Lesson: Mean Value Theorems

Paper : Analysis-II

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Mean Value Theorems

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Mean Value Theorems

1. Learning Outcomes

After you have read this chapter, you should be able to :

- Define and locate the local extremum of a function and obtain a relation between local extremum and derivative of a function.
- State and prove Rolle's theorem.
- State and prove Lagrange's Mean Value Theorem.
- To characterize constant functions in terms of differentiable function.
- Define monotonic functions and find a connection between monotonicity of a function and derivative of a function.
- State and prove Darboux's theorem which is also known as the Intermediate Value Property for derivatives.

"I do not frame hypothesis"

– Isaac Newton (1642 - 1727)

"I do not know"

– J.L. Lagrange (1736 - 1813)

"I have so many ideas that may perhaps be of some use in time if others more penetrating than I go deeply into them some day and join the beauty of their minds to the labor of mine".

– *G.W Leibniz* (1646 - 1716)

2. Introduction:

In this chapter we begin our study of applications of differentiation in understanding various geometric properties of the graph of a function such as local extrema, monotonicity, mean value theorem and intermediate value property. Here, we use derivatives to find extreme values of functions, to predict and analyze the shapes of graphs, to find the zeros of functions numerically etc. The main idea behind these results is the mean value theorem which is due Lagrange. Various corollaries to this theorem provide the gateway to Integral Calculus. Furthermore, we can characterize constant functions, monotonic functions and convex or concave functions in terms of derivative with the help of the mean value theorem. In fact, this chapter shows how to draw conclusions from derivatives.

Thus, the chapter describes the importance of a derivative of a function as one of the most classified topic in calculus as well as in real analysis. In calculus, we give geometric definition of a derivative, which uses the idea of tangents, whereas in real analysis, we define a derivative analytically, using the concept of limits.

Modern mathematics began with two great advances – analytic geometry and the calculus. Analytic geometry took definite form in the year 1637 while the calculus took the definite shape in 1966. Rene Descartes (1596-1650), the great French mathematician of the seventeenth century is generally credited as the founder of analytic geometry. On the other hand, both Newton (1642-1727) and Leibnitz (1646-1716), share the credit for inventing "Calculus" independently in the seventeenth century. The former used physical approach while the later used geometrical approach.

Further, Newton used the term "rate of change in his second law of motion. Thus, the calculus, sometimes, may be defined as mathematics of motion and change. It was inevitable after the work of Cavalieri (1598-1647), Fermat (1601-1665), Wallis (1616-1703), Barrow (1630-1677) and others that the calculus should presently get itself organized as an autonomous discipline. In fact, P. Laplace (1749-1827) considered Fermat the discoverer of differential calculus as Fermat developed a method for finding tangents and solving maximum and minimum problem using a difference quotient, identical to the one which we now use to define derivatives, although he did not have a theory of limit. Calculus grew with the stimulus of applied work that continued through the 18^{th} century, into analysis topics such as the calculus of variations, differential calculus etc. During this period, calculus techniques were applied to approximate discrete problems by continuous one.

Today, calculus and its extension in real analysis which is a part of mathematical analysis are far reaching indeed. Now, every mathematician knew that analysis arose naturally in the nineteenth century out of the calculus, which involves the elementary concepts and techniques of analysis, of the previous two centuries.

Today not only the mathematics but many other subjects – such as Economic, Physics, Chemistry and Biological Sciences are enjoying the fruits of calculus.

We now give the section wise summary of the chapter. We start the chapter with quotations taken from the literature. The chapter is divided into five sections. The section one (3.1) starts with a discussion on methods for determining the high or low points, the peaks or the dips occur in the graph of a real-valued function. For this, we introduce the concept of local extremum of a function which is well-explained through various examples and remarks. Here, we also discuss theorem which connects zero derivatives and local extremum.

In section two (3.2) we discuss Rolle's theorem in detail which forms the ground work for further developments in the theory of differential calculus. In fact, Rolle's theorem is the basis of all mean value theorems due to Lagrange, Cauchy, Taylor and Maclaurin. The section ends with various interpretations of Rolle's theorem.

The section three (3.3) deals mean value theorem which mainly relates the values of a function to those of its derivative. Sometimes it is also called the fundamental theorem of differential calculus. Moreover, we characterize constant function in terms of derivative using the mean value theorem. Well also interpret the mean value theorem geometrically.

In this fourth section (3.4) we study monotonicity which is a geometric property of a real valued function defined on a subset of R. But we notice that in the case of a differentiable function, there is a close connection between derivatives and notion of monotonicity.

Finally, in the fifth section (3.5) we discuss an interesting property of derivatives, namely, intermediate value property. We know that every continuous function on closed and bounded interval possesses intermediate value property. In the similar manner derivative of a function possesses the same property. This important result is due to Darboux. Darboux's theorem simply says that intermediate value property holds even if a function is not continuous, provided it is a derivative. However, not every function is a derivative. The section ends with a result which shows that the monotonic derivatives are necessarily continuous.

Lastly, the chapter ends with a list of exercises (with the answers / hints) and references for further reading.

3. Mean Value Theorems:

We know that a continuous real-valued function defined on a closed and bounded subset of *R* is bounded and attains its bounds. Equivalently, if $A \subset R$ is closed and bounded, and $f : A \to R$ is continuous, then the absolute minimum and the absolute maximum of *f* on *A*, namely

$$M = \sup f(A) \text{ and } m = \inf f(A) \quad (\text{where } f(A) = \{ f(x) : x \in A\}$$

exist, and furthermore, there are $s, t \in A$ such that $M = f(s)$ and $m = f(t)$. Naturally, the following question arises : knowing the function *f*, how does one find the absolute extrema *m* and *M* and the points *s* and *t* where they are attained?

As we proceed, it turns out that we can considerably narrow down the search for the points where the absolute extrema are attained if we concentrate on the derivative of *f*. In order to get the precise answer to the above questions, let us first develop some definition and related concepts.

3.1 Local Extremum:

3.1.1. Definition. Let *f* be a real-valued function defined on a subset *A* of *R* and let $a \in A$. Then *f* is said to a local maximum at *a* if there exists a neighbourhood, namely $N = N_e(a)$ of *a* such that

$$f(x) \le f(a) \text{ for all } x \text{ in } N \cap A.$$

Likewise, if $f(x) \ge f(a)$ for all *x* in $N \cap A$, then *f* is said to have a local minimum at *a*.

If *f* has either a local maximum or a local minimum,, then we say that *f* has a local extremum at *a*.

Mean Value Theorems

Roughly speaking a local maximum and local minimum can be identified with points where the graph of a function has a "little hill" or "peak" and has a "little valley" or a "dip" respectively.

The notion of a local extremum is of great importance in calculus and its applications.

Value Addition: Note
(i) Sometimes, local extremum may be replaced by relative extremum, or turning point. Also, absolute extremum may be replaced by global extremum. (ii) We observe that a local maximum at *a* is the absolute maximum *f* on the subset $N \cap A$ of *R*. Furthermore, if *f* has an absolute maximum at *a*, then *a* also a local maximum. However, *f* can have local maxima at several points in *A* without having an absolute maximum on the whole set *A*.

