Lesson: Numerical Solutions of Algebraic Equations: Iterative Methods

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Numerical Solutions of Algebraic Equations: Iterative Method

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2. Introduction:

We have studied that Gauss elimination method and its variants belongs to the direct methods to solve the system of linear equations. These methods yield the solution after an amount of computation that can be specified in advance. In contrast, in an indirect or iterative method, we start with an approximate solution and if convergent obtain better and better approximations from a computational cycle repeated as often as may be necessary for achieving a desired accuracy. The amount of arithmetic in a iterative method depends upon the accuracy required and varies from case to case. The iterative method is used if the convergence is rapid (if matrices have large main diagonal entries).

In general, one should prefer a direct method for the solution of a linear system but in the case of matrices with a large number of zero elements, it will be advantageous to use iterative methods which preserve these elements.

3. Vector Norms:

Let $X = \left[x_j\right],\ j = 1, 2, \ldots, \mathrm{n}$ is a vector. The distance between a vector and the null vector is a measure of the size or length of the vector. This is also called the norm of the vector.

The norm of a vector X is a real number denoted by $\|X\|$ and satisfies the following axioms:

(I) $\|X\|$ is a non-negative real number.

(II) $\|X\|=0$ if and only if $X=0$.

(III) $\|kX\|=|k|\,\|X\|$ for all k

(IV) $\|X+Y\|\le\|X\|+\|Y\|$ (Triangular Inequality).

3.1. P-norm of a Vector:

The p-norm of a vector $X=(x_1, x_2, \ldots, x_n)$ is denoted by $\|X\|_p$ and defined as

$$\|X\|_p=\left(|x_1|^p+|x_2|^p+\ldots+|x_n|^p\right)^{\frac{1}{p}}$$

where p is a fixed number and $p\ge1$.

3.2. ℓ -norm of a Vector:

The ℓ-norm of a vector $X=(x_1, x_2, \ldots, x_n)$ is denoted by $\|X\|_1$ and defined as

$$\|X\|_1=\left(|x_1|+|x_2|+\ldots+|x_n|\right).$$

3.3. ℓ-norm or Euclidean norm of a Vector:

The Euclidean or ℓ-norm of a vector $X=(x_1, x_2, \ldots, x_n)$ is denoted by $\|X\|_2$ *or* $\|X\|_e$ and defined as

$$\|X\|_e=\left(|x_1|^2+|x_2|^2+\ldots+|x_n|^2\right)^{\frac{1}{2}}=\sqrt{|x_1|^2+|x_2|^2+\ldots+|x_n|^2}\ .$$

$\|X\|_2$ *or* $\|X\|_e$ is called the Euclidean norm because it is just the formula for distance in the three-dimensional Euclidean space.

3.4. ℓ -norm or Maximum Norm of a Vector:

The ℓ -norm or maximum norm of a vector $X=(x_1, x_2, \ldots, x_n)$ is denoted by $\|X\|_\infty$ and defined as

$$\|X\|_\infty = \text{maximum}\left(|x_1|,\ |x_2|, \ldots, |x_n|\right) = \max_j |x_j| .$$

$\|X\|_\infty$ is also called the uniform norm.

4. Matrix Norms:

Let $A=\left[a_{ij}\right]$ is a nxn matrix. The matrix norm is denoted by $\|A\|$ and is a non-negative number which satisfies the following axioms:

(I) $\|A\|>0 \ \textit{if} \ \ A\neq 0 \ \textit{and} \ \|A\|=0 \ \textit{iff} \ A=0$.

(II) $\|kA\|=|k|\ \|A\|$ for any arbitrary number k

(III) $\|A+B\|\leq\|A\|+\|B\|$ (Triangular Inequality).

(IV) $\|AB\|\leq\|A\|\|B\|$

4.1. Euclidean Norm or Frobenius Norm of a Matrix:

Frobenius norm of a matrix A is denoted by $F(A)$ *or* $\|A\|_e$ and defined as

$$\|A\|_e = F(A) = \left[\sum_{i,j=1}^{n} |a_{ij}|^2\right]^{\frac{1}{2}} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

Frobenius norm of a matrix is also called Euclidean norm of the matrix.

