Chapter: Interpolation

Paper: Numerical Methods

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Interpolation

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Chapter: Interpolation

1. Learning Outcomes:

In this chapter reader will learn

(i) Definition and meaning of interpolation.

(ii) How with the help of polynomial we can approximate the value of a function which is not known explicitly.

(iii) There are different methods of constructing a polynomial, like Newton-Gregory method for equally spaced points(abcissaes) and Lagranges method for unequally spaced points.

(iv) Reader will appreciate the error analysis explained in this chapter . Suitability of degree of polynomial and bounds on the error is also explained.

(v) Finite difference operators are also introduced to understand a different approach in interpolation.

2. Introduction:

Interpolation: Interpolation means insertion or filling up intermediate terms of series. Interpolation is the method of estimating the value of a function (dependent variable) for any intermediate value of the independent variable when some values of the function corresponding to the values of the variable are given.

that is, given the set of functional values $(x_0, y_0),(x_1, y_1),(x_2, y_2),...,(x_n, y_n)$ satisfying the relation $y = f(x)$ where the explicit nature of $f(x)$ may not be known, it is required (desired) to find a simpler function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points, such a process is called as interpolation and if $\phi(x)$ happens to be a polynomial than the process is polynomial interpolation. $\phi(x)$ approximates (evaluates) for $f(x)$

3. Methods of Interpolation:

Following are the methods of Interpolation

(a) Graphic Method

(b) Method of Curve fitting

(c) Use of finite difference formulae.

Interpolation

Interpolation or interpolating polynomial are having two main uses.

(i) The first use is in reconstructing the function $f(x)$ when it is not given explicitly and only the values of $f(x)$ and for its certain order derivatives at a set of points, called nodes, tabular points or arguments are known.

(ii) The second use is to replace the function $f(x)$ by an interpolating polynomial $\phi(x)$ so that many common operations such as determination of roots, differentiation, integration etc. may be carried out easily using $\phi(x)$.

Now

As a justification for the approximation of an unknown function $f(x)$ by means of a polynomial $\phi(x)$ we state here without proof a theorem due to Weierstrass:

Statement : If $f(x)$ is continuous in $x_0 \le x \le x_n$, than given any ε > 0, there exists a polynomial $\phi(x)$. (ε = arbitrarily small)

such that

$$|f(x)-\phi(x)|<\varepsilon \quad \text{for all} \quad x\in(x_0,x_n)$$

The above theorem clearly says that there always exists a polynomial $\phi(x)$ which is sufficiently close to $f(x)$.

4. Lagrange and Newton Interpolation:

Let us assume that $f(x)$ is a function defined and continuous on $[a,b]$ and we have n + 1 points.

$a \le x_0 < x_1 < x_2 < ... < x_{n-1} < x_n \le b$, at these $(n+1)$ points values of $f(x)$ are known.

We want to find the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \qquad \text{(i)}$$

which satisfies the conditions

$$P(x_i) = f(x_i) \qquad i = 0,1,2,...n \qquad \text{(ii)}$$

Putting $(n+1)$ point $x_0, x, ..., x_n$ in eqn. (i) & using (ii) we get

$$a_0 + a_1 x_0 + a_2 x_0^2 + ... + a_n x_0^n = P(x_0) = f(x_0)$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + ... + a_n x_1^n = P(x_1) = f(x_1)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_0 + a_1 x_n + a_2 x_n^2 + ... + a_n x_n^n = P(x_n) = f(x_n)$$

This system of equation has a unique solution or polynomial $P(x)$ exists if the Vandermonde's determinant

$$V(x_0, x_1, ..., x_n) = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} \ne 0$$

Uniqueness : The polynomial obtained above is unique.

Suppose that there is another polynomial $P^*(x)$ which also satisfies

$$P^*(x_i) = f(x_i) \qquad i = 0,1,2,...,n$$

Consider the polynomial

$Q(x) = P(x) - P^*(x)$

Since $P(x)$ & $P^*(x)$ are polynomials of degree *n*.

$\therefore$ $Q(x)$ is also a polynomial of degree $\leq n$.

Also at $x_0, x, ..., x_n$

$Q(x_i) = P(x_i) - P^*(x_i)$

$= f(x_i) - f(x_i) \quad i = 0,1,...,n$

$= 0$

$\Rightarrow$ $Q(x)$ is a polynomial of degree $\leq n$ which has *n* + 1 distinct roots $x_0, x, ..., x_n$.

$\Rightarrow$ $Q(x) = 0$ [$\because$ a poly. of degree $\leq n$ cannot have $(n+1)$ roots].

5. Linear Interpolation:

Let $P(x) = a_1 x + a_0$ where a_0, a_1 are constants which satisfies the interpolating conditions

$f(x_0) = P(x_0)$

$f(x_1) = P(x_1)$

$\therefore$ $a_1 x_0 + a_0 = P(x_0) = f(x_0)$ (i)

$a_1 x_1 + a_0 = P(x_1) = f(x_1)$ (ii)

Eliminating (or obtaining) a_0, a_1 from (i), (ii) we obtain the required linear interpolating polynomial.

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0 \qquad \text{(iii)}$$

6. Lagrange's Interpolation:

Expanding the determinant equation $\begin{vmatrix} P(x) & x & 1 \\ f(x) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$ we get

$$P(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0) = 0$$

$$\Rightarrow \qquad P(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{x - x_0}{(x_1 - x_0)} f(x_1)$$

$$P(x) = \ell_0(x) f(x_0) + \ell_1(x) f(x_1) \qquad \text{(iv)}$$

where

$$\ell_0(x) = \frac{x - x_1}{(x_0 - x_1)}, \quad \ell_1(x) = \frac{x - x_0}{(x_1 - x_0)}$$

$\ell_0(x)$ & $\ell_1(x)$ are called the Lagrange fundamental polynomial satisfying

$$\ell_0(x) + \ell_1(x) = 1$$

$$\ell_0(x_0) = 1, \quad \ell_0(x_1) = 0$$

$$\ell_1(x_0) = 0, \quad \ell_1(x_1) = 1$$

In general $\ell_i(x_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

Polynomial represented by equation (iv) is called as Lagrange's Interpolating polynomial.

