Discipline Courses-I
Semester-I
Paper: Calculus-I
Lesson: Modeling Ballistics and Planetary Motion

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1. Learning Outcomes

After you have read this chapter, you should be able to

- Modeling the Motion of a Projectile in a vacuum
- Height, Flight Time and Range for Ideal Projectile Motion
- Ideal Trajectory of Projectile is Parabolic
- Kepler's Laws for Planetary Motion
- Proof of Kepler's second Law
- Applications of Kepler's Laws

2. Introduction:

In this section, we shall discuss modeling the motion of a projectile in a vacuum and Kepler's second law.

3. Modeling the Motion of a Projectile in a vacuum:

In order to derive equations for projectile motion, we assume that projectile behaves like a particle moving in a vertical co-ordinate plane and that the only force acting on the projectile during its flight is the constant force of gravity, which always points straight down.

Let the projectile is launched from the origin at time t = 0 into the first quadrant with an initial velocity $\vec{v_0}$. Let $\vec{v_0}$ makes an angle θ with horizontal, then

$$\vec{v_0} = \left(\left|\vec{v_0}\right|\cos\theta\right)\hat{i} + \left(\left|\vec{v_0}\right|\sin\theta\right)\hat{j}$$

If we use the simpler notion v_0 for the initial speed $\left|\vec{v_0}\right|$, then

$$\vec{v_0} = \left(v_0\cos\theta\right)\hat{i} + \left(v_0\sin\theta\right)\hat{j} \qquad \text{...(1)}$$

The projectile's initial position is

$$\vec{r_0} = 0\hat{i} + 0\hat{j} = \vec{0} \qquad \text{...(2)}$$

Newton's second law of motion states that the force acting on the projectile is equal the projectile's mass m times its acceleration. Let $\vec{r}$ is the projectile's position vector and t is time. If the force is only the gravitation force $-mg\hat{j}$, then

$$m\frac{d^2\vec{r}}{dt^2} = -mg\,\hat{j}$$

$$\Rightarrow \qquad \frac{d^2\vec{r}}{dt^2} = -g\,\hat{j}$$

We have to find $\vec{r}$ as a function of t by solving the following initial value problem

$$\frac{d^2\vec{r}}{dt^2} = -g\,\hat{j}$$

Initial conditions: $\vec{r} = \vec{r_0}$ and $\frac{d\vec{r}}{dt} = \vec{v_0}$, when t = 0.

Integrating above equation, we have

$$\frac{d\vec{r}}{dt} = -g\,t\,\hat{j} + \vec{v_0}$$

Again integrating, we get

$$\vec{r}(t) = -\frac{1}{2}gt^2\hat{j} + \vec{v_0}\,t + \vec{r_0}$$

Substituting the values of $\vec{v_0}$ and $\vec{r_0}$ from equations (1) and (2) gives

$$\vec{r}(t) = -\frac{1}{2}gt^2\hat{j} + (v_0\cos\theta)t\,\hat{i} + (v_0\sin\theta)t\,\hat{j}$$

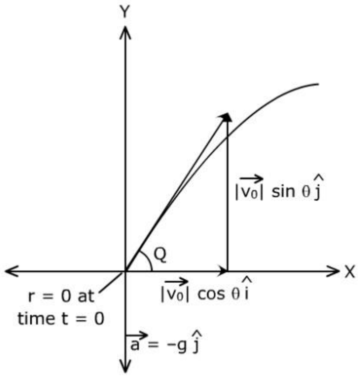

Figure 1

The vector equation for ideal projectile motion is

$$\vec{r}(t) = (v_0 \cos\theta)t\,\hat{i} + \left((v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\hat{j} \qquad \ldots(3)$$

Where θ is the projectile's angle of elevation and v_0 is the projectile's initial speed