4.2. Maximum Norm of a Matrix:

4.2.1. Maximum Absolute Row Sum Norm of a Matrix:

Maximum absolute row sum norm of a matrix is denoted by $\|A\|_\infty$ and defined as

$$\|A\|_\infty = \max_i \sum_j |a_{ij}| .$$

4.2.2. Maximum Absolute Column Sum Norm of a Matrix:

Maximum absolute column sum norm of a matrix is denoted by $\|A\|_1$ and defined as

$$\|A\|_1 = \max_j \sum_i |a_{ij}| .$$

4.3. Hilbert Norm or Spectral Norm of a Matrix:

The Hilbert norm of a matrix A is denoted by $\|A\|_2$ and defined as

$$\|A\|_2 = \sqrt{\lambda}, \quad where \;\; \lambda = \rho(A^* A) .$$

If A is Hermitian or real and symmetric, then

$$\lambda = \rho(A^2) = \rho^2(A)$$

$$\Rightarrow \quad \|A\|_2 = \rho(A) .$$

The choice of a particular norm is dependent mostly on practical considerations. The row-norm is however most widely used because it is easy to compute and at the same time, provides a fairly adequate measure of the size of the matrix.

Example 1: Given the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 6 & 8 & 9 \end{bmatrix}$$

Find $\|A\|_1, \; \|A\|_e \; and \; \|A\|_\infty$.

Solution: We have

$$\begin{aligned} \|A\|_1 &= \max[(1+2+6), \; (3+4+8), \; (5+7+9)] \\ &= \max[9, \; 15, \; 21] = 21. \end{aligned}$$

$$\begin{aligned} \|A\|_e &= \sqrt{1^2+3^2+5^2+2^2+4^2+7^2+6^2+8^2+9^2} \\ &= \sqrt{1+9+25+4+16+49+36+64+81} \\ &= \sqrt{285} \\ &= 16.88 \end{aligned}$$

$$\|A\|_\infty = \max[(1+3+5),\ (2+4+7),\ (6+8+9)]$$
$$= \max[9,\ 13,\ 23] = 23.$$

5. ILL-Conditioned Linear Systems:

In practical applications, if 'small' changes in the input data cause 'large' changes in the solution (the output). Then the computational problem is called ill-conditioned. On the other hand, if the corresponding changes in the solution are also small, then the system is called well-conditioned.

Let

$$A = [a_{ij}]$$

and $s_i = \left[a_{i1}^2 + a_{i2}^2 + \ .\ .\ .\ + a_{in}^2\right]$

Now we define

$$k(A) = \frac{|A|}{s_1 s_2 \ldots s_n}$$

If k(A) is near to unity then the system is called well-conditioned otherwise it is called ill-conditioned.

5.1. Condition Number of a Matrix:

The condition number of non-singular square matrix A is denoted by $k(A)$ and defined as

$$k(A) = \|A\|\|A^{-1}\|.$$

or

$$k(A) = \|A\|\|A^{-1}\| = \sqrt{\frac{\lambda}{\mu}}$$

where $\lambda\ and\ \mu$ are the largest and smallest eigen values in modulus of $A^* A$.

If A is Hermitian or real and symmetric matrix then

$$k(A) = \frac{\lambda^*}{\mu^*}$$

where λ^* and μ^* are the largest and the smallest eigenvalues in modulus of A.

5.2. Residual:

Let $AX = b$ is a system of equations. Then the residual r of an approximate solution $\tilde{X}$ of $AX = b$ is defined as

$$r = b - A\tilde{X}$$

or $$r = A(X - \tilde{X}), \qquad [b = AX]$$

Value Addition: Note
The residual r is small if $\tilde{X}$ has a high accuracy, but the converse may not be true.

Theorem 1: A linear system of equations AX = b whose condition number is small is well-conditioned. A large condition number indicates ill-conditioning.

Proof: Let the system of equations is

$$AX = b \qquad (1)$$

Using the matrix norm properties, we have

$$\|AX\| = \|b\|$$

$$\Rightarrow \quad \|A\|\|X\| \geq \|b\|$$

Let $b \neq 0 \; and \; X \neq 0$

Then we have

$$\frac{1}{\|X\|} \leq \frac{\|A\|}{\|b\|} \qquad (2)$$

We know the residue of an approximate solution $\tilde{X}$ of $AX = b$ is

$$r = A(X - \tilde{X}) \qquad (3)$$

Multiplying (3) by A^{-1} from left and interchanges sides, we have

$$A^{-1}r = A^{-1}A(X - \tilde{X})$$

$$\Rightarrow \quad A^{-1}r = X - \tilde{X}$$

$\Rightarrow \quad X - \tilde{X}$

$\Rightarrow \quad \|X - \tilde{X}\|$

$$= \|A^{-1}\| \|r\|$$

on dividing both sides by $\|X\|$, we have

$$\Rightarrow \quad \frac{\|X - \tilde{X}\|}{\|X\|} \le \frac{1}{\|X\|} \|A^{-1}\| \|r\|$$

$$\le \frac{\|A\|}{\|b\|} \|A^{-1}\| \|r\| \qquad \text{[using equation (2)]}$$

$$\le k(A) \frac{\|r\|}{\|b\|}.$$

Hence if k(A) is small then a small $\frac{\|r\|}{\|b\|}$ implies a small relative error $\frac{\|X - \tilde{X}\|}{\|X\|}$. So that the system is well-conditioned. However if k(A) is large then the system is ill-conditioned.

