Lesson: Limits of Functions

Paper: Analysis II (Real Analysis)

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Chapter: Limits of Functions:

1. Learning Outcomes:

After studying this unit, you will be able to

- understand the concept of cluster point of a set
- define the limit of a function at a cluster point of its domain
- explain the difference between limit of a function at a point and the value of the function at that point
- relate the limit of a function at a point to the limit of a sequence at that point
- tell when a function will fail to have a limit at a point

2. Introduction:

This unit is about the concept of **"limits of functions"**. So far, we have learnt the idea of limits of sequences. We know exactly when a sequence is said to be convergent. Let us recall once again.

2.1 Definition of a Convergent Sequence: A sequence $(x_n)_{n\varepsilon\mathbb{N}}$ is said to converge to a real number x if for every $\epsilon > 0$ there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$, the terms x_n satisfy: $|x_n - x| < \epsilon$.

I.Q.1

Example 1: Show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Solution: We see that

$$2.2.2.2 \ldots..2 \leq 2.3.4.5 \ldots..(n-1)$$

and therefore,

$$2^{n-2} \leq (n-1)! \; \forall \; n \geq 2$$

Hence, we have

$$0 \leq \frac{2^n}{n!} = 4.\frac{2^{n-2}}{n!} \leq 4.\frac{(n-1)!}{n!} = \frac{4}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty;$$

and therefore,

$$\lim_{n\longrightarrow\infty} \frac{2^n}{n!} = 0.$$

Example 2: Prove that $\lim_{n\longrightarrow\infty} \sqrt[n]{n!} = \infty$.

Solution: We have, if n is even, then,

$$n! = 1.2.3.4 \ldots..\frac{n}{2}.\left(\frac{n}{2}+1\right) \ldots.n \geq \frac{n}{2}.\frac{n}{2}.\frac{n}{2} \ldots.\frac{n}{2} = (\frac{n}{2})^{\frac{n}{2}}$$

and if if n is odd, then,

$$n! = 1.2.3.4 \ldots..(\frac{n+1}{2}).\left(\frac{n+3}{2}\right) \ldots.n \geq \frac{(n+1)}{2}.\frac{(n+1)}{2}.\frac{(n+1)}{2} \ldots.\frac{(n+1)}{2} = (\frac{n+1}{2})^{\frac{(n+1)}{2}}$$

so that $\forall \; n \; \varepsilon \; \mathbb{N}$,

$$n! \geq (\frac{n}{2})^{\frac{n}{2}}$$

and taking the nth-root yields $\forall \; n \; \varepsilon \; \mathbb{N}$

$$\sqrt[n]{n!} \geq (\frac{n}{2})^{\frac{1}{2}} \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Therefore, we have

$$\lim_{n\longrightarrow\infty} \sqrt[n]{n!} = \infty.$$

In this unit, we shall extend the idea of convergence of a sequence to the limit of a function. The concept of limits is the very basic of all of calculus and analysis. It took long time for mathematicians to understand and be able to define the concept of limit at the very first place. Finally, the great German mathematician Karl Weierstrass, inspired by Augustin Louis Cauchy, So, let us begin with the concept of cluster point of a set.

3. Cluster Point of a Set:

3.1 Definition of a Cluster Point of a Set: Let $A \subseteq \mathbb{R}$. A point $c \in R$ is said to be a cluster point of the set A if for every $\epsilon > 0$, there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \epsilon$. Thus, a real number c is a cluster point of a set A if every δ-neighbourhood $(c - \delta, c + \delta)$ of c contains a point of the set A different from c.

Let us look at some examples.

Example 3: Find the cluster points of the set $A_1 = [1,2]$.

Solution: Then, the end-point 1 is a cluster point of A_1 because for any $\delta > 0$, the δ –nbd $(1 - \delta, 1 + \delta)$ contains infinitely many points of A_1 and therefore, points of A_1 distinct from 1. Similarly, the end-point 2 is a cluster point of A_1. Further, every interior point of A_1 is a cluster point of A_1. Hence, all the points of the set A_1 are its cluster points.

Example 4: Find the cluster points of the set $A_2 = (0,1)$.