I.Q. 1.

3.1.2 Example (i) The function $f(x) = |x|, x \in R$ has a local minimum at *x* = 0 but no local maximum.

(ii) The function $f(x) = -x^2, x \in R$ has a local maximum at *x* = 0 but no local minimum.

(iii) The function $f(x) = \sin x$ has local maximums at $x = \pi/2 + 2\pi n$, $n \in Z$ and local minimums at $x = 3\pi/2 + 2\pi n,\ n \in Z$.

(iv) The function $f(x) = x^2 - x^4, x \in R$ does not attain an absolute minimum at 0. For example, $f(2) = -12 < 0 = f(0)$.

However, it does attain a local minimum. Let us choose $\varepsilon = 1$.

If we restrict *f* to the interval (–1, 1), then we have $x^4 < x^2$ for all *x* ∈ (–1, 1).

Therefore, $f(x) = x^2 - x^4 \geq 0 = f(0)$ which shows that *f* has a local minimum at 0 in the interval (–1, 1).

(v) The function $f(x) = x, x \in Z$ has no absolute maximum or absolute minimum. In fact, we observe that the function attains both local maximum and local minimum at every integer n.

The next theorem establish a connection between zero derivatives and local extremum at interior points. In fact, it is the basis of various applications of differentiation. Recall that a point *a* ∈ *A* is said to be an interior point if there exists some ε > 0 such that *a* ∈ (a – ε, a + ε) ⊂ A.

3.1.3 Theorem: Let *A* ⊂ *R*, *a* be an interior point of *A*, and *f* : *A* → *R* has a local extremum at *a*. If *f* has a derivative at *a*, then $f'(a) = 0$.

Proof: First, we assume that *f* has a local maximum at *a*.

Suppose that $f'(a) > 0$ and $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. We take $\varepsilon = \frac{f'(a)}{2}$ in the definition of the limit. Thus, we find a number $\delta > 0$ such that if $0 < |x - a| < \delta$ and *x* ∈ *A*, then

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \frac{f'(a)}{2}$$

Therefore, it follows that if *x* ∈ *A* and 0 < | *x* – *a* | < δ then

$$\frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0.$$

That is, if $f'(a)$ > 0, then there exists a neighbourbood *N* ⊆ *A* of a such that

$$\frac{f(x)-f(a)}{x-a} > 0 \text{ for } x \in N,\ x \neq a$$

If we select $x \in N$ and $x > a$, then we have

$$f(x)-f(a) = (x-a)\frac{f(x)-f(a)}{x-a} > 0,$$

that is, $f(x) > f(a),\ x \in N \text{ and } x > a$.

But this is a contradiction to the assumption that f has a local maximum at a. This implies that we cannot have $f'(a) > 0$. In the similar manner, if $f'(a) < 0$, there exists a $\delta > 0$ such that

$$\frac{f(x)-f(a)}{x-a} < 0 \text{ for } x \in N,\ x \neq a.$$

Now, if a take $x \in N$ and $x < a$ then we obtain

$$f(x)-f(a) = (x-a)\frac{f(x)-f(a)}{x-a} > 0$$

Thus, $f(x) > f(a)$, $x \in N$ and $x < a$.

which is again a contradiction that f has a local maximum at a.

Hence, we must have $f'(a) = 0$.

Likewise, the result can be proved for the case $f'(a) < 0$.

I.Q. 2

Value Addition : Remarks	
(i)	The above theorem 3.1.3 is valid only for interior points. For example, let $f(x) = x,\ x \in [0,1]$. Then the point $x = 0$ is the local minimum (unique) and $x = 1$ is the local maximum (unique) on [0,1], but the derivative of f which is not zero at none of

the points. Therefore, we conclude that *f* can have a local extremum at a without $f'(a)$ being zero.

(ii) The function *f* must be differentiable on *A* for the working of the theorem.

(iii) The converse of the above theorem is not true

Let $f(x) = x^3, x \in R$.

We pick *a* = 0. Then $f'(a)$ = 0.

But *f* is increasing in every neighbourhood of 0 and hence no local extremum at *x* = 0.

(iv) Let $f(x) = |x|, x \in R$.

We take *a* = 0, since *f* is not differentiable at 0. Therefore, we cannot apply the above theorem to the function *f*.

However, the function *f* has a local minimum at *x* = 0.

(v) The above theorem is, sometimes, referred as Interior Extremum Theorem.

3.1.4 Corollary: Let *f* : *A* → *R* be continuous on an interval *A* and let *a* ∈ *A* be an interior point. Suppose that *f* has a local extremum at the point *a* of *A*. Then either the derivative of *f* at *a* does not exist, or it is equal to zero.

Proof: If the derivative of *f* at *a* does not exist, then we are done.

Otherwise, assume that the derivative of *f* at *a* exists. Then, by the Theorem 3.1.3, we conclude that $f'(a) = 0$

3.2 Rolle's Theorem:

Till now, we have discussed the relationship between local properties of a function and its derivative.

Mean Value Theorems

The next theorem, famously, known as Rolle's theorem, forms the ground work for further developments in the theory of differentiation and is one of the main theorems in the real analysis. It is the basis of all mean value theorems due to Lagrange, Cauchy, Taylor and Maclaurin.

Theorem 3.2.1 (Rolle's Theorem): Let *f* : [*a, b*] → *R* be continuous on [*a, b*], and differentiation on (*a*, *b*). Suppose also that $f(a)=f(b)$. Then there exists $c\in(a,b)$ such that $f'(c)=0$.

Proof: The theorem becomes obvious if the function *f* vanishes identically on [*a, b*]. In this case, any *c* of (*a, b*), satisfies the conclusion of the theorem. Thus, we assume that *f* does not vanish identically on [*a, b*].

Without any loss of generality, we may suppose that *f* assume some positive values.

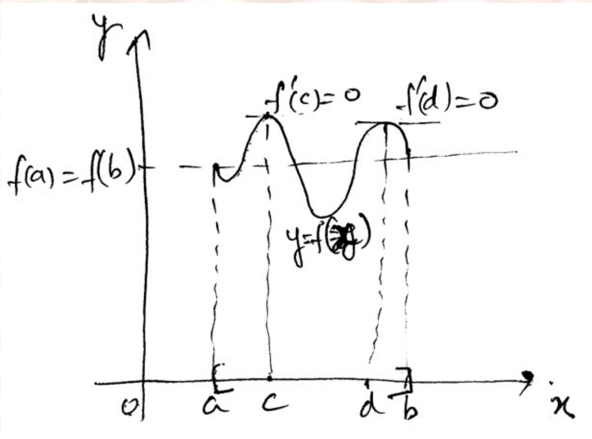


Figure 1

Since *f* is continuous function on the closed and bonded interval [*a, b*], it follows from the Maximum-Minimum Theorem that *f* is bounded and attain its bounds on [*a, b*]. So , there exists $s,t\in[a,b]$ such that

$$f(s)=\sup\{f(x):x\in[a,b]\} \text{ and } f(t)=\inf\{f(x):x\in[a,b]\}.$$

If *s* or *t* is an interior point of [*a, b*], then by Theorem 3.1.3, we obtain $f'(s)=0$ or $f'(t)=0$ and hence the result is established.