7. Newton's Dividend Difference Interpolation:

Consider the determinant Equation.

$$\begin{vmatrix} P(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0 \qquad \text{(i)}$$

Expanding along first how we get

$$P(x)(x_0 - x_1) - x(f(x_0) - f(x_1)) + 1(x_1 f(x_0) - x_0 f(x_1)) = 0$$

$$\Rightarrow \quad P(x)(x_0 - x_1) - xf(x_0) + xf(x_1) + x_1 f(x_0) - x_0 f(x_1) - x_0 f(x_0) + x_0 f(x_0) = 0$$

$$\Rightarrow \quad P(x)(x_0 - x_1) + x(f(x_1) - f(x_0)) - x_0(f(x_1) - f(x_0))$$

$$+(x_1 - x_0) f(x_0) = 0$$

$$\Rightarrow \quad P(x)(x_0 - x_1) = (x_0 - x_1) f(x_0) - (x - x_0)(f(x_1) - f(x))$$

$$\Rightarrow \quad P(x) = \frac{(x_0 - x_1) f(x_0)}{(x_0 - x_1)} - \frac{(x - x_0)(f(x_1) - f(x_0))}{-(x_1 - x_0)}$$

$$\Rightarrow \quad P(x) = f(x_0) + (x - x_0)\frac{(f(x_1) - f(x_0))}{(x_1 - x_0)}$$

$$\Rightarrow \quad P(x) = f(x_0) + (x - x_0) f[x_0, x_1] \qquad \text{(ii)}$$

The ratio $f[x_0, x_1] = \dfrac{f(x_1) - f(x_0)}{(x_1 - x_0)}$ is called as first dividend difference of $f(x)$ relative to x_0 and x_1 Polynomial $P(x)$ represented by (ii) is Newton's Dividend Difference Interpolating Polynomial.

I.Q.1

Interpolation

Example 1: Given $f(2)=4,\ f(2.5)=5.5$, find the linear interpolating polynomial using

(i) Lagrange Interpolation

(ii) Newton's Dividend difference interpolation

Hence find an approximate value of $f(2.2)$

Solution: We have

$$x_0 = 2, \qquad f(x_0)=4$$

$$x_1 = 2.5, \qquad f(x_1)=5.5$$

(i) The Lagrange fundamental polynomials are given by

$$\ell_0(x)=\frac{x-x_1}{x_0-x_1}=\frac{x-2.5}{-0.5}=-2(x-2.5)$$

$$\ell_1(x)=\frac{x-x_0}{x_1-x_0}=\frac{x-2}{0.5}=2(x-2)$$

$$P_1(x)=\ell_0(x)f(x_0)+\ell_1(x)f(x_1)$$

$$=-2(x-2.5)(4)+2(x-2)(5.5)$$

$$=(-2x+5)(4)+(2x-4)(5.5)$$

$$=-8x+20+11x-22$$

$$=3x-2.$$

(ii) Newton's dividend difference interpolation

We have

$$f[x_0,x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{5.5-4}{0.5}=3$$

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

$$= 4 + (x-2)(3) = 4 + 3x - 6 = 3x - 2$$

$$f(2.2) \approx P_1(2.2) = 3 \times (2.2) - 2$$

= 6.6 – 2 = 4.6.

8. Truncation Error Bounds:

The polynomial $P(x)$ coincides with the function $f(x)$ at x_0 and x_1 and it deviates at all other points, in the interval (x_0, x_1) This deviation is called the truncation error.

$$E_1(f, x) = f(x) - P(x) \qquad \text{(i)}$$

we will develop bound on E_1

Useful result (Rolle's Theorem) if $g(x)$ is a continuous function on some interval $[a,b]$ and differentiable on (a,b) and $g(a) = g(b) = (0)$, than there exist at least one point $\xi \in (a,b)$ such that $g'(\xi) = 0$

If $x = x_0$ or $x = x_1$ $E_1(f, x) = 0$

If $x \in (x_0, x_1)$ define a function

$$g(t) = f(t) - P(t) - \left[f(x) - P(x)\right] . \frac{(t - x_0)(t - x_1)}{(x - x_0)(x - x_1)} \qquad \text{(ii)}$$

at $t = x_0, x_1, x$

clearly $g(t) = 0$

$\therefore$ we can say that $g(t)$ satisfies the conditions of Rolle's Theorem.

Applying Rolle's theorem on the intervals (x_0, t) and (t, x_1) separately we get

$$g'(\xi_1) = 0 \qquad x_0 < \xi_1 < t$$

$$g'(\xi_2) = 0 \qquad t < \xi_2 < x_1$$ (By the conclusion of Rolle’s Theorem).

where

$$g'(t) = f'(t) - P'(t) - \left(f(x) - P(x)\right)\frac{\left[2t - (x_0 + x_1)\right]}{(x - x_0)(x - x_1)}$$

$$g''(t) = f''(t) - P''(t) - \left[f(x) - P(x)\right] \times 2/(x - x_0)(x - x_1) \qquad \text{(iii)}$$

Now $g'(t)$ also satisfies the conditions of Rolle’s Theorem.