Solution: Then, the end-point 1 is a cluster point of A_2. This is because, for any $\delta > 0$, the δ–nbd $(1 - \delta, 1 + \delta)$ contains infinitely many points of A_2 and therefore, points of A_2 distinct from 1. Similarly, the end-point 0 is a cluster point of A_2. Further, every interior point of A_2 is a cluster point of A_2. Hence, all the points of the set A_2 are its cluster points.

Example 5: Find the cluster points of the set $A_3 = \{3,4\}$.

Solution: Here, 3 is not a cluster point of A_3. To see this, consider the δ-nbd of 3 for $\delta = 1/2$ given by the interval $V = (3 - \delta, 3 + \delta) = (2.5,3.5)$. Then, this neighbourhood contains only one point of A_3, namely 3. That is, there exists a neighbourhood of 3 containing no point of A_3 distinct from 3. Hence, 3 is not a cluster point of A_3. Similarly, 4 is not a cluster point of A_3. Hence, there are no cluster points of A_3.

Value Addition: Remarks
• A finite set has no cluster point. • A cluster point of a set need not belong to that set.

Let us now study the characterization of cluster points as limits of sequences.

Theorem 1: A number $c \, \varepsilon \, R$ is a cluster point of a set $A \subseteq \mathbb{R}$ if and only if there exists a sequence $(a_n)_{n \varepsilon N}$ in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in N$.

Proof: Firstly, let us assume c to be a cluster point of A.

Therefore, for every $\delta > 0$, the δ-nbd $(c - \delta, c + \delta)$ contains a point of A distinct from c. In particular, for every $n \, \varepsilon \, N$, taking $\delta = 1/n$, we may see that the δ –nbd $(c - 1/n, c + 1/n)$ contains at least one point a_n of A such that $a_n \neq c$. Thus, there exists some $a_n \, \varepsilon \, A$ such that

$a_n \neq c$ and $a_n \, \varepsilon \, (c - 1/n, c + 1/n)$.

That is, there exists some $a_n \, \varepsilon \, A$ such that

$a_n \neq c$ and $|a_n - c| < 1/n \; \forall \, n \in N$.

Hence, there exists a sequence $a_n \, \varepsilon \, A$ such that $a_n \neq c$ and $\lim(a_n) = c$.

Conversely, let there be a sequence $a_n \, \varepsilon \, A$ such that $a_n \neq c$ and $\lim(a_n) = c$.

Clearly, then

$a_n \, \varepsilon \, A\backslash\{c\} \; \forall \, n \in N$.

Since, $\lim(a_n) = c$, we have for every $\delta > 0$, a natural number $K(\delta) = K$ such that

$|a_n - c| < \delta \; \forall \, n \geq K$.

Hence, for each $\delta > 0$, there exists a natural number K such that

$a_n \, \varepsilon \, (c - \delta, c + \delta) \; \forall \, n \geq K$ with $a_n \neq c$.

Hence, for each $\delta > 0$, there exists a nbd $V(c)$ of c containing a point of A distinct from c; and hence, c is a cluster point of A.

The proof is now complete.

Let us look at a few more examples.

Example 6: Show that the set $\mathbb{N}$ of natural numbers has no cluster point.

Solution: The neighbourhood $(n-\frac{1}{2}, n+\frac{1}{2})$ of any natural number n contains no point of $\mathbb{N}$ other than n, and therefore, N has no cluster point.

Example 7: Prove that the set $A_4 = \{1/n : n \,\varepsilon\, \mathbb{N}\}$ has only cluster point 0.

Solution: This is clear since for each $\delta > 0$, the nbd $(-\delta, \delta)$ of 0 contains infinitely many points of A_4. This is because for each $\delta > 0$, there exists a natural number n such that $0 < 1/n < \delta$, that is, $1/n \,\epsilon\, (-\delta, \delta)$. Further, no point of A_4 can be a cluster point of A_4 because any sequence in A_4 converges to 0. Hence, by theorem 1, no point of A_4 is a cluster point of A_4. Hence, 0 is the only cluster point of A_4.

Example 8: Let $x_n = 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 5, 1, 2, 3, 4, 5, \ldots$. Find the cluster points of this sequence x_n.

Solution: The given sequence in set theoretic form corresponds to the finite set $S = \{1, 2, 3, 4, 5\}$. Hence, the only cluster points of the sequence are the members of this set S.

I.Q.2

We are now ready to take up the concept of limit of a function at a point.

