to \infty$.

Working Rule:

(i) Check that $\frac{f(x)}{g(x)}$ is an indeterminate form.

(ii) Differentiate f and g separately

(iii) Find $\lim_{x \to a} \frac{f'(x)}{g'(x)}$, If this limit is finite, +∞ or –∞, then it is equal to $\lim_{x \to a} \frac{f(x)}{g(x)}$.

Example 1: Check in each case that the limit is an indeterminate form of $\frac{0}{0}$ if yes then evaluate it

(i) $\lim_{x \to 1} \frac{x^{10} - 1}{x - 1}$ (ii) $\lim_{x \to 0} \frac{\sin ax}{\cos bx}, ab \neq 0,$

(iii) $\lim_{x \to 0} \frac{x^2 + \sin x^2}{x^2 + x^3}$

Solution: (i) $\lim_{x \to 1} \frac{x^{10} - 1}{x - 1} \left[\frac{0}{0}\right] = \lim_{x \to 0} \frac{10x^9}{1} = 10.0 = 0$

(ii) $\lim_{x \to 0} \frac{\sin ax}{\cos bx}, (ab \neq 0)$ *Here limit of numerator is 0 but that of denominator is not 0; so it is not an indeterminate form*

However, the limit is 0.

(iii) $\lim_{x \to 0} \frac{x^2 + \sin x^2}{x^2 + x^3} \left[\frac{0}{0}\right]$ *form*.

$$= \lim_{x \to 0} \frac{2x + \cos x^2 . (2x)}{2x + 3x^2} \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0} \frac{2 + 2\cos x^2 + \left(-\sin x^2\right)(2x . 2x)}{2}$$

$$= \frac{2 + 2 + 0}{2} = 2.$$

3.2. L' Hôpital's Rules for form $\frac{\infty}{\infty}$**:**

Theorem 2: If f and g are differentiable functions with $g'(x) \neq 0$ on an open interval containing a (except possibly at x = a itself) and $\lim_{x \to a} f(x) = \infty = \lim_{x \to 0} g(x)$ such that $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \text{L}$, where L is a finite number or +∞, or −∞, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \text{L}$$

Furthermore the theorem is still true in case x → a^+, x → a^-, x → +∞ or as x→ −∞.

Example 2: Check that the following are indeterminate forms of type $\frac{\infty}{\infty}$ and evaluate it

(i) $\lim_{x \to \pi/2} \frac{3\sec x}{2 + \tan x}$ (ii) $\lim_{x \to 0^+} \frac{\ln x}{\operatorname{cosec} x}$

Solution: (i) $\lim_{x \to \pi} \frac{3\sec x}{2 + \tan x} \left[\frac{\infty}{\infty}\right]$ *form*

$$= \lim_{x \to \pi} \frac{3\sec x \tan x}{\sec^2 x} \left[\frac{\infty}{\infty}\right] \text{ form}$$

$$= \lim_{x \to \pi} 3\sin x = 0$$

(ii) $\lim_{x \to 0^+} \frac{\ln x}{\operatorname{cosec} x} \left[\frac{\infty}{\infty}\right]$ *form*

$$= \lim_{x \to 0^+} \frac{1/x}{-\operatorname{cosec} x \cot x} \left[\frac{\infty}{\infty}\right] \text{ form}$$

Now further application of L'Hôpital rule will produce further indeterminate form, so we simplify and rewrite

$$= \lim_{x \to 0^+} \left(-\frac{\sin x}{x} \tan x\right)$$

$$= \lim_{x \to 0^+} \left(-\frac{\sin x}{x}\right)\left(\lim_{x \to 0^+} \tan x\right)$$

$$= -1.0 = 0.$$

3.3. Indeterminate forms of type 0.∞:

Example 3: The $\lim_{x \to 0^+} x \ln x$ is an indeterminate form of type 0.∞

Solution: Here, $\lim_{x \to 0^+} \frac{\ln x}{1/x} \left[\frac{\infty}{\infty}\right]$ *form*

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \left[\frac{\infty}{\infty}\right] \text{ form}$$

Now further application of L'Hôpital rule will further result in indeterminate form, so we simplify and rewrite

$$= \lim_{x \to 0^+} -x$$

$$= 0. \qquad \therefore \lim_{x \to 0^+} x \ln x = 0$$

Example 4: Evaluate (i) $\lim_{x \to \pi/2^-} \left(x - \pi/2\right) \tan x$ (ii) $\lim_{x \to \infty} \left(\frac{\ln x}{x^2}\right)$ $\left[\frac{\infty}{\infty}\right]$ *form*.

Solution: (i) The given limit

$$= \lim_{x \to \pi/2^-} \left(x - \pi/2\right) \tan x \quad [0.\infty] \text{ form}.$$

$$= \lim_{x \to \pi/2^-} \left(\frac{x - \pi/2}{\cot x}\right) \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to \pi/2^-} \left(\frac{1}{-\operatorname{cosec}^2 x}\right)$$

$$= \lim_{x \to \pi/2^-} \left(-\sin^2 x\right) = -1.$$

(ii) The given limit

$$= \lim_{x \to \infty} 1/x \left(\frac{-2}{x^3}\right) = \lim_{x \to \infty} \frac{-2}{x^4} = 0.$$

Value Addition: Note
Here limit can also be interpreted as indeterminate form of type $0.\infty$ by writing $\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1}{x^2}.\ln x$

3.4. Indeterminate forms of type $\infty - \infty$:

Example 5: Evaluate $\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Solution: The given limit

$$= \lim_{x \to 0+} \frac{\sin x - x}{x \sin x} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0+} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \lim_{x \to 0+} \frac{-\sin x}{\cos x + \cos x - x\sin x} \quad \left[\frac{0}{0}\right] \text{ form}$$

$$= \frac{0}{2} = 0.$$

Example 6: Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos 3x}{x^2} \right)$.