Now, let us assume that both s and t are end points of [*a, b*] and since $f(a)=f(b)$, then we get $f(s)=f(t)$. Hence, both the maximum and the minimum values of *f* on [*a, b*] coincide. Thus, *f* is a constant function on [*a, b*]. This implies that, $f'(c)=0$ for every $c \in$ (*a, b*).

I.Q.3

Value Addition : Remark	
(i)	We observe that *c* is always an interior point of [*a, b*], since *f* is differentiable on (*a, b*).

3.2.2 Various Interpretations of Rolle's Theorem

(i) **Geometric:** Let the curve *y* = *f*(*x*), which is continuous on [*a, b*] and differentiable on (*a, b*) be drawn.

The theorem simply claims that somewhere between a and *b* with equal ordinates on the graph of *f*, there exists at least one point where the tangent at (*c*, *f*(*c*)) to the graph of *f* is parallel to *x*-axis.

Such points can be traced or located where the local extreme values of *f* are attained as suggested from the proof of the theorem

(ii) **Algebraic:** Between any two zeros, say *a* and *b,* of *f(x)* which is continuous on [*a, b*] and differentiable on (*a, b*) there exists at least one zero of $f'(x)$.

Value Addition: Do you Know	
(i)	We notice the importance of Maximum-Minimum Theorem of continuous function in the proof of Rolle's Theorem.

(ii) Rolle's Theorem guarantees us about the existence of at least one real number *c* such that $f'(c)=0$. It does not say anything about the existence of more than one such number.

(iii) A point *c* at which $f'(c)=0$ is called a stationary point of *f*. We observe that not all stationary points give rise to a local maximum or local minimum.

For example, let $f(x)=x^3, x\in R$. Then, $f'(0)=0$.

But *x* = 0 is neither a local maximum nor a local minimum.

(iv) Rolle's theorem is sometimes stated with the condition (iii) as $f(a)=f(b)=0$ and the proof given above does not require this condition. Thus, the result with the condition $f(a)=f(b)=0$ follows as a corollary to the Theorem 3.2.1.

3.2.3 Example: Let the function $f(x)=(x-2)\sqrt{x},\ x\in[0,2]$. Then, the given function is obviously continuous on [0, 2] and differentiable on (0, 2) and also, $f(0)=f(2)$. Thus, all the condition of Rolle's Theorem are satisfied. Hence, there exists at least one point, say, *c* $\in$ (0, 2) such that $f'(c)=0$.

Now, $f'(x)=\sqrt{x}.1+(x-2).\dfrac{1}{2\sqrt{x}},\ x\in(0,2)$.

This implies that $f'(c)$ = 0, that is, $\sqrt{c}+\dfrac{(c-2)}{2\sqrt{c}}=0$,

that is, $2c+c-2=0$ which gives $c=\dfrac{2}{3}\in(0,2)$.

Thus, we conclude that the above function satisfies the Rolle's Theorem.

The following examples show that the conclusion of the Rolle's theorem does not hold even if anyone of the three conditions on *f* is dropped.

3.2.4 Example : Let $f(x)=\begin{cases}1, & x=0\\ x, & 0<x\le 1\end{cases}$

Then, clearly, *f* is not continuous at *x* = 0. Thus, *f* is continuous on (0, 1]. Also $f(0)=1=f(1)$. But $f'(x)=1\neq 0$ for all $x\in(0,1]$.

It follows that Rolle's theorem does not hold if *f* is not continuous on [0, 1].

3.2.5 Example: Let $f(x)=|x|, x\in[-1,1]$. Then, *f* is continuous on $[-1,1]$ and $f(-1)=1=f(1)$.

We observe that $f'(c)=1$ or –1 for $c \neq 0$.

So we conclude that Rolle's theorem does not hold here since *f* is not differentiable on (–1, 1).

3.2.6 Example : Let $f(x)=x, x\in[0,\ 1]$.

Then *f* is continuous on [0, 1] and differentiable on (0,1).

Also $f(0)=0\neq 1=f(1)$. But $f'(c)=1\neq 0$ for any $c \in (0, 1)$.

Here, Rolle's theorem does not hold since $f(0)\neq f(1)$.

3.2.7 Example : Let $f(x)=1-(x-1)^{2/3}$, $x \in [0, 2]$.

Being an algebraic function, the given function is continuous on [0, 2].

Now, $f'(x)=-\frac{2}{3}(x-1)^{-1/2}=\frac{-2}{3(x-1)^{1/3}}$.

Thus, *f* is not differentiable at *x* = 1 and $1 \in (0, 2)$. But $f(0)=f(2)=0$.

Therefore, *f* does not satisfy all the conditions Rolle's theorem. It is obvious that the conclusion of Rolle's theorem is not valid for the given function.

3.3 Mean Value Theorems:

In this section we study mean value theorems. It may be observed that there are, in fact, several versions of the mean value theorems. Here, we state and prove the most commonly used mean value theorem due to Lagrange which is the first mean value theorem of differential calculus. Sometimes it is also called the fundamental theorem of differential calculus. As we proceed, we may conclude Lagrange's mean value theorem as an important corollary to Rolle's Theorem. Basically, the mean value theorem connects the values of a function to values of its derivative.

3.3.1 Theorem (Lagrange's Mean Value Theorem): Let a function $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f(b)-f(a)=f'(c)(b-a)$$

Proof : Consider the function $g : [a, b] \to R$ defined by

$$g(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a) \quad \text{where } x \in [a, b]$$

We show that g satisfies all the conditions of Rolle's Theorem 3.2.1.

(i) g, being the sum of two continuous functions, is continuous on [a b].

(ii) g, being the sum of two differentiable functions, is differentiable on (a, b).

(iii) $g(a)=f(a)-\frac{f(b)-f(a)}{b-a}(a-a)=f(a)$ and

$$g(b)=f(b)-\frac{f(b)-f(a)}{b-a}(b-a)=f(a)$$

Therefore, $g(a)=g(b)$.

Thus, by Rolle's theorem 3.2.1, there exists at least one point $c \in (a, b)$ such that $g'(c) = 0$. Now, $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$

Hence, $g'(c) = 0$, implies that $f'(c) - \frac{f(b) - f(a)}{b - a}$.

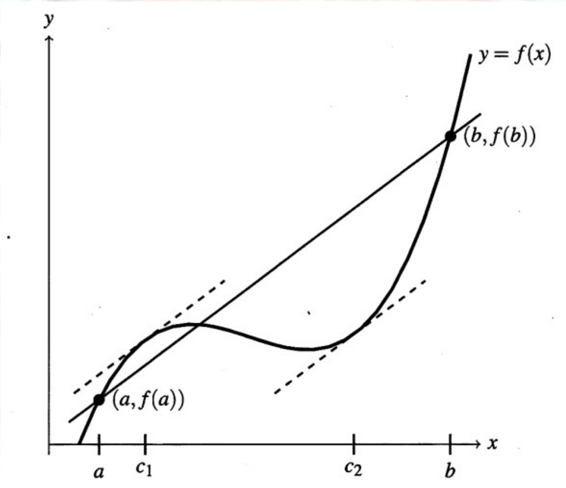


Figure 2

I.Q.4:

3.3.2 Geometrical Interpretation of Lagrange's mean value theorem.

The theorem tells us that between any two points *A* and *B* on the graph of the function *f* there exists at least one point where the tangent is parallel to the chord AB.