Applying Rolle's thm for $g'(t)$ on the interval (ξ_1, ξ_2) we get

$$g''(\xi) = 0 \qquad \xi_1 < \xi < \xi_2 \quad \text{or} \quad x_0 < \xi < x_1$$

$$(\because\ x_0 < \xi_1 < \xi < \xi_2 < x_1)$$

$$\Rightarrow \qquad 0 = f''(\xi) - \frac{2\left(f(x) - P(x)\right)}{(x - x_0)(x - x_1)}$$

$$\Rightarrow \qquad f''(\xi) \times (x - x_0)(x - x_0) \times \frac{1}{2} = f(x) - P(x)$$

$$\Rightarrow \qquad f(x) = P(x) + \frac{1}{2}(x - x_0)(x - x_1) f''(\xi) \qquad \text{(iv)}$$

Using (iv)

Truncation error in linear interpolation is

$$E_1(f, x) = \frac{1}{2}(x - x_0)(x - x_1) f''(\xi) \qquad \left(\because\ E_1(f, x) = f(x) - P(x)\right)$$

$\therefore \quad |f(x)-P(x)| = \left|\frac{1}{2}(x-x_0)(x-x_1)f''(\xi)\right| \le \frac{1}{2}\max_{x_0\le x\le x_1}|(x-x_0)(x-x_1)f''(\xi)|$

$$\le \frac{1}{2}\max_{x_0\le x\le x_1}|(x-x_0)(x-x_1)|M \qquad \text{(v)}$$

where *M* is s.t. $|f''(x)| \le M \quad \text{for all } x\in[x_0,x_1]$

man value of $(x-x_0)(x-x_1)$ occur at $x = \frac{(x_0+x_1)}{2}$

Let $\quad w(x) = |(x-x_0)(x-x_1)| = -(x-x_0)(x-x_1)$

$$w'(x) = (x-x_0)+(x-x_1), \qquad w''(x) = -2$$

$$= -2x+(x_0+x_1)$$

Put $\quad w'(x)=0 \Rightarrow x = \frac{x_0+x_1}{2}$

$\therefore$ maximum value of $|(x-x_0)(x-x_1)|$ occur at $\frac{x_0+x_1}{2}$ and maximum value of $w(x)$ is $w(x) = \frac{(x_1-x_0)^2}{4}$

$\therefore$ From (v) we get

$$|f(x)-P(x)| \le \frac{1}{8}(x_1-x_0)^2 M \qquad \text{(vi)}$$

Value addition
Equation (vi) can be used to construct a table of values for a function $f(x)$ for equally spaced points $x_i = a+ih \quad i=0,1,\ldots,n,\ h=(b-a)n$ This can help us in setting maximum absolute truncation error using the linear interpolating polynomial $P(x)$ is less than a given error tolerance $\varepsilon > 0$ as $\quad x_1-x_0=h$ $\Rightarrow$ from equation (vi)

$$\frac{h^2}{8}\max_{a\le x\le b}\left|f''(x)\right|\le\varepsilon .$$

I.Q.2

I.Q. 3

Example.2 : Using the data sin(0.1) = 0.09983 and sin(0.2) = 0.19867, find an approximate value of sin(0.15) by Lagrange Interpolation. Also obtain a bound on the truncation error.

Solution: Using Lagrange Interpolation

$$P(x)=\frac{x-x_1}{(x_0-x_1)}f(x_0)+\frac{x-x_0}{(x_1-x_0)}f(x_1) \qquad \text{(i)}$$

Here $x_0=0.1,\quad x_1=0.2,\quad$ put x = 0.15, $\quad f(x)=\sin x$

$\therefore$ from (i)

$$P(0.15)=\frac{0.15-0.2}{0.1-0.2}\times(0.09983)+\frac{0.15-0.1}{0.2-0.1}\times(0.19867)$$

$$=(0.5)(0.09983)+(0.5)(0.19867)=0.14925$$

$\therefore$ The truncation error is

$$E_1(f;x)=\frac{(x-0.1)(x-0.2)}{2}(-\sin\xi)$$

$$\left[\because\ E_1(f,x)=\frac{1}{2}(x-x_0)(x-x_1)f''(\xi)\right]$$

where

$$0.1<\xi<0.2$$

As sin x is a increasing function in $\left[0,\frac{\pi}{2}\right]$

$\therefore$ The maximum value of $|-\sin\xi|\, \xi \in [0.1, 0.2]$ is sin (0.2) = 0.19867

Thus $|E_1(f;x)| \le \left|\frac{(0.15-0.1)(0.15-0.2)}{2}\right|(0.19867)$

$$=(0.19867)(0.00125) \approx 0.00025$$

Example 3: Determine the step size *h* that can be used in the tabulation of $f(x)=\sin x$ in the interval [1, 3] so that the linear interpolation will be correct to four decimal places after rounding.

Solution: We have

$$f(x)=\sin x$$

$$f'(x)=\cos x,\ f''(x)=-\sin x$$

and $\max\limits_{1\le x\le 3} |-\sin x| = 1 \qquad \left(\because 1 \le \frac{\pi}{2} \le 3 \ \&\ \sin\frac{\pi}{2}=1\right)$

According Problem

$$\frac{h^2}{8}\max_{1\le x\le 3}|f''(x)| \le \varepsilon$$

Here $\varepsilon = 5\times 10^{-5}$

$\therefore \qquad \frac{h^2}{8}\times 1 \le 5\times 10^{-5}$

$\Rightarrow \qquad h \le 0.02$

9. Quadratic Interpolation:

Here $n=2$, and we want to determine a polynomial $P_2(x)=a_0+a_1x+a_2x^2$

where a_0, a_1, a_2 are arbitrary constants satisfying the interpolatory condition $f(x_0)=P_2(x_0), f(x_1)=P_2(x_1)\ f(x_2)=P_2(x_2)$.