Solution: The given limit.

$$= \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos 3x}{x^2} \right) \ (\infty - \infty) \ \textit{form}.$$

$$= \lim_{x \to 0} \left(\frac{1 - \cos 3x}{x^2} \right) \ \left[\frac{0}{0}\right] \ \textit{form}$$

$$= \lim_{x \to 0} \frac{3\sin 3x}{2x} \ \left[\frac{0}{0}\right] \ \textit{form}.$$

$$= \lim_{x \to 0} \frac{9\cos 3x}{2}$$

$$= \frac{9}{2}.$$

Example 7: Evaluate $\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\tan^{-1} x} \right) \qquad [\infty - \infty] \ \textit{form}$

Solution: The given limit

$$= \lim_{x \to 0+} \left(\frac{\tan^{-1} x - x}{x \tan^{-1} x} \right) \qquad \left[\frac{0}{0}\right] \ \textit{form}$$

$$= \lim_{x \to 0+} \frac{\frac{1}{1 + x^2}}{\tan^{-1} x + \frac{x}{1 + x^2}} \qquad \left[\frac{0}{0}\right] \ \textit{form}$$

$$= \lim_{x \to 0+} \frac{-x^2}{(1 + x^2)\tan^{-1} x + x} \qquad \left[\frac{0}{0}\right] \ \textit{form}$$

$$= \lim_{x \to 0+} \frac{-2x}{(1 + x^2)\frac{1}{(1 + x^2)} + 1}$$

$$= \lim_{x \to 0+} \frac{-2x}{2} = 0 .$$

Example 8: Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ $\quad [1^\infty \ form]$.

Solution: The given limit

$$= \lim_{x \to \infty} e^{x \ln\left(1+\frac{1}{x}\right)}$$

Further, $\lim_{x \to \infty} x \ln\left(1+\frac{1}{x}\right) \quad [0.\infty] \ form$

$$= \lim_{x \to \infty} \frac{\ln\left(1+ 1/x\right)}{1/x} \quad \left[\frac{\infty}{\infty}\right] \ form$$

$$= \lim_{x \to \infty} \frac{\frac{1}{1 + 1/x}\left(-1/x^2\right)}{-1/x^2}$$

$$= \lim_{x \to \infty} \frac{1}{1 + 1/x} = 1 .$$

Since the function f(t) = e^t is continuous

$$= \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e .$$

3.5. Limits of the form 0^0:

Example 9: Evaluate $\lim_{x \to 0+} x^x$

Solution: The given limit

$$= \lim_{x \to 0+} e^{x \ln x}$$

Consider $\lim_{x \to 0+} x \ln x$

$$= \lim_{x \to 0} \frac{\ln x}{1/x} \quad \left[\frac{\infty}{\infty}\right] \textit{ form}$$

$$= \lim_{x \to 0} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0} -x = 0 .$$

Since the function f(t) = e^t is continuous at t = 0.

$$\therefore \quad \lim_{x \to 0} x^x = e^0 = 1$$

3.6. Limits of the form ∞^0:

Example 10: Evaluate $\lim_{x \to 0+} \left(1 + 1/x\right)^x$

Solution: Let $y = \left(1 + \frac{1}{x}\right)^x$, Then

$$\ln y = x \ln\left(1 + \frac{1}{x}\right) .$$

$$\lim_{x \to 0+} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x \to 0+} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} \quad \left[\frac{\infty}{\infty}\right] \textit{ form}$$

$$= \lim_{x \to 0+} \frac{\frac{1}{1 + 1/x}\left(-1/x^2\right)}{-1/x}$$

$$= \lim_{x \to 0+} \frac{1}{1 + 1/x} = 0 .$$

$$\therefore \quad \lim_{x \to 0+} \left(1 + 1/x\right)^x = e^0 = 1 .$$

Evaluate 11: Evaluate $\lim_{x \to 0+} \frac{\tan x - x}{x^2 \tan x} \quad \left[\frac{0}{0}\right] \textit{ form}$

Solution: The given limit

$$= \lim_{x \to 0+} \frac{\tan x - x}{x^3} \cdot \lim_{x \to 0+} \frac{x}{\tan x}$$

$$= \lim_{x \to 0+} \frac{\tan x - x}{x^3} \cdot 1$$

$$= \lim_{x \to 0+} \frac{\sec^2 x - 1}{3x^2} \left[\frac{0}{0}\right] \textit{form}$$

$$= \lim_{x \to 0+} \frac{2\sec^2 x \tan x}{6x} \left[\frac{0}{0}\right] \textit{form}$$

$$= \lim_{x \to 0+} \frac{1}{3} \lim_{x \to 0+} \sec^2 x \,.\, \lim_{x \to 0+} \frac{\tan x}{x}$$

$$= \frac{1}{3} .1.1. = \frac{1}{3} .$$

3.7. Exercise:

1. Evaluate the following limits:

(i) $\lim_{x \to 0} \frac{e^x}{\sin x}$ (ii) $\lim_{x \to 0+} \frac{\sin x}{x^3}$ (iii) $\lim_{x \to 0+} \frac{\tan x - x}{x^3}$

(iv) $\lim_{x \to \infty} \frac{x^2 + 5}{e^x}$ (v) $\lim_{x \to 0+} x^2 \ln x$ (vi) $\lim_{x \to 0} x \ln \sin x$

2. Evaluate the following limits

(i) $\lim_{x \to \infty} x^{1/x}$ (ii) $\lim_{x \to 0+} x^{\sin x}$

3. Limits of the type

$$0/\infty,\ \infty/0,\ \infty.\infty,\ \infty - (-\infty),\ +\infty + (+\infty),\ -\infty - (+\infty),\ -\infty + (-\infty)$$

are not indeterminate forms. Find the following limits by inspection:

(i) $\lim_{x \to 0+} \frac{x^2}{\ln x}$ (ii) $\lim_{x \to +\infty} \frac{x^2}{e^{-x}}$

(iii) $\lim_{x \to 0+} \left(\frac{1}{x^2} - \ln x \right)$ (iv) $\lim_{x \to \pi/2-} (\cos x)^{\tan x}$

4. Evaluate the following limits

(i) $\lim_{x \to 0} \frac{x - \sin x}{(x \sin x)^{3/2}}$ (ii) $\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$

(iii) $\lim_{x \to 0} \frac{x \cos x - \log(1 + x)}{x^2}$ (iv) $\lim_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin x^2}$

4. Application of derivatives to Business:

Recall that the derivative of a real valued function of real variable $f(x)$ at a point x, $f'(x)$ is defined as the value of the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If we write $y = f(x)$, $y + \Delta y = f(x + \Delta x)$, where Δy is the increment in the value of y corresponding to the increment Δx in the value of x, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

For convenience we sometimes write $\frac{dy}{dx}$ for $f'(x)$. The function $f'(x)$ is called the marginal function of $f(x)$.

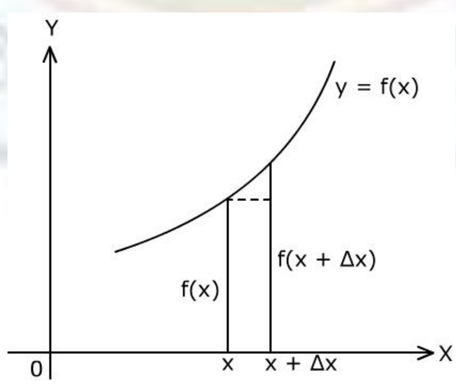

Figure 1

The ratio $\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\Delta y}{\Delta x}$ denotes the average value of y at the level x calculated from Δx units. Therefore, when Δx becomes smaller and smaller, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ shall denote average (approximate) value of y calculated from those Δx units. Hence in the limiting case when Δx tends to zero, $f'(x)$ denotes the approximate (average) value of y when x changes by 1 unit at the level x. Thus, for example, if $C=C(x)$ denotes the total cost function for a product whose x units are produced, the marginal cost $C'(x)$ denotes the approximate cost of producing one additional units at the level x. Similarly, if $R(x)$ is the total revenue function, then $R'(x)$ denotes the approximate revenue that will we generated by selling one more unit at the level x.

Assuming that the output produced is sold, the revenue R may be considered a function of output denoted $R=R(x)$. Similarly, the cast C can be considered as a function of output x written $C=C(x)$. Then the profit P is a function of the output x, and is given by

$$P(x)=R(x)-C(x).$$

For maximum profit $\frac{dP}{dx}=0$ and $\frac{d^2P}{dx^2}<0$ (The second derivative test) Now

$$\frac{dP}{dx}=\frac{dR}{dx}-\frac{dC}{dx}$$

Thus at the output giving maximum profit

Marginal revenue (MR) $=\frac{dR}{dx}=\frac{dC}{dx}=$ Marginal cost (MC).

Thus, at the optimum profit level, the tangents to the total revenue curve and the total cost cure are parallel (see fig. 2)

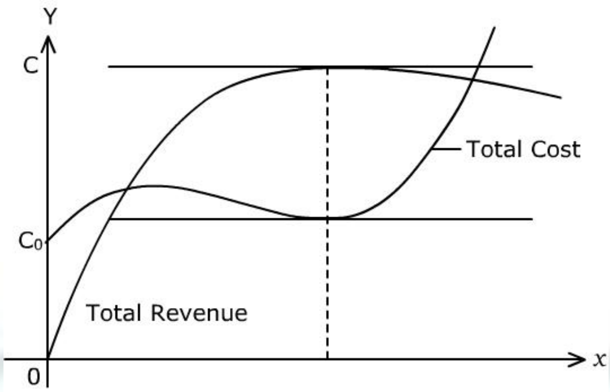

Figure 2

Note that C_0 is the fixed Cost which is the cost incurred even when output is not produced.

Now

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(MR) - \frac{d}{dx}(MC).$$

Thus, $\frac{d^2P}{dx^2} < 0$ when $\frac{d}{dx}(MR) < \frac{d}{dx}(MC)$.

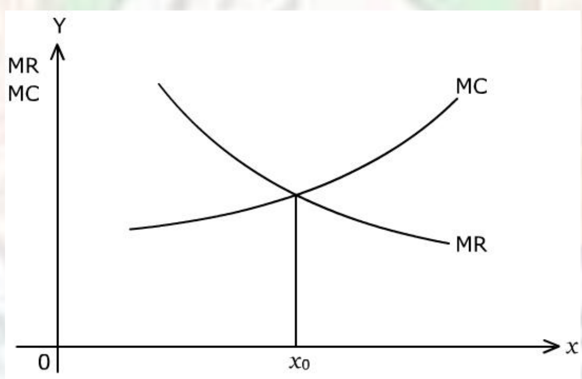

Figure 3: x_0 : point of maximum profit.

This mean that the slope of the marginal revenue curve is less than the slope of the marginal cost curve at the point of maximum profit.

That is the marginal revenue and marginal cost curve must be as shown at the point of maximum profit x_0.

Example 12: A producer estimates that if he spends x thousand rupees on development and y thousand rupees on promotion of a new product. The demand for the product would be $\frac{320y}{y+2} + \frac{160x}{x+4}$ if he sells a unit of the product at rupees 150. The cost of production is rupees 50 per unit. If the producer has a total of rupees 8000 to spend on development and promotion, how should this money be distributed to generate maximum profit.

Solution: The profit function is

$$P = 100\left[\frac{320y}{y+2} + \frac{160x}{x+4}\right] - 1000(x+y),$$

where $1000(x+y) = 8000$. This P can be expressed as a function of single variable x:

$$P = 100\left[320\frac{8-x}{10-x} + 160\frac{x}{x+4}\right] - 8000.$$

$\frac{dP}{dx} = 0$ at $x = 3$.