Physical Interpretation of Lagrange's mean value theorem.

If we consider the number $[f(b) - f(a)](b - a)$ as the average change of *f* over [*a, b*] and *f′* (*c*) as an instantaneous change, then the theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

Value Addition: Note

(i) We may regard the above theorem 3.3.1 as an important corollary to the Rolle's theorem 3.2.1.

(ii) In the above theorem, we have observed that there is always a point on the graph of the function, where the tangent is parallel to the line joining the end points of the graph whereas in Rolle's theorem, we have noticed that somewhere on the graph, the tangent is parallel to the *x*-axis.

(iii) If we take $b = a + h$ in the above theorem, then the conclusion of the mean value theorem may be stated as follows:

$$f(a+h) = f(a) + hf'(a+\theta h) \text{ for some } \theta \in (0, 1).$$

3.3.3. Corollary: (Mean Value Inequality): Let a function $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b), and if $m, M \in R$ are such that $m < f'(x) \le M$ for all $x \in (a, b)$, then

$$m(b-a) \le f(b) - f(a) \le M(b-a).$$

Proof: Using Theorem 3.3.1, we obtain

$$f'(c) = \frac{f(b)-f(a)}{b-a} \text{ for some } c \in (a, b).$$

Since $m \le f'(x) \le M$ for all $x \in (a, b)$, therefore in particular for c we have

$$m \le f'(c) \le M.$$

Thus, $m(b-a) \le f(b) - f(a) \le M(b-a)$.

I.Q.5

The following examples establish that the conditions of the hypothesis of Lagrange's mean value theorem cannot be dropped.

3.3.4 Example: Let us consider the function *f* : [1, 2] defined by

$$f(x)=\begin{cases}2, & x=1\\ x^2, & x\in(1,2)\\ 1, & x=2\end{cases}.$$

Then *f* is continuous and differentiable on (1, 2) but *f* is not continuous at *x* = 1 and *x* = 2. This show that *f* is not continuous on [1, 2] which implies that the first of the two conditions is not satisfied.

Now, $\dfrac{f(2)-f(1)}{2-1}=-1,$ and $f'(x)=2,\ x\ (1,2)$.

Since the derivative $f'(x)$ of the function *f* is positive for all *x* ∈ (1,2), therefore $\dfrac{f(2)-f(1)}{2-1}\neq f'(c)$ for any *c* ∈ (1, 2).

3.3.5 Example: Let us consider the function *f* : [–1, 1] → R defined by

$$f(x)=|x|,x\in[-1,1].$$

We see that *f* is continuous on [-1, 1] and differentiable on (–1, 1) except at the point *x* = 0.

Here, $$f'(x)=\begin{cases}-1, & x\in(-1,\ 0)\\ 1, & x\in(0,1)\end{cases}$$

Now, we have

$$\frac{f(1)-f(-1)}{1-(-1)}=\frac{1-1}{2}=0.$$

Thus, $\dfrac{f(1)-f(-1)}{1-(-1)}\neq f'(c)$ for any *c* ∈ (–1, 1).

3.3.6 Example: Let us consider the function $f(x)=\sqrt{x^2-4},\ x\in[2,4]$. The function *f*, being an algebraic function of *x*, is continuous in [2, 4] and differentiable in (2, 4). Now, $f'(x)=\dfrac{x}{\sqrt{x^2-4}}$ for all *x* ∈ (2, 4).

Thus, by Lagrange's mean value theorem, there exists at least one *c* ∈ (2, 4) such that $f'(c)=\dfrac{f(4)-f(2)}{4-2}$, that is, $2\sqrt{3}-0=2.\dfrac{c}{\sqrt{c^2-4}}$.

On solving the above equation, we obtain

$$c=\sqrt{6}\in(2,4).$$

We conclude that Lagrange's mean value theorem is verified.

3.3.7 Example: Let us consider the function $f(x)=\sin x,\ x\in R$.

Take any two points, say, $x_1,x_2\in R$. Then, *f* is continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2).

By applying Lagrange's mean value theorem 3.3.1, we find at least on *c* ∈ (x_1,x_2) such that

$$f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Thus, we obtain

$$|\sin x_2-\sin x_1|=|\cos c||x_2-x_1|\le|x_2-x_1|, \text{ since } |\cos c|\le 1.$$

This shows that $|\sin x_2-\sin x_1|\le|x_2-x_1|$, where $x_1,x_2\in R$.

3.3.8 Example: Let us consider $f(x)=px^2+qx+r,\ x\in R$ which is any quadratic function.

Then, *f* being a polynomial, is continuous and differentiable on *R*.

Now, we consider $f(x) = px^2 + qx + r,\ x \in [a, a+h]$. We notice that f satisfies all the conditions of the Theorem 3 3.1.

Thus, there exists $\theta,\ 0 < \theta < 1$, such that

$$f(a+h) - f(a) = hf'(a+\theta h),\ \ 0 < \theta < 1$$

That is, $p(a+h)^2 + q(a+b) + r - pa^2 - qs - r = h[2(a+\theta h) + q]$

On solving the above equation, we obtain $\theta = \frac{1}{2} \in (0,1)$.

3.3.9 Corollary: Let *f* be differentiable on [*a, b*] and suppose that *f′* is bounded on [*a, b*]. Then, *f* is uniformly continuous on [*a, b*].

Proof: Since *f′* is bounded on [*a, b*], therefore there exists some *M* > 0 such that

$$|f'(x)| \le M \text{ for all } x \in [a,b].$$

Let x_1, x_2, be the any points of [*a, b*] with $x_1 < x_2$. Then *f* satisfies all the conditions of the Theorem 3.3.1. Using Lagrange's mean value theorem, we obtain $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ for some $c \in (x_1, x_2)$.

This gives $|f(x_2) - f(x_1)| = |x_2 - x_1||f'(c)| \le M|x_2 - x_1|$

Now, we select $\delta = \frac{\varepsilon}{M} > 0$.

Thus, $|f(x_2) - f(x_1)| \le \varepsilon$, where $|x_2 - x_1| < \delta$ for all $x_1, x_2 \in [a, b]$.

It follows that *f* is uniformly continuous on [*a, b*].

I.Q. 6:

Value-Addition : Remarks

(i) If the slope of the graph of a function is bounded, then the function is uniformly continuous.

(ii) The interval [*a, b*] in corollary 3.3.9 may be replaced by [–∞, ∞], that is, there is no restriction on [*a, b*].

(iii) The converse of the corollary 3.3.9 is not true.