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$$

$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2$$

$$f(x_2) = a_2 + a_1 x_2 + a_2 x_2^2.$$

Eliminating a_0, a_1, a_2, we obtain the required quadratic interpolating polynomial as

$$\begin{vmatrix} P_2(x) & 1 & x & x^2 \\ f(x_0) & 1 & x_0 & x_0^2 \\ f(x_1) & 1 & x_1 & x_1^2 \\ f(x_2) & 1 & x_2 & x_2^2 \end{vmatrix} = 0$$

Expanding the determinant we obtain

$$P_2(x) D_0 - f(x_0) D_1 + f(x_1) D_2 - f(x_2) D_3 = 0$$

where

$$D_0 = \begin{vmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x_0 - x_1)(x_1 - x_2)(x_2 - x_0)$$

$$D_1 = \begin{vmatrix} 1 & x & x^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x - x_1)(x_1 - x_2)(x_2 - x)$$

$$D_2 = \begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x - x_0)(x_0 - x_2)(x_2 - x)$$

$$D_3 = \begin{vmatrix} 1 & x & x^2 \\ 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \end{vmatrix} = (x - x_0)(x_0 - x_1)(x_1 - x)$$

$$\therefore \qquad P_2(x) = \frac{D_1}{D_0} f(x_0) - \frac{D_2}{D_0} f(x_1) + \frac{D_3}{D_0} f(x_3)$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \ell_0(x) f(x_0) + \ell_1(x) f(x_1) + \ell_2(x) f(x_2)$$

The truncation error of the Lagrange quadratic interpolating polynomial is

$$E_2(f;x) = f(x) - P_2(x)$$

Using Rolle's Thm we get expression for $f(x)$ estimated by $P_2(x)$ and the error adjustment we get

$$f(x) = P_2(x) + \frac{1}{3!}(x-x_0)(x-x_1)(x-x_2) f''(\xi)$$

where $\xi \in (x_0, x_2)$

$\Rightarrow$ Truncation error

$$E_2(f;x) = f(x) - P_2(x) = \frac{1}{3}(x-x_0)(x-x_1)(x-x_2) f'''(\xi)$$

and bound on the truncation error is

$$\left| f(x) - P_2(x) \right| \le \frac{1}{6} M_3 \left[\max_{x_0 \le x \le x_2} \left| (x-x_0)(x-x_1)(x-x_2) \right| \right]$$

where $M_3 = \max_{x_0 \le x \le x_2} \left| f'''(x) \right|$

10. Higher Order Interpolation:

The Lagrange fundamental polynomials of degree *n* based on *n* + 1 distinct points $a \le x_0 < x_1 < x_2 < \ldots\ldots < x_n \le b$ and which satisfy the condition

$$l_i\left(x_j\right) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \ne j \end{cases}$$

Can be written in the form

$$l_i\left(x\right) = \frac{\left(x - x_0\right)\left(x - x_1\right)\ldots\left(x - x_{i-1}\right)\left(x - x_{i+1}\right)\ldots\left(x - x_n\right)}{\left(x_i - x_0\right)\left(x_i - x_1\right)\ldots\left(x_i - x_{i-1}\right)\left(x_i - x_{i+1}\right)\ldots\left(x_i - x_n\right)}$$

$i = 0, 1, \ldots, n.$ (i)

An alternative way of writing $l_i\left(x\right)$ can be

$$l_i\left(x\right) = \frac{w\left(x\right)}{\left(x - x_0\right)w'\left(x_i\right)}$$

where

$$w\left(x\right) = \left(x - x_0\right)\left(x - x_1\right)\ldots\left(x - x_n\right)$$

$$w'\left(x_i\right) = \left(x_i - x_0\right)\left(x_i - x_1\right)\ldots\left(x_i - x_{i-1}\right)\left(x_i - x_{i+1}\right)\ldots\left(x_i - x_n\right)$$

and $w'\left(x\right)$ is the derivative of $w\left(x\right)$ with respect to *x*.

thus the polynomial

$$P\left(x\right) = \sum_{i=0}^{n} l_i\left(x\right) f\left(x_i\right)$$

is the Lagrange interpolating polynomial of degree *n*.

The truncation error in the Lagrange interpolation is given by

$$E_n\left(f; x\right) = f\left(x\right) - P\left(x\right)$$

Since $E_n(f;x)=$ 0 at $x = x_i$ $\quad i = 0, 1, 2, \ldots, n$

than for $x\in[a,b]$ and $x\neq x_i$, we define a function $g(t)$ as

$$g(t)=f(t)-P(t)-\left[f(x)-P(x)\right]\frac{(t-x_0)(t-x_1)\ldots(t-x_n)}{(x-x_0)(x-x_1)\ldots(x-x_n)} \qquad \text{(ii)}$$

We observe that $g(t)=$ 0 at $t=x$ and $t=x_i,\ i=0,1,\ldots,n$

Applying the Rolle's theorem repeatedly for $g(t),g'(t),\ldots$ and $g^{(n)}(t)$ we obtain $g^{(n+1)}(\xi)=0$ where ξ is some point such that

$$\min(x_0,x_1,\ldots\ldots x_n,x)<\xi<\max(x_0,x_1,\ldots\ldots x_n,x)$$

Differentiating $g(t)$ represented by equation (ii) $(n+1)$ times with respect to *t*, we get

$$g^{(n+1)}(t)=f^{(n+1)}(t)-\frac{(n+1)!\left[f(x)-P(x)\right]}{(x-x_0)(x-x_1)\ldots.(x-x_n)} \qquad \text{(iii)}$$

[$(n+1)$th derivative of $P(x)$ is zero as $P(x)$ is a polynomial of degree *n*]

Setting $g^{(n+1)}(\xi)=0$ and solving equation (iii) for $f(x)$, we get

$$f(x)=P(x)+\frac{w(x)}{(n+1)!}f^{(n+1)}(\xi)$$

Hence, the truncation error in Lagrange interpolation is given by

$$E_n(f;x)=\frac{w(x)}{(n+1)!}f^{(n+1)}(\xi).$$

Example4. Given that $f(0)=1, f(1)=3, f(3)=55$, find the unique polynomial of degree 2 or less, which fits the given data using.

(i) Lagrange Method

(ii) Newton divided difference method

Also find the bound on the error.