One can easily check that $\frac{d^2P}{dx^2} < 0$ at $x = 3$. Then $y = 3$.

Taxes are imposed by the government an producers. The producers try to shift this tax burden onto the consumers. This is not however possible in an environment of pure competition since a producer has no control over the prices. However, a monopolist has the choice to fix the price or the output. The monopolist would add the new tax to the cost of the product and will fix the output (price) in such way that his profit is maximum. It may be remarked that "excise tax" is imposed per unit and the "sales tax" is imposed on the total revenue.

Example 13: Suppose the demand and total cost function for a monopolist are $p = 20 - 4x$ and $C = 4x$ respectively.

(i) Determine the output that maximizes the profit.

(ii) If a tax "t" per unit is imposed, find the output and price that correspond the maximum profit. Determine the tax that maximizes the tax revenue and find the maximum tax revenue.

(iii) Find the tax revenue if 10 % sales tax is imposed.

Solution: The revenue obtained by selling x units is

$$R(x) = p.x = 20x - 4x^2$$

The profit function $P(x) = R(x) - C(x) = 16x - 4x^2$.

For maximum profit $\frac{dP}{dx} = 0$ which gives $x = 2$. Also $\frac{d^2P}{dx^2} = 8 < 0$.

Thus, the profit is maximum when $x = 2$.

Now if a tax of t per unit is imposed, then the new cost function is

$$C_n(x) = C + tx = 4x + tx$$

The new profit function is $P_n = R - C_n = 16x - tx - 4x^2$

The output that maximizes profit is now $x = \frac{16 - t}{\delta}$

The corresponding price

$$p = 12 + \frac{t}{2}$$

The tax revenue is given by

$$T = t.\frac{16 - t}{8} = \frac{16t - t^2}{8}$$

The maximum value of T occurs when t = 8. The corresponding tax revenue if

T = 8

If sales tax of 10% is imposed, this is added to the cost function. That is

$$C_n = 4x + \frac{10}{100}\left(20x - 4x^2\right) = 6x - \frac{2}{5}x^2$$

and so the new profit function

$$P_n(x) = \left(20x - 4x^2\right) - \left(6x - \frac{2}{5}x^2\right)$$

Now the maximum profit will occur at $x = \frac{35}{18}$, and the total tax revenue will be

$$= 2\left[\frac{35}{8}\right] - \frac{2}{5}\left[\frac{35}{8}\right]^2 = \frac{385}{162}.$$

Let $y = f(x)$ be a real valued function of real variable. Then average elasticity of y with respect to the independent variable x is defined as the ratio of percentage change in the dependent variable y to the percentage change in the independent variable x. Thus, if the variable y changes by Δy when the variable x changes by Δx, than the average elasticity y with respect to x is given by

$$\frac{\frac{\Delta y}{y} \times 100}{\frac{\Delta x}{x} \times 100}$$

The elasticity as y with respect to x is the limiting value of the average elasticity. Thus, the elasticity of y with respect to x is given by

$$\eta = \lim_{\Delta x \to 10} \frac{\frac{\Delta y}{y} \times 100}{\frac{\Delta x}{x} \times 100} = \frac{x}{y}\frac{dy}{dx}$$

For example, if $x = f(p)$ is a demand function, then the elasticity of demand with respect to the price is given

$$\eta = \frac{p}{x}\frac{dy}{dx}$$

The revenue function $R = R(x) = xp = p.f(p)$. It is easily seen that

$$\frac{dR}{dx} = \frac{R}{x}\left[1 + \frac{1}{\eta}\right]$$

That is, the marginal revenue is $\left(1 + \frac{1}{\eta}\right)$ times the average revenue.

Note that the elasticity $\eta = \frac{x}{y}\frac{dy}{dx}$ is approximately the percentage change in the dependent variable y when the independent variable changes by one percentage point at the level x. Such information about the behavior of various functions occurring in business and economics such as demand, function, supply functions, consumption functions is of vital importance.

Example 14: A manufacturer finds that the demand function for a product is $x(p) = \frac{73}{\sqrt{p}}$. Should the price p be raised or lowered to increase consumer expenditure. Explain your answer.

Solution: The consumer expenditure R is given by

$$R = px = \mathrm{p}\frac{73}{\sqrt{p}}\ 77\sqrt{p}$$

The consumer expenditure increaser with the increase in p since $\frac{dR}{dp} > 0$.

4.1. Inventory Control:

The process of selling a product usually involves its warehousing and hence associated cost is incurred. The inventory cost is usually of three types: purchasing cost, ordering cost and holding cost. If the demand of the product is uniform throughout the period and the inventory can be replaced immediately when it reaches the level zero then the inventory at any time t assuming the order size is same each time roughly follow the following pattern

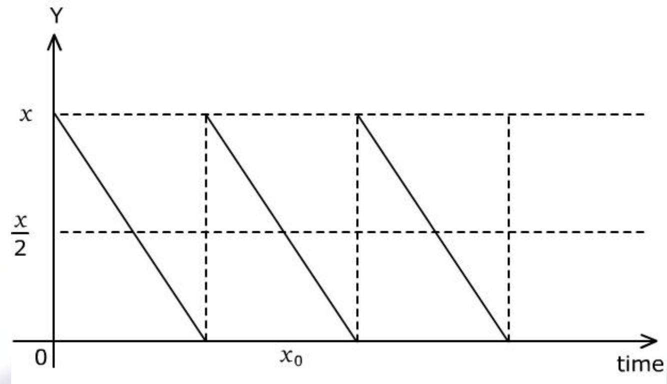

Figure 4

Thus, in the time interval of successive arrivals of inventories, on an average $\frac{x}{2}$ units shall be stored and hence $\frac{x}{2}$ units will be stored throughout the year. Thus the holding cost or the storage cost equals $\frac{x}{2}$ times the cost of holding 1 unit throughout the year. Thus,