Example 2: Determine the Euclidean and the maximum absolute row sum norms of the matrix

$$A = \begin{bmatrix} 1 & 7 & -4 \\ 4 & -4 & 9 \\ 12 & -1 & 3 \end{bmatrix}.$$

Solution: We know that

Euclidean norm $= F(A) = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$

Therefore,

$$[F(A)]^2 = 1 + 49 + 16 + 16 + 16 + 81 + 144 + 1 + 9 = 333$$

or $\quad F(A) = \sqrt{333}$.

Maximum absolute row sum norm = $\max_i \sum_{j=1}^{n} |a_{ij}|$

$$= \max[12,\ 17,\ 16] = 17.$$

Example 3: Show that the matrix

$$A = \begin{bmatrix} 25 & 24 & 10 \\ 66 & 78 & 37 \\ 92 & -73 & -80 \end{bmatrix}$$

is ill-conditioned.

Solution: Given matrix is

$$A = \begin{bmatrix} 25 & 24 & 10 \\ 66 & 78 & 37 \\ 92 & -73 & -80 \end{bmatrix}$$

$\Rightarrow \quad |A| = 1$

Also,

$$s_1 = 36.0694,\ s_2 = 108.6692 \ \ and \ \ 142.1021$$

Using the formula

$$k(A) = \frac{|A|}{s_1 s_2 \ldots s_n}$$

we have,

$$k = 1.7954 \times 10^{-6}$$

Since k is very small therefore matrix A is ill-conditioned.

Example 4: Determine the condition number of the matrix

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$$

using the maximum absolute row sum norm.

Solution: We have

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$$

By finding the inverse of the matrix A we have

$$A^{-1} = -\frac{1}{8}\begin{bmatrix} -31 & 44 & -17 \\ 44 & -56 & 20 \\ -17 & 20 & -7 \end{bmatrix}$$

Maximum absolute row-sum norm of A = $\|A\|_\infty$

$$\|A\|_\infty = \max\{14,\ 29,\ 50\} = 50.$$

Maximum absolute row-sum norm of $A^{-1} = \|A^{-1}\|_\infty$

$$\|A^{-1}\|_\infty = \max\{\frac{92}{8},\ 15,\ \frac{44}{8}\} = 15.$$

Therefore, condition number of the matrix A is

$$k(A) = \|A\|_\infty \|A^{-1}\|_\infty = 50 \times 15 = 750.$$

6. Method to Solve the ILL-Conditioned Systems:

The accuracy of an approximate solution can be improve by an iterative procedure.

Consider the system of equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ - \quad - \quad - \quad - \quad &- \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \qquad (1)$$

Let $x_1^{(1)},\ x_2^{(1)},\ \ldots,\ x_n^{(1)}$ is an approximate solution of the system of equations (1). On substituting these values in (1), we have

$$\begin{aligned}
&a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + \ldots + a_{1n}x_n^{(1)} = b_1^{(1)} \\
&a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + \ldots + a_{2n}x_n^{(1)} = b_2^{(1)} \\
&- \quad - \quad - \quad - \quad - \\
&a_{n1}x_1^{(1)} + a_{n2}x_2^{(1)} + \ldots + a_{nn}x_n^{(1)} = b_n^{(1)}
\end{aligned} \qquad (2)$$

on subtracting (2) from (1), we have

$$\begin{aligned}
&a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n = d_1 \\
&a_{21}e_1 + a_{22}e_2 + \ldots + a_{2n}e_n = d_2 \\
&- \quad - \quad - \quad - \quad - \\
&a_{n1}e_1 + a_{n2}e_2 + \ldots + a_{nn}e_n = d_n
\end{aligned} \qquad (3)$$

where $e_i = x_i - x_i^{(1)}$ *and* $d_i = b_i - b_i^{(1)}$

on solving system of equations (3) we get the values of $e_1, e_2, \ldots, e_n$.

Since

$$e_i = x_i - x_i^{(1)},$$

we have

$$x_i = x_i^{(1)} + e_i$$

which is a latter approximation for x_i. This process can be repeated to improve upon the accuracy.

Example 5: Solve the system of equations

$$\begin{aligned} x + 2y &= 2 \\ 1.01x + 2y &= 2.01 \end{aligned}.$$

Solution: Given system of equations is

$$\begin{aligned} x + 2y &= 2 \\ 1.01x + 2y &= 2.01 \end{aligned} \qquad (1)$$

Let $x^{(1)} = 1$ and $y^{(1)} = 1$

is an approximate solution of the given system.