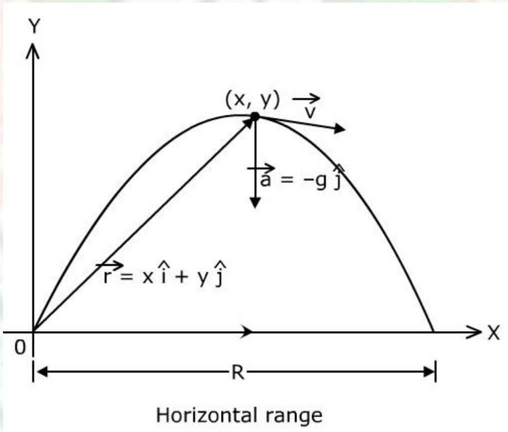

Figure 2

The components of $\vec{r}(t)$ give the parametric equations

$$x(t) = (v_0 \cos\theta)t \text{ and } y(t) = (v_0 \sin\theta)t - \frac{1}{2}gt^2. \qquad \ldots(4)$$

where x(t) is distance downrange and y(t) is the height of the projectile at time t $\geq$ 0

Value Addition: Motion of a projectile in a vacuum
Consider a projectile that travels in a vacuum in a co-ordinate plane, with x axis along level ground. Let the projectile is fired from height of s_0 with initial speed v_0 and angle of elevation θ, then a time t (t $\geq$ 0), it will at the point (x (t), y (t)) where $x(t)=(v_0\cos\theta)t$ and $y(t)=-\frac{1}{2}gt^2+(v_0\sin\theta)t+s_0$.

Example 1: A projectile is fired from the origin over horizontal ground at an initial speed of 400 m/sec and a launch angle 30°. Where will the projectile be 8 second later?

Solution : given, v_0 = 400, θ = 30°, g = 9.8 and t = 8

we have to find the projectile's components 8 seconds after firing.

$$\vec{r}(t) = (v_0\cos\theta)t\,\hat{i} + \left((v_0\sin\theta)t - \frac{1}{2}gt^2\right)\hat{j}$$

$$\Rightarrow \quad \vec{r}(8) = \left(400\cos 30^\circ\right)8\hat{i} + \left(400\left(\sin 30^\circ\right)t - \frac{1}{2}(9.8)8^2\right)\hat{j}$$

$$= (400)\left(\frac{\sqrt{3}}{2}\right)(8)\,\hat{i} + \left((400)\left(\frac{1}{2}\right)8 - \frac{1}{2}(9.8)(64)\right)\hat{j}$$

$$= 8\left(200\sqrt{3}\right)\hat{i} + (1600 - 313.6)\,\hat{j}$$

$$= 1600\sqrt{3}\,\hat{i} + 1286.4\,\hat{j}$$

Therefore, 8 seconds after firing, the projectile is about 1286 m in the air and $1600\sqrt{3}$ m downrange.

3.1. Height, Flight Time and Range:

Equation (3) enables us to answer most questions about the motion for a projectile fired from the origin. The projectile reaches its highest point, when its vertical velocity component is zero.

$$\Rightarrow \qquad \frac{dy}{dt} = v_0 \sin\theta - gt = 0$$

$$\Rightarrow \qquad t = \frac{v_0 \sin\theta}{g}$$

For this value of t, the value of y is

$$y_{\max} = (v_0 \sin\theta)\left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{v_0 \sin\theta}{g}\right)^2$$

$$= \frac{(v_0 \sin\theta)^2}{2g}$$

To find when the projectile lands when fired over horizontal ground, we set the vertical component equal to zero in equation (3) and the time of flight T satisfies

$$(v_0 \sin\theta)\, T - \frac{1}{2} g T^2 = 0$$

$$T\left(v_0 \sin\theta - \frac{1}{2} g T\right) = 0$$

$$\Rightarrow \qquad T = 0\,, \;\; T = \frac{2 v_0 \sin\theta}{g}$$

Since 0 is the time, when the projectile is fired.

$$\Rightarrow \qquad T = \frac{2v_0 \sin\theta}{g}$$

must be time, when the projectile strikes the ground.