The holding cost $= \frac{x}{2} \times c_h$

where c_h = holding cost per unit per year. Now if N = no of units sold in a year the order size is x and the cost of placing an order is c_o, then the total cost of placing orders is given by

ordering cost = $\frac{N}{x} \times c_0$

The total inventory cost

$$C = \frac{x}{2}C_h + \frac{N}{x}C_0 .$$

The optimal order size per economic order quantity (EOQ) is obtained by minimizing C. For minimum C, $\frac{dC}{dx} = 0$, gives

$$x = \sqrt{\frac{2c_0 N}{C_h}}$$

Since $\frac{d^2C}{dx^2} = \frac{2c_0N}{x^3} > 0,$ the EOQ is given by

$$EoQ = \sqrt{\frac{2c_0N}{C_h}}$$

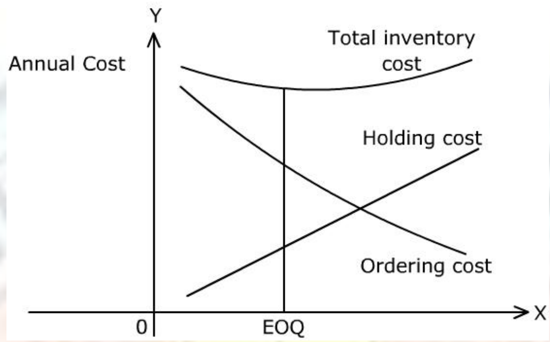

Figure 5

Example 15: A company uses 500 units of a component per month and it cost rupees 12 each. The cost of placing an order is rupees 1200, irrespective of the quantity ordered. The holding cost consists of the capital cost of 10% of the value of the stock, rupees 0.30 per unit per annum for insurance, rupees 0.60 per item per annum for storage and rupees 0.30 per item for maintenance. Find the optimal order size, the number of orders per year and the total annual cost of the inventory.

Solution: If x is the order size, the total annual cost

$$C = \frac{500 \times 12}{x} \times 200 + \left(0.10 \times 12 + 0.30 + 0.60 + 0.30\right)\frac{x}{2} + 6000 \times 12$$

Now $\frac{dC}{dx} = 0$ gives $x = 1000$ and $\frac{d^2C}{dx^2} > 0$. So the inventory cost is minimum when the order size = 1000. The number of orders $= \frac{500 \times 12}{1000} = 6$ and the total annual cost of the inventory = rupees 74,000.

Example 16: A store sells 2500 refrigerators per year, with sales occurring at relatively uniform rate. The annual holding cost per refrigerator is rupees 10. It cost rupees 20 to place an order and rupees 9 as insurance charge for each refrigerator ordered. What is the economic order size and the total inventory cost.

Solution: The annual inventory cost

$$C = \frac{2500}{x} \times 20 + \frac{x}{2} \times 10 + 2500 \times 9$$

The economic order size

$$x = \sqrt{\frac{2 \times 2500 \times 20}{10}} = 100$$

The total inventory cost is rupees 23500.

Example 17: An investor determines the optimal holding time of an asset worth V(t) at time to by the equation

$$\frac{V'(t)}{V(t)} = r,$$

where r is the rate of interest. Show that the optional time determined by this criterion is the same as that found by maximizing the present value of V(t).

Solution: Note that $V'(t)$ is approximately the value addition in the value of the asset in one year after time t. This value is earned from the value V(t) at time t. Thus, earning per rupee during the year is $\frac{V'(t)}{V(t)}$. One can hold the asset till the time the value $\frac{V'(t)}{V(t)} \geq r$ the rate of interest. So the optimum time to sell the money is determined by the equation $\frac{V'(t)}{V(t)} = r$.

The present value of V(t) is given by

$$p(t) = V(t)e^{-vt}.$$

For maximum value of p $p'(t) = 0$. This gives $\frac{V'(t)}{V(t)} = r$

Which is the same criterion as obtained above.

4.2. Applications in Life Sciences:

Calculus can be used to understand a variety of situations in biological and life sciences. The following problem deals with the concentration of a drug in blood injected intramuscularly.

Example 18: Let c(t) be the concentration in the blood at time t of a drug injected into the body intramuscularly

$$c(t) = \frac{k}{b-a}\left(e^{-at} - e^{-bt}\right) \; t \geq 0 , \;\; k, a, b > 0 \;\; and \;\; b > a .$$

At what time c(t) is maximum? Is concentration c(t) tends to zero as t tends to infinity?

Solution: At a maxima or minima $c'(t) = 0$. Now

$$c'(t) = \frac{k}{b-a}\left[-ae^{-at} + be^{-bt}\right].$$

Therefore, $c'(t) = 0$ gives $t = \frac{1}{b-a} \log \frac{b}{a}$. Since $b > a$,

$$c''(t) = at \; t = \frac{1}{b-a} \log \frac{b}{a}$$

$$= \frac{k}{k-a}\left[ab - b^2\right] e^{-\frac{b}{b-a} \log \frac{b}{a}} < 0.$$

Thus, c(t) is maximum at $t = \frac{1}{b-a} \log \frac{b}{a}$.

Now since

$$\lim_{t \to \infty} c(t) = 0$$

It follows that the concentration of the drug in the blood tends to zero as t tends infinity, but it is never zero at a finite time.