Substituting these value in (1), we have

$$\begin{aligned} x^{(1)} + 2y^{(1)} &= 3 \\ 1.01x^{(1)} + 2y^{(1)} &= 3.01 \end{aligned} \qquad (2)$$

On subtracting the equations of (2) from the corresponding equations of (1), we have

$$\begin{aligned} x - x^{(1)} + 2(y - y^{(1)}) &= -1 \\ 1.01(x - x^{(1)}) + 2(y - y^{(1)}) &= -1 \end{aligned} \qquad (3)$$

On solving the system of equations (3), we obtain

$$x - x^{(1)} = -\frac{1}{2} \quad and \quad y - y^{(1)} = 0$$

Hence

$$x = \frac{1}{2} \quad and \quad y = 1$$

is the required solution.

7. Solutions of Linear Systems: Iterative Methods:

In the Iterative methods, we start with an approximate solution and if convergent, derive a sequence of closer approximations. The cycle of computation is repeated so as to obtain the desired accuracy. This means that in a direct method the amount of computation is fixed, while in an iterative method the amount of computation depends on the accuracy required.

7.1. Jacobi-Iterative Method or Gauss-Jacobi Iterative Method:

Let us consider the system of simultaneous linear equation as

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ . \ . \ . + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ . \ . \ . + a_{2n}x_n = b_2 \\ &- \quad - \quad - \quad - \quad - \\ &a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ . \ . \ . + a_{nn}x_n = b_n \end{aligned} \qquad (1)$$

Let the diagonal coefficients a_{ii} in (1) do not vanish. If this condition is not satisfied then rearrange the equation to satisfy this condition.

Now rearrange the equations as

$$\begin{aligned}
x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 - \ . \ . \ . - \frac{a_{1n}}{a_{11}} x_n \\
x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - \frac{a_{23}}{a_{22}} x_3 - \ . \ . \ . - \frac{a_{2n}}{a_{22}} x_n \\
&- \quad - \quad - \quad - \quad - \\
x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1 - \frac{a_{n2}}{a_{nn}} x_2 - \ . \ . \ . - \frac{a_{n(n-1)}}{a_{nn}} x_{(n-1)}
\end{aligned} \tag{2}$$

Let the first approximations to the unknowns $x_1, x_2, x_3, \ldots, x_n$ be $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$. Putting these in the R.H.S. of (2), a system of second approximation is obtained.

$$\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \frac{a_{13}}{a_{11}} x_3^{(1)} - \ . \ . \ . - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \frac{a_{23}}{a_{22}} x_3^{(1)} - \ . \ . \ . - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
&- \quad - \quad - \quad - \quad - \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \frac{a_{n2}}{a_{nn}} x_2^{(1)} - \ . \ . \ . - \frac{a_{n(n-1)}}{a_{nn}} x_{(n-1)}^{(1)}
\end{aligned} \tag{3}$$

Continuing in this way let $x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots, x_n^{(n)}$ be the n^{th} approximations, then the system of next approximations is given by

$$\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - \ . \ . \ . - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - \ . \ . \ . - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
&- \quad - \quad - \quad - \quad - \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \frac{a_{n2}}{a_{nn}} x_2^{(n)} - \ . \ . \ . - \frac{a_{n(n-1)}}{a_{nn}} x_{(n-1)}^{(n)}
\end{aligned} \tag{4}$$

We proceed in this way until we get a result of desired accuracy.

In the matrix form the solution of system of equations can be written as

$$\hat{X} = HX + C$$

where H is called iteration matrix.

Thus, the (n+1)th approximation of iteration formula can be written as

$$X^{(n+1)} = HX^{(n)} + C \,.$$

This method is also called the method of simultaneous displacements.

Example 6: Solve the system of equations

$$27x_1 + 6x_2 - x_3 = 85$$
$$6x_1 + 15x_2 + 2x_3 = 72$$
$$x_1 + x_2 + 54x_3 = 110$$

by Jacobi iterative method.

Solution: Given system of equations is

$$\begin{aligned} 27x_1 + 6x_2 - x_3 &= 85 \\ 6x_1 + 15x_2 + 2x_3 &= 72 \\ x_1 + x_2 + 54x_3 &= 110 \end{aligned} \tag{1}$$

Rearranging the equations for the unknown with the largest coefficient in terms of the remaining unknowns,

$$\begin{aligned} x_1 &= \frac{1}{27}(85 - 6x_2 + x_3) \\ x_2 &= \frac{1}{15}(72 - 6x_1 - 2x_3) \\ x_3 &= \frac{1}{54}(110 - x_1 - x_2) \end{aligned} \tag{2}$$