To find the projectile's range R, we find the value of the horizontal component, when

$$T = \frac{2v_0 \sin\theta}{g}$$

$$\Rightarrow \qquad x = (v_0 cos\theta)\, t$$

$$R = \left(v_0 \cos\theta\right)\left(\frac{2v_0 \sin\theta}{g}\right)$$

$$= \frac{v_0^2}{g}\left(2\sin\theta\, \cos\theta\right) = \frac{v_0^2}{g}\sin 2\theta$$

The range is largest when $\sin 2\theta = 1$

$$\Rightarrow \qquad 2\theta = 90^\circ$$

$$\Rightarrow \qquad \theta = 45^\circ$$

Value Addition: Height, Flight Time and Range for Ideal Projectile Motion
For ideal projectile motion, when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle θ

Maximum height $= y_{\max} = \dfrac{(v_0 \sin\theta)^2}{2g}$

Flight time $= T = \dfrac{2v_0 \sin\theta}{g}$

Range $= R = \dfrac{v_0^2 \sin 2\theta}{g}$

Example 2: Find the maximum height, flight time and range of a projectile fired from the origin over horizontal ground at an initial speed 500 m/sec and a launch angle of 60°

Solution:

Maximum height $= y_{\max} = \dfrac{(v_0 \sin\theta)^2}{2g}$

$$= \frac{(500 \sin 60^\circ)^2}{2(9.8)} = \frac{\left(500 \times \frac{\sqrt{3}}{2}\right)^2}{19.6}$$

= 9566 m (approximately)

Flight time = T $= \dfrac{2v_0 \sin\theta}{g}$

$$= \frac{2(500)\sin 60^\circ}{9.8} = \frac{1000 \times \frac{\sqrt{3}}{2}}{9.8}$$

= 88.4 second (Approximately)

Range = R $= \frac{v_0^2}{g}\sin 2\theta$

$$= \frac{(500)^2 \sin 120^\circ}{9.8} = \frac{250000 \times \frac{\sqrt{3}}{2}}{9.8}$$

= 22,098 m (Approximately)

The position vector of the projectile is

$$\vec{r}(t) = (v_0 \cos\theta) t\,\hat{i} + \left((v_0 \sin\theta) t - \frac{1}{2} g t^2 \right) \hat{j}$$

$$= \left(500 \cos 60^\circ\right) t\,\hat{i} + \left(\left(500 \sin 60^\circ\right) t - \frac{1}{2}(9.8) t^2 \right) \hat{j}$$

$$= 500 t\,\hat{i} + \left(\left(250\sqrt{3}\right) t - 4.9 t^2 \right) \hat{j}$$

4. Ideal Trajectory of Projectile is Parabolic:

The parametric equations for the motion of a projectile provide useful general information about projectile's motion. If $\theta \neq 0$, we can eliminate t by substituting $t = \frac{x}{(v_0 \cos\theta)}$ in equation (4), we obtain the Cartesian equation.

$$y = -\left(\frac{g}{2 v_0^2 \cos^2 \theta} \right) x^2 + (\tan\theta) x$$

The equation is of the form $y = ax^2 + bx$

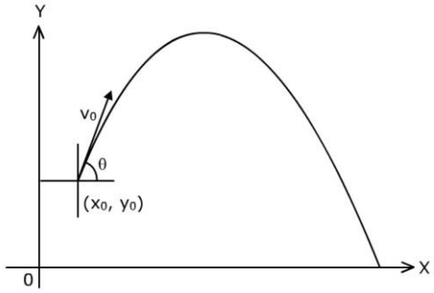

Figure 3

Hence its graph is a parabola

Figure 3 is the path of a projectile fired from (x_0, y_0) with an initial velocity v_0 at an angle of θ with horizontal.

Example 3: A boy standing at the edge of a cliff throws a ball upward at 30° angle with an initial speed of 64 ft/s. Suppose that when the ball leaves the boy's hand, it is 48 ft above the ground at the base of the cliff.

(i) What are the time of flight of the ball and its range?

(ii) What are the velocity of the ball and its speed at impact?

(iii) What is the highest point reached by the ball during its flight?

Solution: given, g = 32 ft/s^2, s_0 = 48 ft.

$v_0 = 64\, ft/s$ and θ = 30°

The parametric equations of ball's trajectory are

$$x(t) = (v_0 \cos\theta)t = (64\cos 30^\circ)t = 32\sqrt{3}t$$

$$y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + s_0$$

$$= -\frac{1}{2}(32)t^2 + \left(64\sin 30^\circ\right)t + 48$$

$$= -16\,t^2 + 32\,t + 48$$

(i) The ball hits the ground, when y = 0

∴ $-16t^2 + 32t + 48 = 0$, for $t \geq 0$

⇒ $2t^2 - 4t - 6 = 0$

⇒ $t^2 - 2t - 3 = 0$

⇒ $t^2 - 3t + t - 3 = 0$

⇒ $t(t - 3) + 1(t - 3) = 0$

$(t - 3)(t + 1) = 0$

⇒ t = 3 and t = -1

So time of flight is T = 3 seconds. The range is

R = x(3) = $32\sqrt{3}\,(3)$ = 166.27688 (Approximately)

⇒ The ball hits the ground about 166 ft from the base of the cliff.