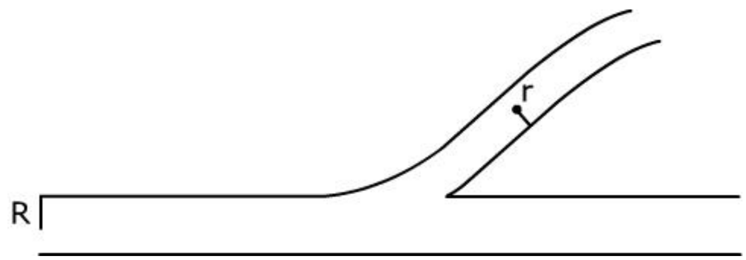

Figure 6

Example 19: (Optimal angle for vascular branching). The blood circulates from the heart, through the organs of the body, and back to the heart in a network of arteries. The question naturally a riser as to how this complex system of arteries is formed. A Commonly accepted principle is that the daughter artery branches out from the parent artery in such a way that the minimum consumption of energy takes place in transporting the blood. This consumption of energy is minimum if the resistance to the flow of blood is minimum. According to French physiologist Poiseuile (1784 – 1869);

"The resistance to the flow of blood in an artery is directly proportional to the artery's length and inversely proportional to the fourth power of its radius".

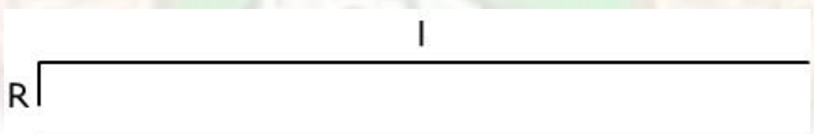

Figure 7

Thus, if f is the resistance to the flow of blood in artery of length l and radius R, then.

$$f \ \alpha \ \frac{l}{R^4}$$

Therefore $f = \frac{kl}{R^4}$ where is k is constant of proportionality called a viscosity constant. The above principle helps us in determining the angel between the parent and the daughter arteries as fallows.

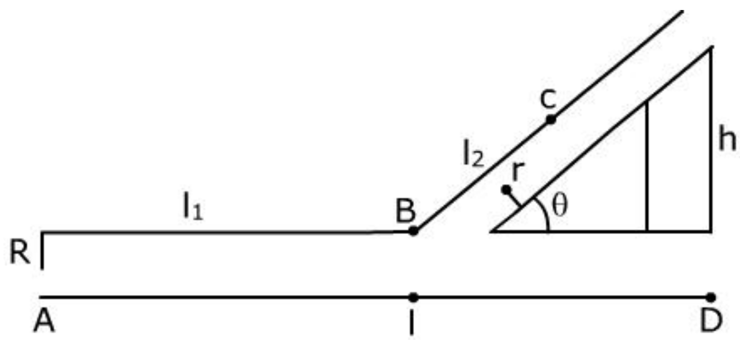

Figure 8

Let AD be an artery of length l and radius R, and BC be the daughter artery of radius r branching out from AD at the pain B. Let AB = l_1 and BC =l_2.

Let f_1, f_2 be the resistance to the flow of blood in the portion AB, BC respectively. Then by Poiseuille law

$$f_1 = \frac{k\,l_1}{R^4}, \quad f_2 = \frac{k\,l_2}{r^4}$$

The total resistance to flow is given by

$$\begin{aligned} f &= f_1 + f_2 \\ &= b\frac{l_1}{R^4} + b\frac{l_2}{R^4} \\ &= b\left[\frac{l - h\cot\theta}{R^4} + \frac{h\operatorname{cosec}\theta}{r^4}\right] \end{aligned}$$

At a minimum value of f, $\frac{df}{d\theta} = 0$ which gives $\frac{r^4}{R^4} = \cos\theta$. It is easily seen when $\cos\theta = \frac{r^4}{R^4}$, $\frac{d^2 f}{d\theta^2} < 0$. Thus, the daughter branch will be at an angel θ_m which satisfies the equation $\frac{r^4}{R^4} = \cos\theta_m$.

Example 20: The number of infected person in an epidemic at time t after its outbreak is given by

$$f(t) = \frac{A}{1 + ce^{-bt}}$$

where A is the number of susceptible person. Prove that the epidemic spreads most rapidly when the half of the susceptible population is infected.

Solution: We need to find when the rate of f, $f'(t)$, is maximum

Now $f'(t) = \dfrac{Akc\, e^{-bt}}{\left(1+ce^{-bt}\right)^2}$

For maximum value of $f'(t)$, $f''(t)=0$ which gives $ce^{-bt}=1$. That is $f(t)=\dfrac{A}{2}$.

We leave it to the reader to show that $f'''(t)<0$ when $f(t)=\dfrac{A}{2}$.

4.3. Applications to optics:

Example 21: Fermat's principle in optics is the following:

"The path followed by the light in traveling from one point to another is one that minimizes the total time".

In a uniform medium the path minimizing the total time is also the path of shortest distance if there is no obstruction. Consider now the following experiment: Let there be a light source A, a mirror M and an observer B as shown in the following figure.

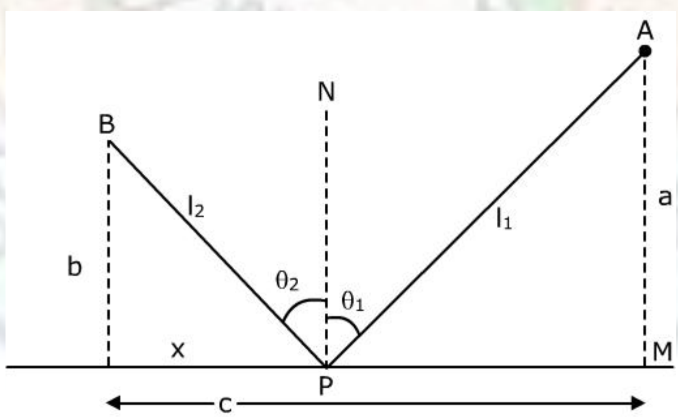

Figure 9

Thus a light ray leaves the source A, bounces off the mirror and travels on to the observer B. Since the x no obstruction the light ray travels from A to P in a straight line and from P to B again in a straight line. Thus, the length of the path APB is $l = l_1 + l_2$. Thus, according to Fermat's principle, the light ray from A that reaches B after bouncing off the mirror will take the path APB that minimizes the time of travel. The time t of travel is a function of the path length l. Now time of travel is minimum if l is minimum. Now

$$l = l_1 + l_2$$

$$= \left[a^2 + (c-x)^2\right]^{1/2} + \left[b^2 + x^2\right]^{1/2}$$

For minimum value of l, $\frac{dl}{dx} = 0$. Which h gives

$$\frac{c-x}{a} = \frac{x}{b}$$

That is tan θ_1 = tan θ_2, thus, $\theta_1 = \theta_2$. Which is the law of refection? Hence, the light ray that reaches B after bouncing off from the mirror is the one that strikes the mirror at the point p having angel of incidence = $\theta_1 = \theta_2$ = angle of reflection.