Starting with the approximations

$$x_1 = 0,\ x_2 = 0 \ \ and \ \ x_3 = 0$$

we have, first approximation

$$x_1^{(1)} = \frac{85}{27} = 3.15,\ x_2^{(1)} = \frac{72}{15} = 4.8 \ \ and \ \ x_3^{(1)} = \frac{110}{54} = 2.04$$

Second approximations to the solution,

$$\begin{aligned} x_1^{(2)} &= \frac{1}{27}\left(85 - 6x_2^{(1)} + x_3^{(1)}\right) \\ x_2^{(2)} &= \frac{1}{15}\left(72 - 6x_1^{(1)} - 2x_3^{(1)}\right) \\ x_3^{(2)} &= \frac{1}{54}\left(110 - x_1^{(1)} - x_2^{(1)}\right) \end{aligned} \tag{3}$$

on putting the values of $x_1^{(1)},\ x_2^{(1)}\ \ and\ \ x_3^{(1)}$ in (3) we have

$$x_1^{(2)} = 2.16,\ x_2^{(2)} = 3.27 \ \ and \ \ x_3^{(2)} = 1.89$$

Third approximations to the solution,

$$\begin{aligned} x_1^{(3)} &= \frac{1}{27}\left(85 - 6x_2^{(2)} + x_3^{(2)}\right) \\ x_2^{(3)} &= \frac{1}{15}\left(72 - 6x_1^{(2)} - 2x_3^{(2)}\right) \\ x_3^{(3)} &= \frac{1}{54}\left(110 - x_1^{(2)} - x_2^{(2)}\right) \end{aligned} \qquad (4)$$

on putting the values of $x_1^{(2)},\ x_2^{(2)}\ \ and\ \ x_3^{(2)}$ in (4) we have

$$x_1^{(3)} = 2.426,\ x_2^{(2)} = 3.572 \ \ and \ \ x_3^{(2)} = 1.926$$

Fourth approximations to the solution,

$$\begin{aligned} x_1^{(4)} &= \frac{1}{27}\left(85 - 6x_2^{(3)} + x_3^{(3)}\right) \\ x_2^{(4)} &= \frac{1}{15}\left(72 - 6x_1^{(3)} - 2x_3^{(3)}\right) \\ x_3^{(4)} &= \frac{1}{54}\left(110 - x_1^{(3)} - x_2^{(3)}\right) \end{aligned} \qquad (5)$$

on putting the values of $x_1^{(3)},\ x_2^{(3)}\ \ and\ \ x_3^{(3)}$ from (5) we have

$$x_1^{(4)} = 2.4257,\ x_2^{(4)} = 3.5728 \ \ and \ \ x_3^{(4)} = 1.9259$$

Since the values of $x_1^{(4)}, x_2^{(4)}\ and\ x_3^{(4)}$ are sufficiently close to $x_1^{(3)},\ x_2^{(3)}\ \ and\ \ x_3^{(3)}$ respectively. Hence the values $x_1^{(4)} = 2.4257,\ x_2^{(4)} = 3.5728\ \ and\ \ x_3^{(4)} = 1.9259$ can be considered as the solution of the given system.

Example 7: Solve the system of equations

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 20 \\ x_1 + 4x_2 - x_3 &= 6 \\ x_1 - x_2 + 5x_3 &= 7 \end{aligned}$$

by Jacobi iterative method.

Solution: Given system of equations is

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 20 \\ x_1 + 4x_2 - x_3 &= 6 \\ x_1 - x_2 + 5x_3 &= 7 \end{aligned} \qquad (1)$$

Rearranging the equations for the unknown with the largest coefficient in terms of the remaining unknowns,

$$\begin{aligned} x_1 &= \frac{1}{6}(20 - x_2 - x_3) \\ x_2 &= \frac{1}{4}(6 - x_1 + x_3) \\ x_3 &= \frac{1}{5}(7 - x_1 + x_2) \end{aligned} \qquad (2)$$

In matrix form, the above system can be written as

$$\hat{X} = C + HX$$

where

$$C = \begin{bmatrix} 3.3333 \\ 1.5 \\ 1.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -0.1667 & -0.1667 \\ -0.25 & 0 & 0.25 \\ -0.2 & 0.2 & 0 \end{bmatrix} \quad and \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let the initial approximation is

$$X^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

First approximation is

$$X^{(1)} = C + HX^{(0)}$$

$$\Rightarrow \quad X^{(1)} = \begin{bmatrix} 3.3333 \\ 1.5 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 & -0.1667 & -0.1667 \\ -0.25 & 0 & 0.25 \\ -0.2 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.3333 \\ 1.5 \\ 1.4 \end{bmatrix}$$