4. Limit of a function at a point:

4.1 Definition: Let $A \subseteq \mathbb{R}$. Let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a limit of f at c, if given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \,\epsilon\, A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Value Addition: Remarks

- The inequality $0 < |x - c| < \delta$ simply says: $x \neq c$.
- The value of δ usually depends upon the value of ϵ.
- When L is the limit of f at c, we also say that f converges to L at c; and we write this as:

  $$\lim_{x \to c} f(x) = L .$$

- If, however, limit of f at c does not exist, we say that f **diverges** at c.

I.Q.3

I.Q.4

First of all, we show that the value L of the limit is unique at each point c, whenever it exists.

Theorem 2: If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.

Proof: Suppose, there are two limits L and L' of f at c. We shall show that

$L = L'$.

Since, L and L' are limits f at c, therefore, for any $\epsilon > 0$, there exist a $\delta > 0$ and a $\delta' > 0$ such that for $x \varepsilon A$,

whenever $0 < |x - c| < \delta$, we have: $|f(x) - L| < \epsilon /2$;

and

whenever $0 < |x - c| < \delta'$, we have: $|f(x) - L'| < \epsilon /2$.

So, let

$\delta'' = min\{\delta, \delta'\}$.

Then, for $x \varepsilon A$, whenever $0 < |x - c| < \delta''$, one has by the triangle inequality:

$$|L - L'| \leq |L - f(x)| + |f(x) - L'|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

and since $\epsilon > 0$ was arbitrary, we conclude that $L = L'$.

This completes the proof.

Let us illustrate the definition of limit using some basic examples.

Example 9: Show that $\forall\, c\, \varepsilon\, \mathbb{R},\ \lim_{x\to c} b = b$.

Solution: Let $f(x) = b$ for all $x\, \varepsilon\, \mathbb{R}$. Let $\epsilon > 0$ be arbitrary. Then, for $\delta = 1$, we see that whenever

$$0 < |x - c| < \delta = 1,$$

we have,

$$|f(x) - b| = |b - b| = 0 < \epsilon.$$

Since, $\epsilon > 0$ was arbitrary, we conclude that $\forall\, c\, \varepsilon\, \mathbb{R}, \lim_{x\to c} f(x) = \lim_{x\to c} b = b$.

Example 10: Show that $\forall\, c\, \varepsilon\, \mathbb{R}, \lim_{x\to c} x^2 = c^2$.

Solution: Let $h(x) = x^2$ for all $c\, \varepsilon\, \mathbb{R}$. Let $\epsilon > 0$ be arbitrary. Now, if

$$|x - c| < 1,$$

then, we have,

$$|x| \le |c| + 1$$

and therefore,

$$|x + c| \le |x| + |c| \le |c| + 1 + |c| = 2|c| + 1.$$

We now observe that

$$|h(x) - c^2| = |x^2 - c^2| = |x + c||x - c| \le (2|c| + 1)(|x - c|)$$

and

$$(2|c| + 1)(|x - c|) < \epsilon \text{ , if } \quad |x - c| < \epsilon / (2|c| + 1).$$

So, we take $\delta = \frac{\epsilon}{2|c|+1} > 0$.

Hence, we see that given $\epsilon > 0$, there exists a $\delta = \frac{\epsilon}{2|c|+1} > 0$ such that whenever

$$0 < |x - c| < \delta,$$

we have,

$$|h(x) - c^2| < \epsilon.$$

Since, $\epsilon > 0$ was arbitrary, we conclude that

$$\forall\, c\, \varepsilon\, \mathbb{R}, \lim_{x\to c} x^2 = c^2.$$

Example 11: Let $f:[0,\infty)\backslash\{9\} \longrightarrow \mathbb{R}$ be $f(x) = \frac{x-9}{\sqrt{x}-3}$. Then, show that

$\lim_{x\to 9} f(x) = 6.$

Solution: Let $\epsilon > 0$ be arbitrary. We observe that $\forall\, x\, \varepsilon\, A = [0,\infty)\backslash\{9\}$,

$$|f(x) - 6| = |\frac{x-9}{\sqrt{x}-3} - 6|$$

$$= |\frac{(\sqrt{x}+3)(\sqrt{x}-3)}{\sqrt{x}-3} - 6|$$

$$= |\sqrt{x} + 3 - 6|$$

$$= |\sqrt{x} - 3|$$

$$= |\frac{x-9}{\sqrt{x}+3}| \leq \frac{1}{3}|x - 9| < \epsilon, \text{ if } |x - 9| < 3\epsilon.$$

So, we take $\delta = 3\epsilon > 0$. Hence, we see that given $\epsilon > 0$, there exists a $\delta = 3\epsilon > 0$ such that whenever $0 < |x - 9| < \delta$, we have

$$|f(x) - 6| < \epsilon$$

so that $\lim_{x\to 9} f(x) = 6$.