4.4. The Fermat's principle also explains the snell's law of refraction:

Suppose that a light ray travels from a pain A in Medium-I to a point B in Medium-II. In a medium the speed of the light is constant. (It is slower in a dense medium than in a thin medium.)

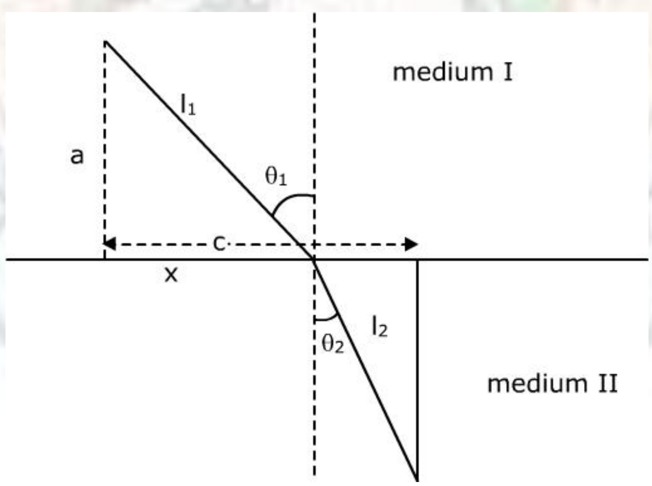

Figure 10

Let v_1 be the speed of the light in medium-I and v_2 be the speed in medium-II. The snell's law states that the pasth of the light ray from A to B will be such that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}. \qquad \text{-------------- (1)}$$

By Fermat's principle the path of the light ray will be such that the time $t = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{\left[a^2 + x^2\right]^{1/2}}{v_1} + \frac{\left[b^2 + (c - x)^2\right]^{1/2}}{v_2}$ is minimum. Now $\frac{dt}{dx} = 0$ gives the relation (1).

Example 22: A pigeon is released from a boat floating on a Lake 3 kilometer away from the shore and it reaches the pigeon's loft which is at a distance of 10 kilometer as shown on the figure.

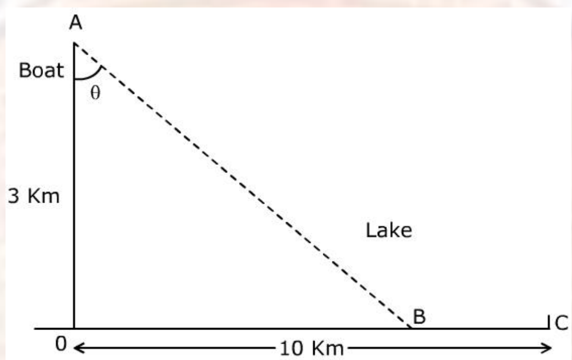

Figure 11

Assuming the pigeon requires twice as such energy to over water as over land, it follows the path that minimizes the energy consumption. Find the angle θ at which it leaves the boat.

Solution: Let e = energy consumed per kilometer by pigeon during the flight over land. Thus the total energy consumed along the path ABC

$$F = 2\ e\ 3 \sec\theta + (10 - 3\tan\theta)\ e$$

For minimum valued of E, $\frac{dE}{d\theta} = 0$

This gives

$$\left[6e\sec\theta\tan\theta - 3e\sec^2\theta\right] = 0$$

$$\Rightarrow \quad \sec\theta\left[6\tan\theta - \sec\theta\right] = 0$$

$$\Rightarrow \quad \frac{6\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} = 0$$

$$\Rightarrow \quad \frac{6\sin\theta - 1}{\cos\theta} = 0$$

$$\Rightarrow \quad \sin\theta = \frac{1}{6}.$$

$$\frac{d^2E}{d\theta^2} = e\sec\theta\cos\theta\left[6\tan\theta - \sec\theta\right] + e\sec\theta\left[6\sec^2\theta - \sec\theta\tan\theta\right]$$

$$= e\sec\theta\tan\theta\left[6\tan\theta - \sec\theta\right] + e\sec^2\theta\left[6\sec\theta - \tan\theta\right]$$

$$= e\sec\theta\tan\theta\left[6\tan\theta - \sec\theta\right] + e\sec^2\theta\left[\frac{6 - \sin\theta}{\cos\theta}\right]$$

$$> 0 \ \ at \ \sin\theta = \frac{1}{6}.$$

This E is minimum when $\sin\theta = \frac{1}{6}$.

4.5. Exercise:

1. Let $c = ax^2 + bx + c$ and $p = \beta - \alpha x$, where a, b, c, α and β is are positive constants, be the total cost function and demand function of a monopolist, respectively. If an excise tax of t per unit of output is imposed, show that the total excise tax is maximum when $t = \frac{\beta - b}{2}$.

2. A machinery cost rupees 15,000. The cost of operating the machine up to time t is $30t^2 + 20t$. the scrap value of the machine at time t is $8520 - 50t^2$. Find the optimum time for the replacement of the machine.