Second approximation is

$$X^{(2)} = C + HX^{(1)}$$

$$\Rightarrow \quad X^{(2)} = \begin{bmatrix} 3.3333 \\ 1.5 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 & -0.1667 & -0.1667 \\ -0.25 & 0 & 0.25 \\ -0.2 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 3.3333 \\ 1.5 \\ 1.4 \end{bmatrix} = \begin{bmatrix} 2.8499 \\ 1.0167 \\ 1.0333 \end{bmatrix}$$

Third approximation is

$$X^{(3)} = C + HX^{(1)}$$

$$\Rightarrow \quad X^{(3)} = \begin{bmatrix} 3.3333 \\ 1.5 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 & -0.1667 & -0.1667 \\ -0.25 & 0 & 0.25 \\ -0.2 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 2.8499 \\ 1.0167 \\ 1.0333 \end{bmatrix} = \begin{bmatrix} 2.9647 \\ 1.0458 \\ 1.0656 \end{bmatrix}$$

continuing in this way, we obtain

$$X^{(9)} = \begin{bmatrix} 2.9991 \\ 1.0012 \\ 1.0010 \end{bmatrix}$$

and $X^{(10)} = \begin{bmatrix} 2.9995 \\ 1.0005 \\ 1.0004 \end{bmatrix}$

Since the values of $X^{(9)}$ are sufficiently close to $X^{(10)}$. Hence the values $x_1 = 2.99$ [illegible] $1.000 = 1$ *and* $x_3 = 1.00 = 1$ can be considered as the solution of the given system.

Example 8: Solve the system of equations

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

by Jacobi iterative method.

Solution: Given system of equations is

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned} \qquad (1)$$

Rearranging the equations for the unknown with the largest coefficient in terms of the remaining unknowns,

$$\begin{aligned}
x_1 &= \frac{1}{10}\left(3+2x_2+x_3+x_4\right)\\
x_2 &= \frac{1}{10}\left(15+2x_1+x_3+x_4\right)\\
x_3 &= \frac{1}{10}\left(27+x_1+x_2+2x_4\right)\\
x_4 &= \frac{1}{10}\left(-9+x_1+x_2+2x_3\right)
\end{aligned} \qquad (2)$$

In matrix form, the above system can be written as

$$\hat{X} = C + HX$$

where

$$C = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \quad and \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Let the initial approximation is

$$X^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

First approximation is

$$X^{(1)} = C + HX^{(0)}$$

$$\Rightarrow \qquad X^{(1)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix}$$

Second approximation is

$$X^{(2)} = C + HX^{(1)}$$

$$\Rightarrow \quad X^{(2)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} = \begin{bmatrix} 0.78 \\ 1.74 \\ 2.7 \\ -0.18 \end{bmatrix}$$

Third approximation is

$$X^{(3)} = C + HX^{(2)}$$

$$\Rightarrow \quad X^{(3)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.78 \\ 1.74 \\ 2.7 \\ -0.18 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 1.908 \\ 2.916 \\ -0.108 \end{bmatrix}$$

Fourth approximation is

$$X^{(4)} = C + HX^{(3)}$$

$$\Rightarrow \quad X^{(4)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9 \\ 1.908 \\ 2.916 \\ -0.108 \end{bmatrix} = \begin{bmatrix} 0.9624 \\ 1.9608 \\ 2.9592 \\ -0.036 \end{bmatrix}$$

Fifth approximation is

$$X^{(5)} = C + HX^{(4)}$$

$$\Rightarrow \quad X^{(5)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9624 \\ 1.9608 \\ 2.9592 \\ -0.036 \end{bmatrix} = \begin{bmatrix} 0.9845 \\ 1.9848 \\ 2.9851 \\ -0.0158 \end{bmatrix}$$

Sixth approximation is

$$X^{(6)} = C + HX^{(5)}$$

$$\Rightarrow \quad X^{(6)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9845 \\ 1.9848 \\ 2.9851 \\ -0.0158 \end{bmatrix} = \begin{bmatrix} 0.9939 \\ 1.9938 \\ 2.9938 \\ -0.006 \end{bmatrix}$$

Seventh approximation is

$$X^{(7)} = C + HX^{(6)}$$

$$\Rightarrow \quad X^{(7)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9939 \\ 1.9938 \\ 2.9938 \\ -0.006 \end{bmatrix} = \begin{bmatrix} 0.9975 \\ 1.9975 \\ 2.9976 \\ -0.0025 \end{bmatrix}$$

Eighth approximation is

$$X^{(8)} = C + HX^{(7)}$$

$$\Rightarrow \quad X^{(8)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9975 \\ 1.9975 \\ 2.9976 \\ -0.0025 \end{bmatrix} = \begin{bmatrix} 0.9990 \\ 1.9990 \\ 2.9990 \\ -0.001 \end{bmatrix}$$