I.Q.5

I.Q.6

I.Q.7

We shall now study the sequential criterion for limits.

5. The Sequential Criterion:

5.1 Theorem 2 (The Sequential Criterion): Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then the following are equivalent:

(i) $\lim_{x\to c} f(x) = L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in N$, the sequence $(f(x_n))$ converges to L.

Proof: $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$**:**

Here, we show that $f(x_n) \to L$ as $n \to \infty$.

So, let $\epsilon > 0$ be arbitrary but fixed.

Since, $\lim_{x\to c} f(x) = L$, there exists a $\delta > 0$ such that whenever any $x\, \varepsilon\, A$ satisfies

$$0 < |x - c| < \delta,$$

then, $f(x)$ satisfies

$$|f(x) - L| < \epsilon.$$

Since the sequence x_n converges to c, we have the following:

For $\delta > 0$, there exists a natural number $K(\delta) = K$ such that for $n \geq K$:

$$|x_n - c| < \delta$$

and, since $x_n \neq c$ for all $n \,\varepsilon\, \mathbb{N}$, one has:

$$0 < |x_n - c| < \delta \text{ for all } n \geq K.$$

But, for each such x_n, we have

$$|f(x_n) - L| < \epsilon.$$

That is, for any $\delta > 0$, there exists a natural number K such that

$$|f(x_n) - L| < \epsilon \text{ for all } n \geq K.$$

Hence, the sequence $(f(x_n))$ converges to L.

$(ii) \Rightarrow (i)$:

We shall show the contra-positive argument: If (i) is not true, then, (ii) is not true. So, let (i) does not hold. Then,

$$\lim_{x \to c} f(x) \neq L.$$

This means that there exists an $\epsilon > 0$ such that for every $\delta > 0$, one has some $x \,\varepsilon\, A$ and $x \neq c$ such that

$$0 < |x - c| < \delta \text{ but } |f(x) - L| \geq \epsilon.$$

Hence, for every natural number n, there exists some x_n in A with $x_n \neq c$ such that

$$0 < |x_n - c| < 1/n$$

but

$$|f(x_n) - L| \geq \epsilon \,\forall\, n \,\varepsilon\, \mathbb{N}.$$

Hence, there exists an $\epsilon > 0$ such that $\forall\, n \,\varepsilon\, \mathbb{N}$, one has: $|f(x_n) - L| \geq \epsilon$.

This implies that the sequence $f(x_n)$ does not converge to L; that is (ii) fails to be true.

This completes the contra-positive argument.

Hence, $(\boldsymbol{ii}) \Rightarrow (\boldsymbol{i})$.

The proof is now complete.

I.Q.8

We shall now investigate the conditions for the divergence of a function f at a cluster point of its domain.

6. The Divergence Criteria:

Sometimes it becomes important to show that a certain number is not the limit of the function at a point. For this purpose, we have the divergence criteria. (The proof is omitted.)

Theorem 4 (The Divergence Criterion): Let $f : A \to \mathbb{R}$ and c be a cluster point of A.

(a) If $L \,\varepsilon\, \mathbb{R}$, then f does not have a limit L at c if, and only if, there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \,\varepsilon\, \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge to L.

(b) The function f does not have a limit at c if, and only if, there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \,\varepsilon\, \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge in $\mathbb{R}$.

Let us illustrate the divergence criterion in some examples.

Example 12: Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Solution: Let $f(x) = 1/x$. Here, $c = 0$. We consider the sequence

$$x_n = \frac{1}{n}, \forall\, n \,\varepsilon\, \mathbb{N}.$$

Then,

$$x_n \neq c \;\forall\, n \,\varepsilon\, N$$

and

$$x_n \to 0$$

, that is,

$$x_n \to c.$$

But, the sequence $f(x_n) = n$ does not converge in $\mathbb{R}$.

Hence, by the theorem 4(b), the limit $\lim_{x\to 0}\frac{1}{x}$ does not exist.