Ans (i) We have $x_0 = 0, x_1 = 1, x_2 = 3, f_0 = 1, f_1 = 3$ and $f_2 = 55$. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}(x^2 - 4x + 3)$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{x(x-3)}{(1)(-2)} = \frac{1}{2}(3x + x^2)$$

$$l_2(x) = \frac{(x - x_0)(x - x_2)}{(x_2 - x_0)(x_2 - x_1)} = \frac{x(x-1)}{3(2)} = \frac{1}{6}(x^2 - x).$$

Hence, the Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2$$

$$= \frac{1}{3}(x^2 - 4x + 3) + \frac{3}{2}(3x - x^2) + \frac{55}{6}(x^2 - x)$$

$$= 8x^2 - 6x + 1.$$

(ii) The divided differences are given by

$$f[0,1] = \frac{3-1}{1-0} = 2,\ f[1,3] = \frac{55-3}{3-1} = 26,$$

$$f[0,1,3] = \frac{26-2}{3-0} = 8.$$

The Newton divided difference interpolating polynomial becomes

$$P_2(x) = f[0] + (x-0) f[0,1] + (x-0)(x-1) f[0,1,3]$$

$$= 1 + 2x + 8x(x-1) = 8x^2 - 6x + 1.$$

Bound on the Error :

We have, $\left|E_2(f;x)\right| \le \frac{1}{6} M_3 \left[\max\limits_{0 \le x \le 3} \left|x(x-1)(x-3)\right|\right]$

$$= \frac{1}{6}(2.1126) M_3 = 0.3521\ M_3$$

where $M_3 = \max\limits_{0 \le x \le 3} \left|f'''(x)\right|$ and since the maximum of $\left|x(x-1)(x-3)\right|$ occurs at x = 2.2152.

Example 5. The following values of the function $f(x) = \sin x + \cos x,$ are given

x	10°	20°	30°
$f(x)$	1.1585	1.2817	1.3660

Construct the quadratic interpolating polynomial that fits the data. Hence, find $f(\pi/12)$. Compare with the exact value.

Since the value of f at π/12 radians is required, we convert the data into radian measure. We have

$$x_0 = 10° = \frac{\pi}{18} = 0.1745,\ x_1 = 20° = \frac{\pi}{9} = 0.3491,$$

$$x_2 = 30° = \frac{\pi}{6} = 0.5236.$$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.3491)(x-0.5236)}{(-0.1746)(-0.3491)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.1745)(x-0.5236)}{(-0.1746)(-0.1745)}$$

$$= -32.8616\left(x^2 - 06981\,x + 0.0914\right)$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0.1745)(x - 0.3491)}{(0.3491)(0.1745)}$$

$$= 16.4155\left(x^2 - 0.5236\,x + 0.0609\right).$$

Hence, the Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = 16.4061\left(x^2 - 0.8727\,x + 0.1828\right)(1.1585)$$

$$-32.8616\left(x^2 - 0.6981\,x + 0.0914\right)(1.2817)$$

$$+16.4155\left(x^2 - 0.5236\,x + 0.0609\right)(1.3660)$$

$$= -0.6887\,x^2 + 1.0751\,x + 0.9903$$

Hence, $f(\pi/12) = f(0.2618) = 1.2246$,

The exact value is $f(0.2618) = \sin(0.2618) + \cos(0.2618) = 1.2247$.

Example 6. Construct the divided difference table for the data

x	0.5	1.5	3.0	5.0	6.5	8.0
f (*x*)	1.625	5.875	31.0	131.0	282.125	521.0

Hence, find the interpolating polynomial and an approximation to the value of *f* (7). We have the following divided difference table

x	*f (x)*	*first order d.d.*	*second order d.d.*	*third order d.d.*	*fourth order d.d.*
0.5	1.625				
		4.25			
1.5	5.875		5.0		
		16.75		1.0	
3.0	31.000		9.5		0
		50.00		1.0	
5.0	131.000		14.5		0
		100.75		1.0	

6.5	282.125		19.5	
		159.25		
8.0	521.000			

We write the divided difference interpolating polynomial as

$$f(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$$+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$= 1.625 + (x - 0.5)(4.25) + 5(x - 0.5)(x - 1.5)$$

$$+ (x - 0.5)(x - 1.5)(x - 3.0)$$

$$= (1.625 - 2.125 + 3.75 - 2.25) + x(4.25 - 10.0 + 6.75)$$

$$+ x^2(5 - 5) + x^3$$

$$= x^3 + x + 1.$$

Hence, $f(7.0) = 351.$

11. Finite Difference Operator:

Let the points $x_1, x_2, x_3, ..., x_n$ be equally spaced

$\therefore \quad x_i = x_0 + ih, \quad i = 0, 1, 2, ..., n, \quad h = x_1 - x_0 = x_2 - x_1 = x_n - x_{n-1}$

We define the following operators

(1) Shift Operator $Ef(x_i) = f(x_i + h)$

(2) Forward difference operator $\Delta f(x_i) = f(x_i + h) - f(x_i)$

(3) Backward difference operator $\nabla f(x_i) = f(x_i) - f(x_i - h)$

(4) Central difference operator $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

(5) The average operator $\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2} \right) + f\left(x_i - \frac{h}{2} \right) \right]$

Relation between the operators (take h = 1).