Ninth approximation is

$$X^{(9)} = C + HX^{(8)}$$

$$\Rightarrow \quad X^{(9)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9990 \\ 1.9990 \\ 2.9990 \\ -0.001 \end{bmatrix} = \begin{bmatrix} 0.9996 \\ 1.9996 \\ 2.9996 \\ -0.0004 \end{bmatrix}$$

Tenth approximation is

$$X^{(10)} = C + HX^{(9)}$$

$$\Rightarrow \quad X^{(10)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9996 \\ 1.9996 \\ 2.9996 \\ -0.0004 \end{bmatrix} = \begin{bmatrix} 0.9998 \\ 1.9998 \\ 2.9998 \\ -0.0002 \end{bmatrix}$$

Eleventh approximation is

$$X^{(11)} = C + HX^{(10)}$$

$$\Rightarrow \quad X^{(11)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9998 \\ 1.9998 \\ 2.9998 \\ -0.0002 \end{bmatrix} = \begin{bmatrix} 0.9999 \\ 1.9999 \\ 2.9999 \\ -0.0001 \end{bmatrix}$$

Twelve approximation is

$$X^{(12)} = C + HX^{(11)}$$

$$\Rightarrow \qquad X^{(12)} = \begin{bmatrix} 0.3 \\ 1.5 \\ 2.7 \\ -0.9 \end{bmatrix} + \begin{bmatrix} 0 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 0.9999 \\ 1.9999 \\ 2.9999 \\ -0.0001 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 2.0 \\ 3.0 \\ 0 \end{bmatrix}$$

Hence the required solution is $x_1 = 1,\ x_2 = 2,\ x_3 = 3\ \ and\ \ x_4 = 0$.

7.2. Gauss-Seidel Iterative Method:

Gauss-Seidel iterative method is of great practical importance. This method is a modification to Jacobi iteration method.

Let us consider the system of simultaneous linear equation as

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ . \ . \ . + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ . \ . \ . + a_{2n}x_n = b_2 \\ &- \quad - \quad - \quad - \quad - \\ &a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ . \ . \ . + a_{nn}x_n = b_n \end{aligned} \qquad (1)$$

Let the diagonal coefficients a_{ii} in (1) do not vanish. If this condition is not satisfied then rearrange the equation to satisfy this condition.

Now rearrange the equations as

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 - \ . \ . \ . - \frac{a_{1n}}{a_{11}}x_n \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 - \ . \ . \ . - \frac{a_{2n}}{a_{22}}x_n \\ &- \quad - \quad - \quad - \quad - \\ x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \frac{a_{n2}}{a_{nn}}x_2 - \ . \ . \ . - \frac{a_{n(n-1)}}{a_{nn}}x_{(n-1)} \end{aligned} \qquad (2)$$

Let the first approximations to the unknowns $x_1, x_2, x_3, \ldots, x_n$ be $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$. Then the second approximation system of next approximations is given by

$$
\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \frac{a_{13}}{a_{11}} x_3^{(1)} - \ . \ . \ . - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(2)} - \frac{a_{23}}{a_{22}} x_3^{(1)} - \ . \ . \ . - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
x_3^{(2)} &= \frac{b_2}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(2)} - \frac{a_{32}}{a_{33}} x_2^{(2)} - \ . \ . \ . - \frac{a_{3n}}{a_{33}} x_n^{(1)} \\
- & \quad - \quad - \quad - \quad - \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(2)} - \frac{a_{n2}}{a_{nn}} x_2^{(2)} - \ . \ . \ . - \frac{a_{n(n-1)}}{a_{nn}} x_{(n-1)}^{(2)}
\end{aligned}
\qquad (3)
$$

Continuing in this way let $x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots, x_n^{(n)}$ be the n[th] approximations, then the system of (n+1)[th] approximations is given by

$$
\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - \ . \ . \ . - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n+1)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - \ . \ . \ . - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
x_3^{(n+1)} &= \frac{b_2}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(n+1)} - \frac{a_{32}}{a_{33}} x_2^{(n+1)} - \ . \ . \ . - \frac{a_{3n}}{a_{33}} x_n^{(n)} \\
- & \quad - \quad - \quad - \quad - \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(n+1)} - \ . \ . \ . - \frac{a_{n(n-1)}}{a_{nn}} x_{(n-1)}^{(n+1)}
\end{aligned}
\qquad (4)
$$

This arrangement may also be written as

$$
\begin{aligned}
a_{11}x_1^{(n+1)} &= -\sum_{i=2}^{n} a_{1i}x_i^{(n)} + b_1 \\
a_{21}x_1^{(n+1)} + a_{22}x_2^{(n+1)} &= -\sum_{i=3}^{n} a_{2i}x_i^{(n)} + b_2 \\
- \quad - \quad - & \quad - \quad - \\
a_{n1}x_1^{(n+1)} + a_{n2}x_2^{(n+1)} + \ . \ . \ . + a_{nn}x_n^{(n+1)} &= b_n
\end{aligned}
\qquad (5)
$$

In matrix notation, the system of equations (3) can be written as

$$(D+L)X^{(n+1)} = -UX^{(n)} + b$$

or
$$
\begin{aligned}
X^{(n+1)} &= -(D+L)^{-1}UX^{(n)} + (D+L)^{-1}b \\
&= HX^{(n)} + C, \ \ n = 0, 1, 2, \ldots
\end{aligned}
\qquad (6)
$$

where $H = -(D+L)^{-1}U$ *and* $C = (D+L)^{-1}b$.