(ii) We find that $x'(t) = 32\sqrt{3}$

$$y'(t) = -32t + 32$$

Hence the velocity at time t is

$$\vec{v}(t) = 32\sqrt{3}\hat{i} + (-32t + 32)\hat{j}$$

Thus, at impact, the velocity is

$$\vec{v}(3) = 32\sqrt{3}\hat{i} - 64\hat{j}$$

Its speed is $\left|\vec{v}(3)\right| = \sqrt{\left(32\sqrt{3}\right)^2 + (-64)^2}$

= 84.664042 (Approximately)

⇒ The speed at impact is about 85 ft/s

(iii) The ball attains its maximum height, when the upward component of its velocity $\vec{v}(t) = 0$

⇒ y′ (t) = 0

⇒ –32t + 32 = 0

⇒ t = 1

Therefore, the maximum height attained by the ball is

$$y_{max} = y(1) = -16(1)^2 + 32(1) + 48$$

$$x_{max} = x(1) = 32\sqrt{3}(1) = 55.425626 \text{ (Approximately)}$$

Hence the highest point reached by the ball has co-ordinates (rounded to the nearest foot)

$$= (55, 64)$$

Example 4: A projectile is fired from the ground level at an angle of 40° with muzzle speed of 110 ft/s. Find the time of flight and the range.

Solution. Given, v_0 = 110 ft/s, θ = 40° and g = 32 ft/s^2

The time of flight is given by

$$T = \frac{2}{g} v_0 \quad \sin\theta = \frac{2}{32}(110) \sin 40^\circ$$

$$= 4.4191648 \text{ (Approximately)}$$

The range R is given by

$$R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(110)^2 \sin 80^\circ}{32}$$

$$= 372.38043 \text{ (approximately)}$$

Hence, the projectile travels about 372 ft horizontally before it hits the ground and the flight takes a little more than 4 seconds.

Exercise-I

1. A projectile is fired at a speed of 840 m/sec at an angle of 60°. How long will it take to get 21 km downrange?

2. A shell fired from ground level at an angle of 45° hits the ground 2000 m away. What is the muzzle speed of the shell?

3. A base ball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20° with the horizontal. At the instant the ball is hit, an instaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of -8.8 $\hat{i}$ (ft/sec) to the ball's initial velocity (8.8 ft/sec = 6 mph)

(i) Find the vector equation for the path of the baseball.

(ii) How high does the baseball go, and when does it reach maximum height?

(iii) Assuming that the ball is not caught, find its range and flight time.

4. At what angle (to the nearest tenth of a degree) should a projectile be fired from ground level if its muzzle speed is 167.1 ft/s and the desired range is 600 ft?

5. A shell is fired at ground level with a muzzle speed of 280 ft/s and at an elevation of 45° from ground level.

(i) Find the maximum height attained by the shell.

(ii) Find the time of flight and the range of the shell.

(iii) Find the velocity and speed of the shell at impact.

6. If a shot putter throws a shot from a height of 5 ft with an angle of 46° and initial speed of 25 ft/s, what is the horizontal distance of the throw?

5. Kepler's Laws:

In the seventeenth century, the German astronomer Johannes Kepler formulated three laws for describing planetary motion which are given below.

1. The planets move about the sun in elliptical orbits, with the sun at one focus.
2. The radius vector joining a planet to the sun sweeps out equal areas in equal intervals of time.

3. The square of the time of one complete revolution of a planet about its orbit is proportional to the cube of the length of the semi major axis of its orbit.

We shall prove the second law using vector methods. The other two laws can be proved similarly. Let $\hat{u}_r$ and $\hat{u}_\theta$ denote unit vectors along the radial axis and orthogonal to that axis respectively.