(i) $\Delta f(x_i) = \nabla f(x_i + 1) = \delta f\left(x_i + \frac{1}{2} \right) = f(x_i + 1) - f(x_i)$

(ii) (a) $\Delta f(x_i) = f(x_i + 1) - f(x_i)$

$= Ef(x_i) - f(x_i)$

$= (E - 1) f(x_i)$

$\Rightarrow$ $\boxed{\Delta = E - 1}$

(b) $\nabla f(x_i) = f(x_i) - f(x_i - 1)$

$= f(x_i) - E^{-1} f(x_i - 1)$

$= (1 - E^{-1}) f(x_i)$

$\Rightarrow$ $\boxed{\nabla = 1 - E^{-1}}$

$\Rightarrow$ $\boxed{E^{-1} = 1 - \nabla}$

(iii) We can also have

$\Delta^n = (E - 1)^n =$

$\nabla^n = (1 - E^{-1})^n =$

Table showing relationship between the operators

	E	Δ	∇	δ
E	E	$\Delta + 1$	$(1-\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$
Δ	E – 1	Δ	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

∇	1–E⁻¹ → $1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$
δ	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ
μ	$\frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}})$	$\left(1+\frac{1}{2}\Delta\right)(1+\Delta)^{\frac{1}{2}}$	$\left(1-\frac{1}{2}\nabla\right)(1-\Delta)^{-\frac{1}{2}}$	$\sqrt{1+\frac{1}{4}\delta^2}$

Example 7. Show that

(i) $\delta=\nabla(1-\nabla)^{-\frac{1}{2}}$ (ii) $\mu=\left(1+\frac{\delta^2}{4}\right)^{1/2}$

(iii) $E=1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$ (iv) $\nabla=-\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$

Proof:

(1) R.H.S. = $\nabla(1-\nabla)^{-\frac{1}{2}}$

$$=(1-E^{-1})[1-(1-E^{-1})]^{-\frac{1}{2}} \qquad (\therefore \nabla=1-E^{-1})$$

$$=(1-E^{-1})(E^{-1})^{-\frac{1}{2}}=(1-E^{-1})(E^{\frac{1}{2}})$$

$$=E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

δ = L.H.S.

(ii) R.H.S. = $\left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2}}$

$$=\left[1+\frac{1}{4}(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2\right]^{\frac{1}{2}} \qquad \because \delta=E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

$$=\left[1+\frac{1}{4}(E-E^{-1}-2)\right]^{\frac{1}{2}} \qquad \delta^2=(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2$$

$$= E-E^{-1}-2$$

$$=\left[\frac{4+E+E^{-1}-2}{4}\right]^{1/2}$$

$$=\frac{1}{2}\left[E+E^{-1}+2\right]^{1/2}$$

$$=\frac{1}{2}\left[(E^{\frac{1}{2}}+E^{-\frac{1}{2}})^2\right]^{1/2}$$

$$= \frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}})=\mu=L.H.S.$$

(iii) R.H.S. = $1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$

$$= 1+\frac{1}{2}(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2+(E^{\frac{1}{2}}-E^{-\frac{1}{2}})\times\frac{(E^{\frac{1}{2}}+E^{-\frac{1}{2}})}{2}$$

$$\left[\because \mu=\frac{1}{2}(E^{\frac{1}{2}}+E^{-\frac{1}{2}}) = \sqrt{1+\frac{\delta^2}{4}}\ \right]$$

$$=1+\frac{1}{2}(E-E^{-1}-2)+\frac{1}{2}(E-E^{-1})$$

$$=\frac{1}{2}[2+E+E^{-1}-2+E-E^{-1}]$$

$$=\frac{1}{2}\times 2E=E=L.H.S.$$

(iv) R.H.S.= $-\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{1}{4}\delta^2}$

$$=-\frac{1}{2}(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2+(E^{\frac{1}{2}}-E^{-\frac{1}{2}})\frac{(E^{\frac{1}{2}}+E^{-\frac{1}{2}})}{2} \quad \text{(using relation (ii))}$$

$$=-\frac{1}{2}(E+E^{-1}-2)+\frac{1}{2}(E-E^{-1})$$

$$=\frac{1}{2}[-E-E^{-1}-2+E-E^{-1}]$$

$$=\frac{1}{2}[2-2E^{-1}]= [1-E^{-1}]=\nabla=L.H.S.$$

Similarly we can prove other relations given in the table of operators.

Example 8. Construct the difference table for the sequence of values $f(x) = (0, 0, 0, \varepsilon, 0, 0, 0)$ where ε is an error. Also show that

(i) The error spreads and increases in magnitude as the order of differences is increased.

(ii) The error in each columns have binomial coeff.

Solution:

f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
0	0					
0	0	0	ε	–4ε		
0	ε	ε	–3ε	6ε	10ε	–20ε
ε	–ε	–2ε	3ε	–4ε	–10ε	
0	0	ε	–ε			
0	0	0				
0						

Clearly the table shows

(i) How the error is effected by order of differences

(ii) Difference columns have binomial coefficient.

(iii) Maximum error occurs directly opposite the entry where the function value is in error.

I.Q.4

I.Q.5

12. Relation between Newton's divided differences in terms of forward, backward and central difference operators:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{h} = \frac{1}{h}\Delta f_0$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{1}{h}\Delta f_1 - \frac{1}{h}\Delta f_0}{2h}$$

$$= \frac{1}{2!h^2}\Delta^2 f_0 .$$

We use principle of Mathematical Induction. Assume that

$$f[x_0, x_1, x_2, ..., x_{n-1}] = \frac{1}{(n-1)!h^{n-1}} \Delta^{n-1} f_0$$

$$f[x_0, x_1, x_2, ..., x_{n-1}, x_n] = \frac{f[x_1, x_2, ..., x_n] - f[x_0, x_1, ..., x_{n-1}]}{x_n - x_0}$$

$$= \frac{\frac{1}{(n-1)^1 h^{n-1}} \Delta^{n-1} f_1 - \frac{1}{(n-1)^1 h^{n-1}} \Delta^{n-1} f_0}{nh}$$

$$= \frac{\frac{\Delta^{n-1}}{(n-1)!h^{n-1}} [f_1 - f_0]}{nh} = \frac{\Delta^{n-1}.\Delta f_0}{(n-1)!h^n \times n} = \frac{\Delta^n f_0}{n!h^n}$$