I.Q. 7:

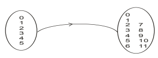
The following results of this section are some of the consequences of Lagrange's mean value theorem 3.3.1. In fact they describe the nature of a function *f* from information given about its derivative.

3.3.10 Theorem : Let $f: I \to R$ be any function with the interval *I* containing more than one point and *I* = [*a, b*]. Then *f* is a constant function on *AI* if and only if *f′* exists and is identically zero on *I*.

Proof: Firstly, we assume that *f* is a constant function. Then, *f* is differentiable on *I* and $f'(x) = 0$ for all $x \in I$. For the converse part, let us suppose that f' exists and $f'(x) = 0$ for all $x \in I$.

We have to show that *f* is a constant function. For this, let us pick $x_1, x_2 \in I$ with $x_1 < x_2$.

Since *f* is differentiable on *I* and $[x_1, x_2] \in I$, therefore *f* is differentiable on $[x_1, x_2]$. By using Lagrange's mean value theorem 3.3.1, we have

$f(x_2) - f(x_1) = f'(c)(x_2, x_1)$ for some .

Now, we are given that $f'(c) = 0$.

Therefore, $f(x_2) = f(x_1)$ which implies that *f* is a constant function on *I*.

3.3.11 Corollary : Let *f* and *g* be continuous function on [*a, b*] and have equal derivatives in (*a, b*). Then, there exists a constant *C* such that *f* = *g* + *C* on *I*.

Proof : Let us consider a function *h* : [*a, b*] → *R* defined by

$$h(x) = f(x) - g(x) \text{ for all } x \in [a,b].$$

Then, by the hypothesis, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in [a, b]$.

Thus, $h'(x) = 0$ for all $x \in [a\ b]$.

Therefore, by applying Theorem 3.3.9, we obtain

$h(x) = C$ for all $x \in [a, b]$ where *C* is a constant.

Hence, $f(x) = g(x) + C$ for all $x \in [a, b]$.

3.4. Monotonicity

This section deals with the monotonic function. Here, we define and study monotonic functions. A relation between monotonicity of a function and the derivative of a function has been established.

3.4.4 Definitions :

(i) A function *f* : [*a, b*] → *R* is said to be an increasing function on [*a, b*] if for every pair x_1 and x_2 of [*a, b*] with $x_1 < x_2$, we have $f(x_1) \le f(x_2)$.

(ii) A function *f* defined on [*a, b*] is said to be strictly increasing on [*a, b*] if for every pair x_1 and x_2 of [*a, b*] with x_1, x_2, we have

$$f(x_1) < f(x_2)$$

(iii) A function *f* defined on [*a, b*] is said to be decreasing on [*a, b*] if for every pair x_1 and x_2 of [*a, b*] with $x_1 < x_2$, we have

$$f(x_1) \ge f(x_2)$$

(iv) A function *f* defined on [*a, b*] is said to be strictly decreasing on [*a, b*] if for every pair x_1 and x_2 of [*a, b*] with $x_1 < x_2$, we have

$$f(x_1) > f(x_2)$$

(v) A function *f* defined on [*a, b*] is said to be monotone or monotonic on [*a, b*] if it is either increasing on I or decreasing on I.

(vi) A function *f* defined on [*a, b*] is said to be strictly monotone on [*a, b*] if it is either strictly increasing on I or strictly decreasing on I.

I.Q. 8:

3.4.2 Examples:

(i) Let us consider the function $f(x): R \to R$ defined by $f(x) = x^3$ is strictly increasing on *R*.

(ii) Let us consider the function $f:(0,\infty) \to R$ defined by

$$f(x) = \frac{1}{x}, \qquad x \in (0,\infty)$$

is strictly decreasing on (0, ∞).

I.Q. 9:

Value Addition : Remarks :
If *f* is an increasing function, then –*f* is decreasing function. Because of this simple fact, in many situations involving monotone functions, it is sufficient to consider only one case either of increasing function or decreasing function.

Our next theorem characterize monotonic functions in terms of sign of derivative of a function and is an immediate consequence of Theorem 3.3.1.

3.4.3 Theorem: Suppose that f is continuous on [*a, b*] and differentiable on (*a, b*). Then

(i) f is increasing on [*a, b*] if and oly if $f'(x) \geq 0$ for all $x \in$ [*a, b*].

(ii) f is decreasing on [*a, b*] if and only if $f'(x) \leq 0$ for all $x \in$ [*a, b*].

Proof: (i) First, we assume that $f'(x) \geq 0$ for all $x \in I$ = [*a, b*].

Let us pick, $x_1, x_2 \in I$, with $x_1 < x_2$.

Then, f satisfies all the conditions of Theorem 3.3.1 on the interval $[x_1, x_2]$.

Now, by using Lagrange's mean value theorem to f on $[x_1, x_2]$. we obtain

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1), \text{ for some } c \in (x_1, x_2)$$

Since $x_2 - x_1 > 0$ and $f'(c) \geq 0$, therefore we have $f(x_2) - f(x_1) \geq 0$.

Thus, $f(x_2) \geq f(x_1)$.

Since x_1, x_2 with $x_1 < x_2$ are chosen arbitrarily in I, therefore, it follows that f is increasing on I. For the converse part, let us assume that f is increasing on I.

Now, pick x, $c \in I$ with $x < c$. Since f is increasing on I, therefore

$$f(c) \geq f(x)$$

which implies that $\dfrac{f(x) - f(c)}{x - c} \geq 0$.

This, $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\geq 0$ by using the fact that if $f(x)\geq 0$, then $\lim_{x\to c} f(x)\geq 0$

But $f'(c)=\frac{f(x)-f(c)}{x-c}\geq 0$.

Hence, we conclude that $f'(x)\geq 0$ for all $x\in I$.

(ii) Let us define a function $g: I\to R$ by $g(x)=-f(x), \quad x\in I$.

Then, g is an increasing function on I and $g'(x)=-f'(x)$, $x\in I$.

Thus, by part (i), we obtain that *g* is an increasing function on I if and only if $g'(x)\geq 0$ for all $x\in I$.

This implies that *f* is a decreasing function if and only if $f'(x)\leq 0$ for all $x \in I$.

3.4.4. Corollary: Suppose that *f* is continuous on [*a, b*] and differentiable on (*a, b*). Then

(i) *f* is strictly increasing on I if $f'(x) > 0$ for all $x\in I$, where $I = [a, b]$

(ii) *f* is strictly decreasing on I if if $f'(x) < 0$ for all $x\in I$, where $I = [a, b]$

Proof: Similar arguments as that of the theorem 3.4.3 may be given in order to prove this assertion.

Value Addition : Remarks
(i) The converse of the Corollary 3.4.4 is not true. For example, let us consider the function $f: R\to R$ defined by $f(x)=x^3, \quad x\in R$. Then, *f* is strictly increasing on *R* but $f'(0)=0$. This shows

that a strictly increasing differentiable function may have a derivative that vanishes at certain points. Similar situation holds for strictly decreasing function.