3. During a cough the trachea contracts. The velocity v of the air in the trachea during the cough is given by $v = Ar^2(r_0 - r), 0 \le r \le r_0$ and r_0 and r are the radii of the trachea in a relaxed state and during the cough respectively an A is a constant. Find the radius of the trachea when the velocity in it is maximum.

4. Suppose a company supplies N units/month at a uniform rate. The storage cost is rupees s_1/month and the set up cost is rupees s_2. The production is at uniform rate of in m units/month (with no units left over in the inventory at the end of the month). Find the number of items that should be produced in each run in order to minimize the total average cost. (Economist refer the above problem as the problem of finding the Economic production quantity (EPQ.)

5. A bird flying at a constant speed v expends the energy E given by

$$E = \frac{1}{v}\left[a(v-b)^2 + c\right],\ a, b, c > 0,\ v.\varepsilon[16, 60]$$

At which speed the energy consumed is minimum

6. The production of blood cells in a leukemia patient is given by

$$P(x) = Ax^s\ e^{-sx/r}\ ,\ \ A, s, r > 0$$

and x is the number of granulocytes (a type of white blood cell)

(i) Find x for which $P(x)$ is maximum.

(ii) If s > 1, show that there are two inflection points

(iii) Sketch the graph of $P(x)$ when 0 < s < 1.

7. Oil from an offshore rig located 3km from the shore is to be pumped to a location L the edge of the shore that is 8km GP shown in the figure. The cost of constructing a pipe in the ocean from the rig to the shore is 1.5 times as expensive as the cost of construction on land. Set up and analyze a model to determine how the pipe should be laid to minimize cost.

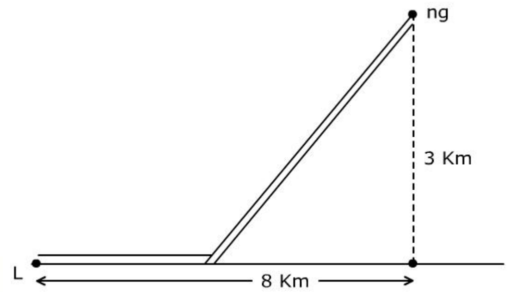

Figure 12

8. A cell of a beehive is a regular hexagonal prism open at one end as shown in the figure below. The surface area of the cell is given by

$$s(\theta) = 6sh + 1.5s^2\left(-\cot\theta + \sqrt{3}\ \text{cosec}\,\theta\right),\ 0 < \theta < \frac{\pi}{2}.$$

What is the angel θ to the nearest degree that minimizes the surface area a of the cell (assuming s and h are fixed)

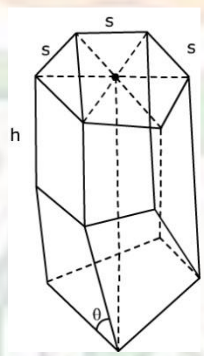

Figure 13: Beehive cell

9. A manufacturer receives an order for 5,000 items. There are 12 machines, each of which produces 25 items per hour. The cost of setting up a machine for a production run is rupees 50. Once the machines are ready for production runt the procedure is fully automated and can be supervised by a single worker earning rupees 20 per hour. Find the number of machines used to minimize the total cost. State any assumption that should be made.

10. The beach of a lake follows contours that are approximated by the curve $4x^2 + x^2 = 1$and a nearby road lies along the curve $y = \frac{1}{x}, x > 0$.

Using your graphing calculator or computer software, determine the closest approach of road to the lake in the north south direction.

11. Suppose a transporter runs each truck of his fleet at a constant speed of x miles per hour $(15 \le x \le 55)$. The gas consumption of the trucks gallons per mile (gal/mi) is given by

$$c = \frac{1}{250}\left[\frac{750}{x} + x\right]$$

Using a graphing and differentiation program, answer the following questions.

(a) If gas cost $1.70/gal. Find the steady speed that will minimize the cost of fuel for 500 miles trip.

(b) Estimate the steady speed that minimizes the cost if the driver is paid $28 per hour and the price of the gasoline remains constant at $ 1.70/gal.

12. Let the total output x of a chemical is related to the defective output y of the same chemical by $y = 0.01x + 0.00003x^2$. If the profit per unit is rupees 100 per unit of nondirective chemical product and loss is rupee 20 per unit of defective chemical, find the number of units of that chemical that must be produced to maximize the profit.

13. The total cost function for x units of output of a chemical is given by $c(x) = 100,00 + 50x + 0.0025x^2$

If the price of the chemical is rupees 100 and not more than 7000 units of the chemical can be produced, what should be the output that gives maximum profit? Would it be beneficial to increase the production capacity? What will be the approximate cost per unit if the production is increased from 100 units to 101.

14. The cost in rupees per hour to run a ship at a constant speed v kim/hr is given by c = a + b v^n where a, b are positive constant and n > 1 is a positive integer. Find the speed needed to make cheapest 500 km.

15. Suppose that the pollution (concentration in the air of particulate matter) at a point is inversely proportional to its distance from the source of the pollution and directly proportional to the amount of pollution released. Suppose that two plants are 10 km apart are releasing 60 and 240 ppm respectively. At what point between the plants the pollution is minimum and at which point it is maximum.

16. Suppose the value of a piece of land after t years is given by $v = 200 \log \sqrt{2t}$. If the interest on money is 10% compounded continuously when it is appropriate to sell the land.

17. A commercial cattle ranch allows 20 cows per acre of grazing land. On the average its cows weighs 200 kgs at market. Estimates by the aquiculture department shows that average market weight per cow will be reduced by 8 kg. How many cows per acre should be allowed to get the maximum market weight for the ranch?

Summary:

This chapter deals with several kinds of indeterminate forms which arise in the study of limits of functions. Besides, the optimization problems in business, science and biology that can be solved using the concept of derivatives in calculus are dealt with. The reader is encouraged to find new problems and solve them using calculus

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