Similarly we can prove that

(b) $f[x_0, x_1, ..., x_n] = \frac{1}{n!h^n} \nabla^n f_n$

(c) $f[x_0, x_1, ..., x_{2n+1}] = \frac{1}{(2m)!h^{2m}} \delta^{2\text{m}} . f_m$ when $n = 2m$

$f[x_0, x_1, ..., x_{2m+1}] = \frac{1}{(2n+1)!h^{2m+1}} \delta^{2\text{m}+1} . f_{m+\frac{1}{2}}$ when $n = 2m+1$

13. Relations between Difference operators and differential operators:

$$\Delta f(x) = f(x+h) - f(x)$$

$$= \left[f(x) + hf'(x) + \frac{h^2}{2} f''(x) + ... \right] - f(x) \quad \text{(Using Taylor's theorem)}$$

$$= hf'(x) + \frac{h^2}{2} f''(x)$$

$\Rightarrow \quad \Delta f(x) \approx hf'(x)$ (where h is so small that terms containing h^2 and higher powers of h can be neglected).

$\Rightarrow \quad f'(x) \approx \frac{\Delta f(x)}{h}$ error is of $O(h)$.

Similarly

$$\Delta^2 f(x) \approx h^2 f''(x)$$

Again $\nabla f(x) = f(x) - f(x-h)$

$$= f(x) - \left[f(x) - hf'(x) + \frac{h^2}{2} f''(x) + ... \right]$$

$$= f(x) - f(x) + hf'(x) + \frac{h^2}{2} f''(x) + ...$$

$$= hf'(x) + O(h^2)$$

Neglecting the terms containing h^2 and higher powers of h.

$$\therefore \quad \nabla f(x) \approx hf'(x) \Rightarrow f'(x) \approx \frac{\nabla f(x)}{h}$$

$$\nabla^2 f(x) \approx h^2 f''(x) \Rightarrow f''(x) \approx \frac{\nabla^2 f(x)}{h^2}$$

14. Interpolating Polynomials using Finite Differences:

14.1. Gregory-Newton Forward Difference Interpolation:

Relation between divided difference and forward difference operator is

$$f[x_0, x_1, ..., x_n] = \frac{1}{n! h^n} \Delta^n f_0 \qquad (1)$$

Divided difference interpolating polynomial is written as

$$P_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + ... + (x - x_0(x - x_1)....(x - x_{n-1}) \; f[x_{0,} x_1, ..., x_n] \qquad (2)$$

Using (1) in (2), Interpolating polynomial can be written as

$$P(x) = P_n(x) = f_0 + (x - x_0)\frac{\Delta f_0}{h} + \frac{(x - x_0(x - x_1)\Delta^2 f_0}{2! h^2} + ...$$

$$+ \frac{(x - x_0)(x - x_1) + ...(x - x_{n-1})}{n! h^n} \Delta^n f_0 \qquad (3)$$

Polynomial P(x) expressed by equation (3) is known as Gregory-Newton forward differences interpolating polynomial.

Now if we put $u = \frac{(x - x_0)}{h} \Rightarrow hu = (x - x_0)$ and since $x_0, x_1, ..., x_n$ are equally spaced Points that is $x_i = x_0 + ih$

$\Rightarrow \quad (x - x_i) = (x - (x_0 + ih)) = [(x - x_0) - ih] = (uh - ih) = (u - i)h$.

$\therefore$ Equation (3) and the truncation error can be written as

$$P(x_0 + hu) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + ... + \frac{u(u-1)....(u-n+1)}{n!}\Delta^n f_0 \qquad (4)$$

$$= \sum_{i=0}^{n} \binom{u}{i} \Delta^i f_0 \quad \text{where } \binom{u}{i} = {}^uC_i$$

and truncation error

$$E_n(f; x) = \frac{u(u-1)....(u-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi) \qquad (5)$$

Alternative Method for the derivation of interpolating polynomial expressed by equation (4) as

$$f(x) = f\left(x_0 + \left(\frac{x - x_0}{h}\right) \times h\right)$$

$$= f(x_0 + uh) \qquad \left(\because \frac{x - x_0}{h} = u\right)$$

$$= E^u f(x_0)$$

$$= (1 + \Delta)^u f(x_0) \qquad (\because E = 1 + \Delta)$$

$$= \left[1 + u\Delta + \frac{u(u-1)}{2!}\Delta^2 + ... + \frac{u(u-1)...(u-n+1)\Delta^n}{n!} + ...\right] f_0$$

$$f(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_0 + ... + \frac{u(u-1)...(u-n+1)}{n!}\Delta^n f_0 + ... \quad (6)$$

Neglecting the $\Delta^{n+1} f_0$ and higher order differences in equation (6) we get.

$$f(x) = f_0 + u\Delta f_0 + \frac{u(u-1)}{2!}\Delta^2 f_1 + ... + \frac{u(u-1)...(u-n+1)}{n!}\Delta^n f_0 \quad (7)$$

equation (7) is again the form of Gregory Newton forward difference interpolating polynomial.

14.2. Gregory-Newton Backward Difference Interpolation:

We observe that Newton-interpolation with divided differences in terms of backward differences should be in terms of the differences at the end point x_n.

$$f(x) = f\left(x_n + \frac{(x - x_n)}{h} \times h\right) = f(x_n + hu)$$

as we take $\frac{(x - x_n)}{h}$ = $u \Rightarrow x - x_n = hu$

$$\Rightarrow \quad f(x) = f(x_n + uh)$$

$$= E^u f(x_n)$$

$$= (1-\nabla)^{-u} f(x_n) \qquad [\because E = (1-\nabla)^{-1}]$$

$$= 1 + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + ... + \frac{u(u+1) - (u+n-1)}{n!}\nabla^n f(x_n) + ... \quad (8)$$

Neglecting the difference $\nabla^{n+1} f(x_n)$ and higher order difference we get the interpolating polynomial as

$$P(x) = P(x_n + hu) = f(x)$$

$$= f_n + u\nabla f_n + \frac{u(u+1)}{2!}\nabla^2 f_n + ... + \frac{u(u+1)....(u+n-1)}{n!} \times \nabla^n f_n.$$

$$= \sum_{i=0}^{n} (-1)^i \binom{-u}{i} \nabla^i f_n \qquad (9)$$

where $\binom{-u}{i} = {}^{-u}C_i$.