(ii) We have to assume that *f* is differentiable at *x* = *c.* There exists monotone functions which are not always differentiable. For example let $f(x) = |x|, x \in [-1,1]$. Then *f* is continuous and monotone, but *f* is not differentiable at *x* = 0.

Thus, we cannot apply Theorem 3.4.1 to the function *f* because *f* is not differentiable at *x* = 0.

3.5 Intermediate Value Theorem:

This last section of the chapter deals with an interesting property of derivatives. We know that continuous functions on a closed and bounded interval possess Intermediate Value Property (IVP, in short), which states that a continuous function assumes every value between its maximum and its minimum on the interval. We have already, seen that a function may have a derivative *f′* which exist at every point, but is not continuous at some point. However, not every function is a derivative. Nevertheless, like a continuous function, *f′* also possesses IVP. This important result in due to Darbou*x* and is called the IVP for derivatives. Basically, Darboux's theorem says that IVP holds even if a function is not continuous, provided it is a derivative .

3.5.1 Lemma : Let $I \subseteq R$ be an interval and let $c \in I$. Suppose that $f : I \to R$ be a differentiable function at *c.* Then

(i) If $f'(c) > 0$, then there exist $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ satisfying $c < x < c + \delta$.

(ii) If $f'(c) < 0$, then there exists $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ satisfying $c - \delta < x < c$.

Proof: (i) Since *f* is differentiable at *c,* therefore we have

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)>0$$

We take $\varepsilon=\frac{f'(c)}{2}>0$. Then, by the definition of limit, there exist $\delta > 0$ such that $\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\varepsilon$ whenever $x \in I$ and $0 < | x - c | < \delta$.

It follows that $\frac{f(x)-f(c)}{x-c} > 0$, $x \in I$ and $0 < | x - c | < \delta$.

Thus, $f(x)>f(c)$ for $x \in I$ and $c < x < c + \delta$ for some $\delta > 0$.

(ii) Given that *f* is differentiable at *c,* therefore we obtain

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)<0$$

Here, we take $\varepsilon=-\frac{f'(c)}{2}>0$. Thus, there exists $\delta >0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right|<\varepsilon, \text{ whenever } x \in I \text{ and } 0 < | x - c | < \delta.$$

We conclude that

$$\frac{f(x)-f(c)}{x-c}<0 \quad \text{for } x \in I \text{ and } 0 < | x - c | < \delta.$$

Now, let $x\in I$ with $x < c$, then we have

$$f(x)-f(c)=(x-c).\frac{f(x)-f(c)}{x-c}>0.$$

Hence, $f(x)>f(c)$ for $x \in$ I and $c - \delta < x < c$ for some $\delta > 0$ which completes the proof of the lemma.

Our next result deals with Darboux's theorem which is also called IVP for derivatives.

3.5.2 Theorem (Darboux): Let $I = [a, b]$ and let $f : [a, b] \to R$ be a differentiable function. Suppose that λ is a number between $f'(a)$ and $f'(b)$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = \lambda$.

Proof : We assume that $f'(a) < \lambda < f'(b)$.

Consider the function $g : [a, b] \to R$ defined by

$$g(x) = f(x) - \lambda x, \quad x \in [a, b].$$

Then g, being a difference of the differentiable function on $[a, b]$, is differentiable on $[a, b]$. Since every differentiable function is continuous, therefore g is also continuous on $[a, b]$.

It follows that g attains a maximum value on I. We notice that

$$g'(a) = f'(a) - \lambda < 0.$$

By using Lemma 3.5.1 (i), we obtain that the maximum of g does not occur at $x = a$. Likewise, since $g'(b) = f'(b) - \lambda > 0$, we have from Lemma 3.5.2 (ii) that the maximum of g does not occur at $x = b$.

Thus, g attain its maximum at some $c \in (a, b)$.

Therefore, by applying Theorem 3.1.3, we obtain $g'(c) = 0$

which implies that $f'(c) = \lambda$. Hence, f has the IVP on I.

Value Addition :Note	
(i)	In the above theorem 3.5.2, the domain of f, that is, $[a, b]$ may replaced by any interval I.
(ii)	The IVP theorem 3.5.2, for derivatives establishes that a

derivative cannot change sign in an interval without taking the value 0.

Darboux's theorem is sometimes useful in order to show that a given function cannot be the derivative of any other function.

3.5.3 Example : Let us consider the function $f : [-1, 1] \to R$ defined by

$$f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \\ -1, & -1 \leq x < 0 \end{cases}$$

Then, it is obvious that f attains the values -1, 0 and 1 but none in between.

Consequently f fails to satisfy the IVP theorem 3.5.2

Thus, by Darboux's theorem 3.5.2., we obtain $f \neq g'$ for any differentiable function $g : [-1, 1] \to R$, or Equivalently we can say that f is not the derivative of any function on the interval $[-1, 1]$.

Darboux' theorem also shows that the monotonic derivative are necessarily continuous.

3.5.4 Theorem: Let $f : (a,b) \to R$ be a monotonic function and let f' exists on (a, b). Then f' is continuous on (a, b).

Proof: We prove the result by contradiction.

Let us assume that f' is discontinuous at some point, say, c of (a, b).

We select a closed sub-interval, say $[x_1, x_2]$ of (a, b) which contains c in its interior. Since f' is monotonic on $[x_1, x_2]$,. therefore the discontinuity at c must be a jump discontinuity. Thus, f' omits some value between $f'(x_1)$ and $f'(x_2)$ which contradicts the IVP theorem 3.5.2. Hence, our assumption was wrong and consequently f' is continuous on (a, b) and this completes the proof of the theorem.

I.Q. 10:

3.5.5 Example : Consider the function $f : R \to R$ defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Firstly, we let $x \in R$ such that $x \neq 0$.

Then,

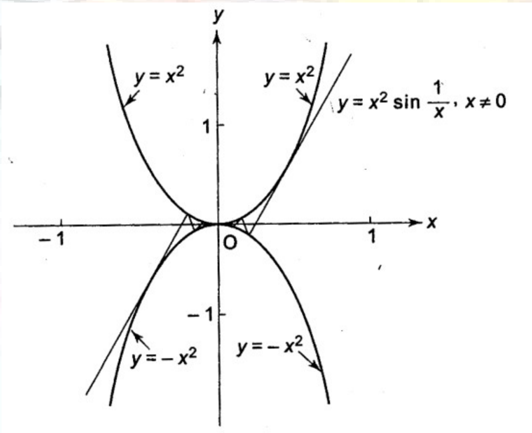


Figure 3

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}, \qquad x \neq 0\,.$$

Now, let us consider the case when $x = 0$.

Then, $Rf'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0^+} \frac{h^2 \sin\frac{1}{h}}{h}$$

$$= \lim_{h \to 0^+} h \sin \frac{1}{h}$$

$= 0$, since $\sin \frac{1}{h}$ is bounded for all *h*.

Also, $Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$

$$= \lim_{h \to 0^+} \frac{h^2 \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2 \sin \frac{1}{h}}{h} = 0$$

Since $Rf'(0) = Lf'(0)$, therefore *f* is differentiable at *x* = 0 and $f'(0) = 0$.