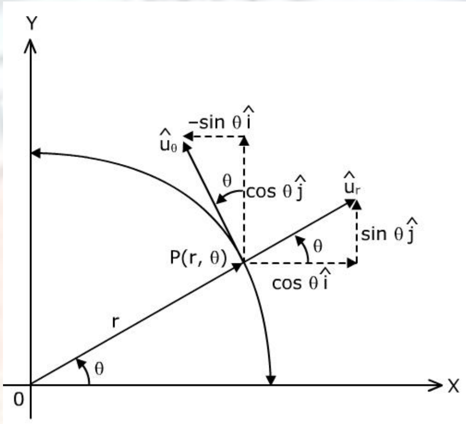

Figure 4: Description of the motion of a particle along a curve.

Then, in terms of unit vectors $\hat{i}$ and $\hat{j}$,

We have

$$\hat{u}_r = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$

$$\hat{u}_\theta = (-\sin\theta)\hat{i} + (\cos\theta)\hat{j}$$

On differentiating above equations w.r.t. θ, we have

$$\frac{d\hat{u}_r}{d\theta} = (-\sin\theta)\hat{i} + (\cos\theta)\hat{j} = \hat{u}_\theta$$

$$\frac{d\hat{u}_\theta}{d\theta} = (-\cos\theta)\hat{i} + (-\sin\theta)\hat{j} = -\hat{u}_r$$

Now, suppose the sun S is at the origin (pole) of a polar co-ordinate system and consider the motion of a body B about S.

The radial vector $\vec{r} = \overrightarrow{SB}$ can be written as

$$\vec{r} = r\hat{u}_r = (r\cos\theta)\,\hat{i} + (r\sin\theta)\,\hat{j}$$

Where r = $r = \left|\vec{r}\right|$ and the velocity $\vec{v}$ satisfies

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{dt}$$

$$= \frac{dr}{dt}\hat{u}_r + r\frac{d\hat{u}_r}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta$$

We summarize these formulas in the following box.

Value Addition: Note
Polar formulas for velocity and Acceleration $$\vec{v}(t) = \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta$$ $$\vec{a}(t) = \frac{\vec{v}(t)}{dt} = \frac{d^2\vec{r}}{dt^2}$$ $$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\hat{u}_\theta$$

Example 5: The position vector of a moving body is $\vec{r}(t) = 2t\hat{i} - t^2\hat{j}$ for t $\geq$ 0. Express $\vec{r}$ and the velocity vector $\vec{v}(t)$ in terms of $\hat{u}_r$ and $\hat{u}_\theta$.

Solution: $\vec{r}(t) = 2t\hat{i} - t^2\hat{j}$

$$\Rightarrow \quad r = \left|\vec{r}(t)\right| = \sqrt{(2t)^2 + \left(-t^2\right)^2} = \sqrt{4t^2 + t^4}$$

$$= t\sqrt{t^2 + 4}$$

Here, x = 2t, y = –t^2

We know that $\vec{r} = r\hat{u}_r = t\sqrt{t^2 + 4}\hat{u}_r$

Since,

$$\vec{v}(t) = \frac{dr}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta$$, we need $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$

We find that

$$\frac{dr}{dt} = \sqrt{t^2+4} + t\left(\frac{1}{2}\right)\left(t^2+4\right)^{-\frac{1}{2}}(2t)$$

$$= \frac{2t^2 + 4}{\sqrt{t^2 + 4}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{t^2}{2t}\right) = \tan^{-1}\left(-\frac{t}{2}\right)$$

$$\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{1+\left(-\frac{t}{2}\right)^2}\left(-\frac{1}{2}\right) = -\frac{2}{t^2+4}$$

Thus, $$\vec{v}(t) = \frac{2t^2+4}{\sqrt{t^2+4}}\,\hat{u}_r + t\sqrt{t^2+4}\left(-\frac{2}{t^2+4}\right)\hat{u}_\theta$$

$$= \frac{\left(2t^2+4\right)\hat{u}_r - 2t\hat{u}_\theta}{\sqrt{t^2+4}}$$

5.1. Kepler's Second Law:

The radius vector from the sun to a planet in its orbit sweeps out equal areas in equal intervals of time.

Proof. We will assume that the only force acting on a planet is the gravitational attraction of the sun.