Polynomial expressed by relation (9) is known as Gregory-Newton backward difference interpolating polynomial and the truncation error is

$$E_n(f;x) = \frac{u(u+1)....(u+n)}{(n+1)!} h^{n+1} f^{n+1}(\xi)$$

I.Q. 6

Example9. For the following data, calculate the differences and obtain the forward and backward difference polynomials. Interpolate at x = 0.25 and x = 0.35

x	0.1	0.2	0.3	0.4	0.5
f(x)	1.40	1.56	1.76	2.00	2.28

Solution: The difference table is obtained as

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

The forward difference polynomial is given by

$$P(x) = 1.4 + (x - 0.1)\frac{0.16}{0.1} + \frac{(x-0.1)(x-0.2)}{2} \times \frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28.$$

The backward difference polynomial is obtained as

$$P(x) = 2.28 + (x - 0.5)\frac{0.28}{0.1} + \frac{(x-0.5)(x-0.4)}{2}\frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28.$$

Both the polynomials are same.

$\therefore \quad f(0.25) = 1.655, \qquad f(0.35) = 1.875.$

We can obtain the interpolated values directly also. So for x = 0.25 we choose x_0 = 0.2 and write

$$u = \frac{x - x_0}{h} = \frac{0.25 - 0.2}{0.1} = 0.5$$

$$\Rightarrow \quad f(0.25) = f(0.2) + (0.5)\Delta f(0.2) + \frac{1}{2}(0.5)(-0.5)\Delta^2 f(0.2)$$

= 1.56 +(0.5(0.20) – (0.125)(0.04) = 1.655

For x = 0.35 we choose x_n = 0.4 and in backward differences as

$$u = \frac{x - x_n}{h} = \frac{0.35 - 0.4}{0.1} = -0.5$$

and $f(0.35) = f(0.4) + (-0.5)\nabla f(0.4) + \frac{1}{2}(0.5)(0.5)\nabla f(0.2)$

= 2.00 – (0.5)(0.24) – (0.125) (0.04)

= 1.875.

Hence the solution.

Interpolation

Exercise:

Q.1 Let $f(x)=\log(1+x)$ $x_0=1$ and $x_1=1.1$.

Use linear interpolation to calculate an approximate value for $f(1.04)$ and obtain a bound on the truncation error.

Q.2 In the following problems, find the maximum value of the step size *h* that can be used to tabulate $f(x)$ on $[a,b]$, using linear interpolation such that |Error| $\le \varepsilon$.

(a) $f(x)=(1+x)^6$ $[a,b]=[0,1],\ \varepsilon=5\times10^{-5}$

(b) $f(x)=\dfrac{1}{(1+x^2)}$ $[a,b]=[1,2],\ \varepsilon=1\times10^{-4}$

(c) $f(x)=2^x$ $[a,b]=[0,1],\ \varepsilon=1\times10^{-5}$

Q.3 Find the unique polynomial $P(x)$ of degree 2 or less such that $P(1)=1,\ P(3)=27, P(4)=64$ using each of the following methods.

(i) Lagrange interpolating polynomial

(ii) Newton divided difference formulae.

Estimate *P*(1.5)

Q.4 In the following problems, the values of a function $f(x)$ are given. Find the interpolating polynomial that fits the data. Find an approximation to $f(x)$ at the indicated points using this polynomial.

(i)

x	−2	−1	0	1	3	4
$f(x)$	9	16	17	18	44	81

(ii)

x	−1	2	4	5

f (*x*)	−5	13	255	625

Q.5 If $f\left(x\right)=\frac{1}{x^2}$, find the dividend difference $f\left[x_1, x_2, x_3, x_4\right]$

Q.6 Prove the following relations

(i) $\nabla - \Delta = -\nabla\Delta$

(ii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

(iii) $\Delta\left(\frac{1}{f_i}\right) = -\Delta\, f_i \,/\, f_i\, f_{i+1}$

Q.7 Construct the interpolating polynomial that fits the data

x	0	0.1	02	0.3	0.4	0.5
f (*x*)	−1.5	−1.27	−0.98	−0.63	−0.22	0.25

Using the Gregory – Newton forward or backward difference interpolation. Hence or otherwise estimate the values of $f\left(x\right)$ at *x* = 0.15, 0.25 and 0.45.

Q.8 Using the Newton's backward difference interpolation construct the interpolating polynomial that fits the data

x	0.1	0.3	0.5	0.7	0.9	1.1
f (*x*)	−1.699	−1.073	−0.375	0.443	1.429	2.631

Estimate the value of $f\left(x\right)$ at *x* = 0.6 and *x* = 1.0

Summary:

In this chapter we studied how different methods of interpolation helps in the evaluation and approximation of a function which may not be known explicitly. We studied methods like Lagranges interpolation , Newton divided difference methods etc. We also derived Gregory – Newton forward and backward interpolation formulae. Interpolation as a procedure finds its use in mainly numerical differentiation, numerical integration , numerical differential equation etc.

Application of Interpolation : An application of interpolation that we see every day is in weather forcasting . When you watch the weather forcasts on television, you may wonder where these usually correct projections come from. The weather service people collect information on temperature wind speed and direction , humidity and barometric pressure from hundreds of weather stations around the region. Added to these are cloud data from satellites that are in elevated orbits above the earth. All of these data items are entered into a massive computer program that models the weather using interpolation . Upto a million pieces of data are involved.

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