Example 13: Prove that the $\lim_{x\to 0} sgn\,(x)$ does not exist.

Solution: Here, the symbol $sgn\,(x)$ stands for the famous signum function defined by

$$f(x) = sgn(x) = \begin{cases} 1, for\ x > 0 \\ 0, for\ x = 0 \\ -1, for\ x < 0 \end{cases}$$

We consider the sequence

$$x_n = \frac{(-1)^n}{n}, \forall\, n\, \varepsilon\, \mathbb{N}\ .$$

Then,

$$\lim(x_n) = 0.$$

However, since

$$f(x_n) = sgn(x_n) = (-1)^n, \forall\, n\, \varepsilon\, \mathbb{N},$$

we see that the sequence $(f(x_n))$ obviously fails to converge in $\mathbb{R}$.

Hence, by theorem 4(b), the limit $\lim_{x\to 0} sgn\,(x)$ does not exist.

The next example is a very famous and an important counterexample in all of analysis.

Example 14: Prove that $\forall\, x\, \varepsilon\, \mathbb{R}\backslash\{0\}$, $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

Solution: Let

$$g(x) = sin(1/x)\ \forall\, x\, \varepsilon\, \mathbb{R}\backslash\{0\}.$$

To show that the given limit does not exist, we produce two sequences (x_n) and (y_n) in $\mathbb{R}\backslash\{0\}$ such that $lim(x_n) = lim(y_n)$ but $lim((g(x_n) \neq lim(g(y_n))$.

We consider the sequences

$$x_n = \frac{1}{n\pi} \text{ and } y_n = \frac{1}{\frac{\pi}{2}+2n\pi}\ \forall\, n\, \varepsilon\, \mathbb{N}.$$

Then, clearly,

$$x_n \neq 0 \text{ and } y_n \neq 0 \ \forall \, n \, \varepsilon \, \mathbb{N}$$

and

$$lim(x_n) = lim(y_n) = 0.$$

But,

$$lim((g(x_n)) = lim(\sin(n\pi)) = lim(0) = 0$$

and

$$lim((g(y_n)) = lim(\sin(\frac{\pi}{2} + 2n\pi)) = lim(\sin(\frac{\pi}{2})) = \lim(1) = 1.$$

Hence, the limit $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist for any $x \, \varepsilon \, \mathbb{R}\backslash\{0\}$.

Example 15: Show that $\forall \, x \, \varepsilon \, \mathbb{R}$, $\lim_{x\to\infty} \cos(\frac{\pi}{3}x)$ does not exist.

Solution: The divergence of the corresponding sequence $u_n = \cos(\frac{\pi}{3}n)$ proves the desired non-existence of the limit. To see that the sequence u_n is divergent, it is enough to produce two subsequences converging to different limits. Here are the two subsequences:

$$u_{6n+1} = \cos\left(\frac{\pi}{3}(6n+1)\right) = \cos\left(2n\pi + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \longrightarrow \frac{1}{2} \text{ as } n \longrightarrow \infty;$$

and

$$u_{6n+3} = \cos\left(\frac{\pi}{3}(6n+3)\right) = \cos(2n\pi + \pi) = \cos(\pi) = -1 \longrightarrow -1 \text{ as } n \longrightarrow \infty.$$

Example 16: Prove that the limit $\lim_{x\to\infty} e^{-x} + 2\cos(3x)$ does not exist.

Solution: Let $f(x) = e^{-x} + 2\cos(3x) \ \forall \, x \, \varepsilon \, \mathbb{R}$. Once again, to show that the given limit does not exist, we produce two sequences (x_n) and (y_n) in $\mathbb{R}$ such that $lim(x_n) = lim(y_n)$ but $lim((f(x_n) \neq lim(f(y_n))$.

We consider the sequences

$$x_n = 2n\pi \text{ and } y_n = 2n\pi + \pi \ \forall \, n \, \varepsilon \, \mathbb{N}.$$

Then,

$$lim(x_n) = lim(y_n) = \infty;$$

but

$$lim((f(x_n)) = lim[e^{-2n\pi} + 2\cos(6n\pi)] = 2$$

and

$$lim((f(y_n)) = lim[e^{-2n\pi-2\pi} + 2\cos(6n\pi + 3\pi)] = -2.$$

Hence, the limit $\lim_{x\to\infty} e^{-x} + 2\cos(3x)$ does not exist.