We notice that $\lim_{x \to 0} x \sin \frac{1}{x}$ exists and equals zero, but $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist.

Thus, $\lim_{x \to 0} f'(x)$ does not exist. Hence, f' is not continuous at *x* = 0.

3.5.6 Example: Let *f* be defined on *R* and assume that

$|f(x) - f(y)| \le (x - y)^2$ for all *x*, *y* $\in$ *R*.

Then show that *f* is a constant function.

Solution: Let $c \in R$. Then for $x \neq c$, we obtain

$\left| \frac{f(x) - f(c)}{x - c} \right| \le |x - c|$, using the given condition.

Let $\varepsilon > 0$ be given. Now, we select $\delta = \varepsilon$. Then

$\left| \frac{f(x) - f(c)}{x - c} - 0 \right| < \varepsilon$, whenever $0 < |x - c| < \delta$.

Thus, $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=0$ which implies that $f'(c)=0$ for all $c \in R$.

or equivalently, $f'(x)=0$ for all $x \in R$.

Therefore, by using Theorem 3.3.10, we obtain

$f(x)=C$ for all $x \in R$ where C is a constant.

3.5.7 Example: Show that tan $x > x$ for $0 < x < \pi/2$.

Solution: Consider the function $f(x)=\tan x-x, \quad x\in[0,a]$, where a < $\pi/2$.

Then, the function f is continuous on [0, a] and differentiable on (0, a) with $f'(x)=\sec^2 x-1,\ x\in(0,a)$. By using Lagrange's mean value theorem, we obtain

$$\frac{f(a)-f(0)}{a-0}=f'(c)\ ,\ c\in(0,a)\ .$$

It implies that $\frac{\tan a-a}{a-0}=\sec^2 c-1$,.

or $\quad \tan a-a=a(\sec^2 c-1),\ c\in(0,a) > 0$, for all $a \in (0, \pi / 2)$

or equivalently, $\tan x > x$ for $0 < x < \pi/2$ which is completes the proof.

3.5.8 Example: Find an approximate value of $\sqrt{104}$ using Lagrange's mean value theorem.

Solution: Lt us consider the function $f(x)=\sqrt{x}, x\in[100,104]$. Then, f is continuous on [100, 104] and differentiable on (100, 104).

Thus, by using Lagrange's Mean Value Theorem, we obtain

$$\sqrt{104}-\sqrt{100}=(104-100)\frac{1}{2\sqrt{c}}=\frac{2}{\sqrt{c}} \text{ for some } c \in (100, 104)$$

Now we have $10 < \sqrt{c} < \sqrt{104}$. It implies that

$$2|\sqrt{104} < 2|\sqrt{c} < 2|10,$$

that is, $$2|\sqrt{104} < \sqrt{104} - \sqrt{100} < 2|10$$

that is, $$0.1961 < \sqrt{104} - 10 < 0.2$$

that is, $$10.1961 < \sqrt{104} < 10.20$$

Hence, we conclude that any value in the interval [10.1961, 10.2000] may be regarded as an estimate to $\sqrt{104}$.

3.5.9 Example : Let *f* : *R* → *R* be defined by $f(x) = x + 2x^2 \sin\frac{1}{x}$ for *x* ≠ 0 and f(0) = 0. Show that (i) $f'(0) > 0$ and (ii) *f* is not monotonic on any open interval containing zero.

Solution: (i) Let us compute

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x + 2x^2 \sin\frac{1}{x} - 0}{x} = 1 + 0 = 1 > 0$$

Thus $f'(0) > 0$. Also, we have $f'(x) = 1 + 4x \sin\frac{1}{x} - 2\cos\frac{1}{x},\ x \neq 0$.

(ii) Let $\frac{1}{2n\pi} = x_1$. Then $f'\left(\frac{1}{2n\pi}\right) = 1 + 0 - 2 = -1 < 0$

and $\frac{2}{(4n+1)\pi} = x_2$ and also we have

$$f'\left(\frac{2}{(4n+1)\pi}\right) = 1 + \frac{8}{(4n+1)\pi}.1 - 0 > 0$$ for all *n* ∈ *N*.

Thus, f is not monotonic on any open interval containing zero.

3.5.10 Example: Let I be an interval and let $g : I \to R$ be any differentiable function on I. Prove that if the derivative g' is never zero on I, then either $g'(x) > 0$ for all $x \in I$ or $g'(x) < 0$ for all $x \in I$.

Solution: We prove it by contradiction.

Let us assume that $g'(a) > 0$ and $g'(b) < 0$ for some $a, b \in I$ with $a < b$. It shows that $g'(b) < 0 < g'(a)$.

By using Darboux's theorem for $\lambda = 0$, we obtain

$$g'(c) = 0 \text{ for some } c \in (a, b).$$

which contradicts the hypothesis. Thus, our assumption was wrong.

Hence either $g'(x) > 0$ for all $x \in I$ for $g'(x) < 0$ for all $x \in I$.

3.5.11 Example: Let f be a twice differentiable function on an open interval I and that $f''(x) = 0$ for all $x \in I$. Show that f has the form $f(x) = ax + b$ for suitable constants a and b.

Solution: Applying theorem 3.3.10 to obtain

$$f'(x) = \text{constant} = a \text{ (say)}, \quad x \in I.$$

Let us define $g : I \to R$ by the rule : $g(x) = f(x) - ax, \quad x \in I$.

Then, $g'(x) = f'(x) - a = 0, \quad x \in I$.

Again, by applying Theorem 3.3.10 we have

$$g(x) = \text{constant} = b \text{ (say)}, \quad x \in I.$$

Thus, $f(x) = ax + b, x \in I$ which proves the assertion.

3.5.12 Example: Let *f* be continuous on [*a, b*], differentiable on (*a, b*) and let *f(a)*= 0 and $f'(x) \neq 0$ for all *x* ∈ (*a, b*). Prove that *x* = *a* is the only zero of *f(x)* on [*a, b*].

Solution: We show it by contradiction. Let, if possible, *f* has two zeros, namely, *x* = *a* and *x* = *c* with a < *c* and *a* ≠ *c.* So that we have $f(a) = 0 = f(c)$.

By using Rolle's Theorem we obtain $f'(d) = 0$ for some *d* ∈ (*a, c*).

But this is a contradiction to the fact that $f'(x) \neq 0$ for any *x* ∈ (*a, b*).

Thus, there is only one zero of *f* on [*a, b*] namely *x* = *a.*

Exercises

1. Determine all local maximums and local minimums of the following function:
   (i) $f(x) = x^{35} - 35x$
   (ii) $f(x) = x^{35} - 160x^{28}$
   (iii) $f(x) = x + \frac{1}{x},\ x \neq 0$
2. Locate the points of local extrema for each of the following functions on the specified intervals
   (i) $f(x) = |x^2 - 4|$ on [0, 3]
   (ii) $f(x) = \frac{x}{x^2 + 1}$ on [0, 2]
   (iii) $f(x) = |x^2 - 4x + 3|$ on [0, 4]
3. Determine the intervals on which the following functions are increasing and those on which they are decreasing
   (i) $f(x) = \frac{x - a}{x - b},\ a \neq b$

(ii) $f(x) = \frac{1}{x^2 + a^2}$

(iii) $f(x) = x^2 - 3x + 5$

4. Let *f* : [–1, 1] → *R* be defined by

$$f(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 1 < x \le 1 \end{cases}$$

Does there exist a function g such that $g'(x) = f(x)$ for all *x* ∈ [–1, 1].