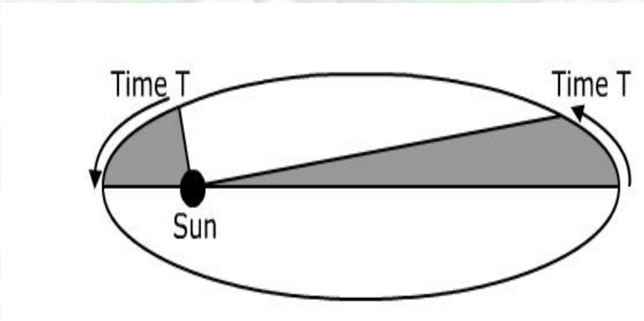

Figure 5: The radial line sweeps out equal area in equal time.

According to the universal law of gravitation, the force of attraction is given by –

$$\vec{F} = -\frac{GmM}{r^2}\hat{u}_r \qquad \text{...(1)}$$

Where G is gravitational constant and m and M are the masses of the planet and the sun respectively.

By Newton's second law of motion

$$\vec{F} = m\,\vec{a} \qquad \text{...(2)}$$

Where $\vec{a}$ is the acceleration of the planet in its orbit from equations (1) and (2), we have

$$m\,\vec{a} = -G\frac{mM}{r^2}\hat{u}_r$$

$$\vec{a} = -\frac{GM}{r^2}\hat{u}_r$$

$\Rightarrow$ The acceleration of a planet in its orbit has only a radial component

This means that the $\hat{u}_\theta$ component of the planet's acceleration is 0. By examining the polar formula for acceleration, we see that this condition is equivalent to the differential equation.

$$r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0 \qquad \text{...(i)}$$

We know that $\omega = \frac{d\theta}{dt} \Rightarrow \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$ ...(ii)

From equations (i) and (ii), we have

$$r\frac{d\omega}{dt} + 2\omega\frac{dr}{dt} = 0$$

$$\Rightarrow \quad \omega^{-1}\frac{d\omega}{dt} = -2r^{-1}\frac{dr}{dt}$$

$$\Rightarrow \quad \int\frac{1}{\omega}d\omega = -2\int\frac{1}{r}dr$$

$\Rightarrow \quad \log|\omega| = -2\log|r| + \log c$

$$= \log\left(cr^{-2}\right)$$

$\Rightarrow \quad \omega = cr^{-2}$

$\Rightarrow \quad \dfrac{d\theta}{dt} = cr^{-2}$

$\Rightarrow \quad r^2 \dfrac{d\theta}{dt} = c$

Let $[t_1, t_2]$ and $[t_3, t_4]$ be two time intervals of equal length.

$\Rightarrow \quad t_2 - t_1 = t_4 - t_3$

The area swept out in the time period $[t_1, t_2]$ is

$$S_1 = \int_{t_1}^{t_2} \frac{1}{2} r^2 d\theta = \int_{t_1}^{t_2} \frac{1}{2}\left(r^2 \frac{d\theta}{dt}\right) dt$$

$$= \int_{t_1}^{t_2} \frac{1}{2} c dt \quad \left(since \ r^2 \frac{d\theta}{dt} = c\right)$$

$$= \frac{1}{2} c [t]_{t_1}^{t_2} = \frac{1}{2} c\left(t_2 - t_1\right)$$

Similarly, the area swept out in time period $[t_3, t_4]$ is

$$S_2 = \frac{1}{2} c\left(t_4 - t_3\right)$$

$\Rightarrow \quad S_1 = \dfrac{1}{2} c\left(t_2 - t_1\right) = \dfrac{1}{2} c\left(t_4 - t_3\right) = S_2$

Hence equal area is swept out in equal time as claimed by Kepler.

Example 6: If the central acceleration is equal to $\frac{\mu}{r^2}$, find the orbit.

Solution: We know that

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

Integrating both side, we have

$$\frac{h^2}{p^2} = \frac{2\mu}{r} + c$$

Comparing with $\frac{b^2}{p^2} = \frac{2a}{r} \mp l$ which is an ellipse or hyperbola referred to the focus, we get

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{c}{\mp l}$$

$\Rightarrow \quad h = \sqrt{\frac{\mu b^2}{a}} = \sqrt{\mu l},$ where l is the semi-latus rectum

$\Rightarrow \quad c = \mp\frac{\mu}{a}$

Hence, the orbit is an ellipse, parabola or hyperbola according as c is negative, zero or positive.