I.Q.9

I.Q.10

I.Q.11

7. Summary:

1. The concept of limit of functions can be easily understood in terms of cluster point of a set.
2. A real number c is a **cluster point** of a set $A \subseteq \mathbb{R}$ if every neighbourhood of the point c contains a point of A distinct from c.
3. Thus, a real number c is a cluster point of set A if for every $\delta > 0$, the δ-neighborhood $(c - \delta, c + \delta)$ of c contains a point of the set A different from c.
4. A cluster point of a set need not belong to that set.
5. A number $c \varepsilon \mathbb{R}$ is a cluster point of a set $A \subseteq \mathbb{R}$ if, and only if, there exists a sequence (a_n) in A such that $lim(a_n) = c$ and $a_n \neq c \; \forall \, n \, \varepsilon \, \mathbb{N}$.
6. **Limit of a function at a point**: Let c be a cluster point of the set A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a limit of f at c, if given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \, \varepsilon \, A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.
7. Limit, if exists, is unique at cluster points: If $f: A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.
8. **The Sequential Criterion**: Let $f: A \to \mathbb{R}$ and c be a cluster point of A. Then the following are equivalent:

   (i) $\lim_{x\to c} f(x) = L$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c \ \forall \, n \, \varepsilon \, \mathbb{N}$, the sequence $(f(x_n))$ converges to L.

9. **The Divergence Criterion**: Let $f: A \to \mathbb{R}$ and c be a cluster point of A.

(a) If $L \, \varepsilon \, \mathbb{R}$, then f does not have a limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c \ \forall \, n \, \varepsilon \, \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge to L.

(b) The function f does not have a limit at c if, and only if, there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in N$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does not converge in R.

Exercises

1. Find the set of cluster points of each of the following subsets of $\mathbb{R}$:

(i) $S = \{x | x \notin \mathbb{Q}, \sqrt{2} < x < 100\}$

(ii) $S = \left\{ x \middle| x \notin \mathbb{Q}, \sqrt{2} \leq x < 109 \right\} \cup \ \{x \,| x \, \varepsilon \, \mathbb{Z}, 50 < x < 1999\}$

(iii) $S = \left\{ (-1)^n \frac{2n-1}{n} \middle| n \, \varepsilon \, \mathbb{N} \right\}$

(iv) $S = (3,6) \cup (6,8)$

2. Using the definition of limits or the sequential criterion, show that:

(i) $lim_{x \to 2} \, x^2 + 4x = 12$

(ii) $lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$

(iii) $lim_{x \to c} \, x^3 = c^3$

(iv) $lim_{x \to 2} \frac{1}{1-x} = -1$

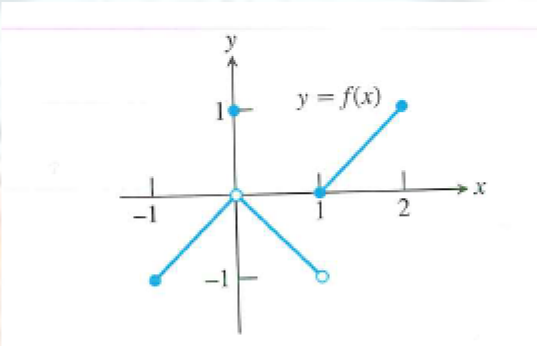
3. Using the divergence criterion, prove that the following limits do not exist.

(i) $lim_{x \to 0} \frac{1}{x^2}$

(ii) $lim_{x \to 0} \frac{1}{\sqrt{x}}$

(iii) $lim_{x \to 0} \sin(\frac{1}{x^2})$

4. The function $y = f(x)$ is graphed as shown.



Conclude in true/false about the following assertions regarding this function.

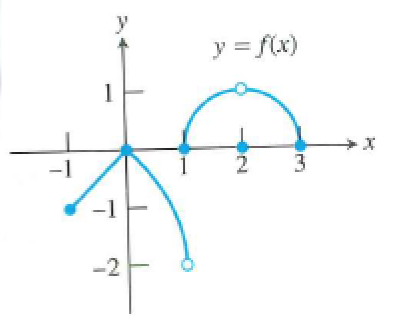
(i) $\lim_{x \to 0} f(x) = 0$.