5. Give an example of a function *f* : (–1, 1) → *R* which is continuous and attains absolute maximum at 0, but which is not differentiable at 0.

6. Give an example of a function *f* : (–1, 1) → *R* which is differentiable and whose derivative equals 0 at 0, but such that 0 is neither a local maximum nor a local minimum.

7. Prove that $|\cos x - \cos y| \le |x - y|$ for all *x*, *y* ∈ *R*.

8. Prove that if *f* and g are differentiable on *R*, if $f(0) = g(0)$ and if $f'(x) \le g'(x)$ for all *x* ∈ *R*, then $f(x) \le g(x)$ for all *x* ≥ 0.

9. Prove that there is no *k* ∈ *R* for which the following equation $x^3 - 12x + k = 0$ has different roots in [0, 2].

.

10. Find '*c*' of the Lagrange's mean value theorem if

$$f(x) = x(x-1)(x-2), \qquad x \in [0, 1/2]$$

11. Suppose that *f* is differentiable on *R* and let $f(0) = 0, f(1) = 1$ and $f(2) = 1$. Show that $f'(x) = \frac{1}{2}$ for some *x* ∈ (0, 2).

12. Show that $\sin x \le x$ for all *x* ≥ 0.

13. Show that $\frac{x-1}{x} < \log x < x - 1$ for *x* ≥ 1.

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14. Prove that $f(x) = x^3 + px + q$ for $x \in R$, where $p, q \in R$ and $p > 0$ has a unique real root.

15. Examine the validity of the hypothesis and the conclusion of Rolle';s theorem for the following functions.

(i) $f(x) = \cos, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $f(x) = 2 + (x-1)^{2/3} \quad x \in [0,2]$

(iii) $f(x) = \frac{1}{x^2+1}, \quad x \in [-3,3]$

(iv) $f(x) = (x-2)\sqrt{x}, \ x \in [0,2]$

(v) $f(x) = x^{5/2}, \quad x \in [1,5]$.

16. Examine the validity of the hypothesis and the conclusion of Lagrange's mean value theorem for the following functions.

(i) $f(x) = 1/x, \quad x \in [1,4]$

(ii) $f(x) = 1 + x^{2/3}, \quad x \in [-8,1]$

(iii) $f(x) = (x-1)(x-2)(x-3), \ x \in [0,4]$

(iv) $f(x) = \log x, \ x \in [1/2, 2]$

(v) $f(x) = |x|, x \in [-2,1]$

Solution:

1. (i) Local minimum at (1, –34) and local maximum at (–1, 34).

(ii) Local minimum at x = 2 and local maximum at x = 0.

(iii) Local minimum at x = 1 and local maximum at x = –1.

2. (i) Minimum at x = 2 and maximum at x = 3.

(ii) Minimum at x = 0 and maximum at x = 1.

(iii) Minimum at x = 1, x = 3 and maximum at x = 0, x = 4.

3. (i) Increasing on $(-\infty, b)$ and (b, ∞) if $a > b$ and decreasing on the same intervals if $b > a$

   (ii) Increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

   (iii) Increasing on $(3/2, \infty)$ and decreasing on $(-\infty, 3/2)$

4. No. Hint : Use Darboux's theorem.

5. Hint : Consider the function $f : (-1, 1) \to R$ defined by

$$f(x) = -|x|, \ x \in (-1,1)$$

   Then, f is continuous and attains absolute maximum at 0 but f is not differentiable at $x = 0$.

6. Hint : Consider the function $f : (-1, 1) \to R$ defined by

$$f(x) = x^3, \ x \in (-1,1).$$

7. Hint : Use Lagrange's Mean Value Theorem.

8. Hint : Consider the function $h(x) = f(x) - g(x), x \in R$. Then, show that h is an increasing function for all $x \geq 0$.

9. Hint : Apply Rolle's theorem, to the function

$$f(x) = x^3 - 12x + k, \ \ x \in [0,2]$$

10. $c = (c - \sqrt{21})/6$.

11. Hint : Apply Lagrange's Mean Value Theorem to the function f on the interval [0, 2].

12. Hint : Consider the function $f(x) = x - \sin x$ and show that it is increasing on $[0, \infty)$.

13. Hint : Use the Lagrang'e Mean Value Theorem.

14. Hint : Use the Rolle's Theorem.

Summary:

In this chapter we have emphasized on the followings:

- Definition and examples of the local extremum of a function.
- Statement and proof of Rolle's theorem.
- Statement and proof of Lagrange's Mean Value Theorem.
- Characterization of constant function in terms of a derivative of a function.
- Definition and examples of monotonic function and obtained a connection between monotonicity of a function and derivative of a function.
- Statement and proof of Darboux's theorem and noticed that it is the intermediate value property for derivation.

References

1. Aliprantis, C.D., and Burkinshaw Owen, *"Principles of Real Analysis",* Third Edition, Academic Press, 2011.

2. Apostal, T.M., "*Mathematical Analysis*", Second Edition, Addison-Wesley, Reading, MA, 1974.

3. Bartle, R.G., and Sherbert, D.R., *"Introduction to Real Analysis",* Third Edition, John Wiley, New York, 2007.

4. Binmore, K.G., *"Mathematical Analysis",* Second Edition, Cambridge University Press, 1999.

5. Courant, R., and John, F., *"Introduction to Calculus and Analysis",* Vols. I and II, Springer-Verlag, New York, 1980.

6. Ghorpade, Sudhir R., and Limaye, Balmohan V., *"A course in Calculus and Analysis",* Springer-Verlag, New York, 2006.

7. Goffman, C., *"Introduction to Real Analysis",* Harper and Row, New York, London, 1966.

8. Goldberg, R., *"Methods of Real Analysis":,* Second Edition, John-Wiley, New York, 1976.

9. Halmos, P.R, *"Naive Set Theory"*, Springer-Verlag, New York, 1974.

10. Hewitt, E., and Stromberg, K., "Real and Abstract Analysis, Springer-Verlag, New York, 1965.

11. Protter, M.H., and Morrey, C.B., *"A First Course 'm Real Analysis",* Second Edition, Springer-Verlag, New York, 1991.

12. Ross, K.A., *"Elementary Analysis: The Theory of Calculus"*, Springer-Verlag, New York, 1980.

13. Roydon, H.L., *"Real Analysis",* Third Edition, Prentice Hall, Englewood Cliffs, NJ, 1988.

14. Rudin, Walter, *"Principles of Mathematical Analysis",* Third Edition, McGraw-Hill, New York, 1976.

15. Simmons, G.F., *"Introduction to Topology and Modern Analysis"*, McGraw-Hill, Macmillan, New York, 1963.