Also $v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + c = \mu\left(\frac{2}{r} \mp \frac{1}{a}\right)$

Hence, for elliptic orbit, $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

For hyperbolic orbit, $v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$

For parabolic orbit, $v^2 = \frac{2\mu}{r}$

Value Addition : Deduction

Since $\frac{1}{2}h$ is the areal velocity.

$\Rightarrow \quad \frac{1}{2}$h. T = Area of ellipse = πab

$\Rightarrow \quad$ periodic time $T = \frac{2\pi\, ab}{h}$

$$= \frac{2\pi ab}{\sqrt{\frac{\mu b^2}{a}}} = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}}$$

Where μ is constant.

Example 7: If v_1 and v_2 are the velocities of a planet, when it is respectively nearest and farthest from the sun, prove that

$$(1-e)v_1 = (1-e)v_2$$

Solution: For an elliptic orbit, we have

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\Rightarrow \quad v_1^2 = \mu\left\{\frac{2}{a(1-e)} - \frac{1}{a}\right\} = \frac{\mu}{a}\left(\frac{1+e}{1-e}\right)$$

$$v_2^2 = \mu\left\{\frac{2}{a(1+e)} - \frac{1}{a}\right\} = \frac{\mu}{a}\left(\frac{1-e}{1+e}\right)$$

Hence, $$\frac{v_1^2}{v_2^2} = \frac{(1+e)^2}{(1-e)^2}$$

$$\Rightarrow \quad \frac{v_1}{v_2} = \frac{1+e}{1-e}$$

$$\Rightarrow \quad (1-e)v_1 = (1+e)v_2$$

Example 8: If ω be the angular velocity of a planet at the nearer end of the major axis, prove that its period is $\frac{2\pi}{\omega}\sqrt{\frac{1+e}{(1-e)^3}}$.

Solution: By Kepler's second law of motion

$$r^2\theta' = h$$

$\therefore \quad a^2(1-e)^2 w = h$ and $h^2 = \mu l = a\mu\left(1-e^2\right)$

$$\Rightarrow \quad a^4(1-e)^4\omega^2 = h^2 = \mu a\left(1-e^2\right)$$

$$\Rightarrow \quad \mu = a^3\frac{(1-e)^3\omega^2}{1+e}$$

Hence, $$T = \frac{2\pi}{\sqrt{\mu}}a^{\frac{3}{2}} = \frac{2\pi}{\omega}\sqrt{\frac{1+e}{(1-e)^3}}$$

Exercise-II

1. A particle moves along the polar path (r, θ), more $\vec{r}(t) = 3+2\sin t$, θ(t) = t^3. Find $\vec{v}(t)$ and $\vec{a}(t)$ in terms of $\hat{u}_r$ and $\hat{u}_\theta$.

2. If the velocity of the earth at any point of its orbit, assumed circular, were increased by about one-half, prove that it would describe a parabola about the sun as focus. Show about that if a body were projected from the earth with a velocity exceeding 7 miles per second, it will not return to the earth.

3. If a planet was suddenly stopped in its orbit supposed circular, show that it would fall into the sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution.

4. Using the formula

$$\vec{v}(t) = \frac{d\vec{r}}{dt}\hat{u}_r + r\frac{d\theta}{dt}\hat{u}_\theta,$$

Prove that polar acceleration

$$\vec{a}(t) = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{u}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}.\frac{d\theta}{dt}\right]\hat{u}_\theta$$

5. A particle is projected from earth's surface with velocity $\vec{v}$. Show that if the dimension of gravity is taken into account, but the resistance of the air is neglected, the path is an ellipse of major axis. $\frac{2ga^2}{2ga - v^2}$, where a is the earth's radius.

Summary:

In this lesson, we have emphasize on the following topics:

- Modeling the Motion of a Projectile in a vacuum
- Height, Flight Time and Range for Ideal Projectile Motion
- Ideal Trajectory of Projectile is Parabolic

- Kepler's Laws for Planetary Motion
- Proof of Kepler's second Law
- Applications of Kepler's Laws

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