(ii) $\lim_{x \to 0} f(x) = 1$.

(iii) $\lim_{x \to 1} f(x) = 1$.

(iv) $\lim_{x \to 1} f(x) = 0$.

5. The function $y = f(x)$ is graphed as shown.



Conclude in true/false about the following assertions regarding this function.

(i) $\lim_{x\to 2} f(x)$ does not exist.

(ii) $\lim_{x\to 2} f(x) = 2$.

(iii) $\lim_{x\to 1} f(x)$ does not exist.

(iv) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(-1, 1)$.

(v) $\lim_{x\to x_0} f(x)$ exists at every point x_0 in $(1, 3)$.

Glossary:

Cluster Point, Limit, Sequence, Convergence Criterion, Divergence Criterion, Signum Function

Further Reading:

It is always welcome to practice more exercises from various books available in libraries and elsewhere. Here are a few more references for this purpose.

(i) Introduction to Analysis(5/e) by Edward D. Gaughan, American Mathematical Society.

(ii) Mathematical Analysis(2/e) by T.M. Apostol, Narosa Publishing House.

(iii) A Course in Calculus and Real Analysis by Sudhir R. Ghorpade & Balmohan V. Limaye, Springer-Verlag.

Hints & Solutions For Exercises

1. (i) The set of cluster points is the closed interval $[\sqrt{2}, 100]$.

   (ii) The set of cluster points is the closed interval $[\sqrt{2}, 109]$.

   (iii) The set of cluster points is the finit set $\{2, -2\}$.

   (iv) The set of cluster points is the closed interval $[3, 8]$.

2. (i) Let $0 < |x - 2| < 1$. Then,

$$|x^2 + 4x - 12| = |(x + 6)(x - 2)| < 9|x - 2| < \epsilon,$$

if

$0 < |x-2| < \delta = \epsilon/9.$

So, let $\epsilon > 0$ be given to be arbitrary. Then, for $\delta = \epsilon/9$, we observe that,

$| x^2 + 4x - 12| < \epsilon,$

whenever, $0 < |x-2| < \delta.$

Hence, $lim_{x\to 2} x^2 + 4x = 12.$

(ii) Let $0 < |x-1| < 1$. Then,

$1 < x < 2,$

so that

$|(x/1+x) - 1/2| = |x-1|/4 < \epsilon,$

if

$0 < |x-1| < \delta = \epsilon/4.$

So, let $\epsilon > 0$ be given to be arbitrary. Then, for $\delta = \epsilon/4$, we observe that,

$|(x/1+x) - 1/2| < \epsilon,$

whenever, $0 < |x-1| < \delta.$

Hence, $lim_{x\to 1} \frac{x}{1+x} = \frac{1}{2}.$

(iii) Let $0 < |x-c| < 1$. Then,

$|x| < 1 + |c|.$

Hence,

$$|x^3 - c^3| = |(x-c)(x^2 + cx + c^2|$$
$$= |x-c||\{(x-c)^2 + 3cx\}|$$
$$\leq |x-c|\{(x-c)^2 + 3|cx|\}$$
$$< |x-c|(1 + 3|c||x|)$$
$$< |x-c|(1 + 3|c| + 3|c|^2)$$
$$< \epsilon,$$

if $0 < |x-c| < \delta = 1/(1 + 3|c| + 3|c|^2).$

So, let $\epsilon > 0$ be given to be arbitrary. Then, for

$\delta = 1/(1 + 3|c| + 3|c|^2)$,

we observe that

$|x^3 - c^3| < \epsilon$, whenever, $0 < |x - c| < \delta$.

Hence, $lim_{x \to c} x^3 = c^3$.

3. (i) Hint: Take $x_n = 1/n^2$, $\forall\, n\, \varepsilon\, \mathbb{N}$.

   (iii) Hint: Take $x_n = (2n\pi)^{-\frac{1}{2}}$ and $y_n = \left(\frac{\pi}{2} + 2n\pi\right)^{-\frac{1}{2}}$, $\forall\, n\, \varepsilon\, \mathbb{N}$. Now, follow example 11.

4. (i) True

   (ii) False

   (iii) False. In fact, the limit does not even exist.

   (iv) False

5. (i) False, because the limit exists and equals 1.

   (ii) True

   (iii) True

   (iv) False, because at $x_0 = -1$, only the right hand limit exists.

   (v) True