This solution can also be written as

$$\begin{aligned} X^{(n+1)} &= X^{(n)} - \left[I + (D+L)^{-1}U \right] X^{(n)} + (D+L)^{-1}b \\ &= X^{(n)} - (D+L)^{-1}(D+L+U)X^{(n)} + (D+L)^{-1}b \\ &= X^{(n)} - (D+L)^{-1}AX^{(n)} + (D+L)^{-1}b \\ &= X^{(n)} + (D+L)^{-1}\left(b - AX^{(n)}\right). \end{aligned}$$

$$\Rightarrow \qquad X^{(n+1)} - X^{(n)} = (D+L)^{-1}\left(b - AX^{(n)}\right)$$

$$\Rightarrow \qquad (D+L)\left(X^{(n+1)} - X^{(n)}\right) = \left(b - AX^{(n)}\right)$$

or we may write it as

$$(D+L)V^{(n)} = r^{(n)} \tag{7}$$

where $V^{(n)} = X^{(n+1)} - X^{(n)}$ *and* $r^{(n)} = b - AX^{(n)}$

Solve the equation (7) for $V^{(n)}$ by forward substitution. The solution is then found from

$$X^{(n+1)} = X^{(n)} + V^{(n)}.$$

This gives the final solution of the Gauss-Seidel method.

We proceed in this way until we get a result of desired accuracy. Gauss-Seidel method is also called the method of successive displacements.

Example 9: Solve the system of equations

$$\begin{aligned} 27x_1 + 6x_2 - x_3 &= 85 \\ 6x_1 + 15x_2 + 2x_3 &= 72 \\ x_1 + x_2 + 54x_3 &= 110 \end{aligned}$$

by Gauss-Seidel iterative method.

Solution: Given system of equations is

$$\begin{aligned} 27x_1 + 6x_2 - x_3 &= 85 \\ 6x_1 + 15x_2 + 2x_3 &= 72 \\ x_1 + x_2 + 54x_3 &= 110 \end{aligned} \tag{1}$$

Rearranging the equations for the unknown with the largest coefficient in terms of the remaining unknowns,

$$x_1 = \frac{1}{27}\left(85 - 6x_2 + x_3\right)$$

$$x_2 = \frac{1}{15}\left(72 - 6x_1 - 2x_3\right) \qquad (2)$$

$$x_3 = \frac{1}{54}\left(110 - x_1 - x_2\right)$$

let $x_1^{(n)}, x_2^{(n)}$ and $x_3^{(n)}$ be the n^{th} approximations, then the $(n+1)^{th}$ approximations is given by

$$x_1^{(n+1)} = \frac{1}{27}\left(85 - 6x_2^{(n)} + x_3^{(n)}\right) \qquad (3)$$

$$x_2^{(n+1)} = \frac{1}{15}\left(72 - 6x_1^{(n+1)} - 2x_3^{(n)}\right) \qquad (4)$$

$$x_3^{(n+1)} = \frac{1}{54}\left(110 - x_1^{(n+1)} - x_2^{(n+1)}\right) \qquad (5)$$

Starting with the initial approximations

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0 \ \ and \ \ x_3^{(0)} = 0$$

first approximation to x_1 *i.e.* $x_1^{(1)}$, using equation (3) we have

$$x_1^{(1)} = \frac{1}{27}\left(85 - 6x_2^{(0)} + x_3^{(0)}\right)$$

$$x_1^{(1)} = \frac{85}{27} = 3.15$$

first approximation to x_2 *i.e.* $x_2^{(1)}$, using equation (4) we have

$$x_2^{(1)} = \frac{1}{15}\left(72 - 6x_1^{(1)} - 2x_3^{(0)}\right)$$

$$\Rightarrow \qquad x_2^{(1)} = \frac{1}{15}\left(72 - 6\times 3.15 - 2\times 0\right) = 3.54$$

first approximation to x_3 *i.e.* $x_3^{(1)}$, using equation (5) we have

$$x_3^{(1)} = \frac{1}{54}\left(110 - x_1^{(1)} - x_2^{(1)}\right)$$

$$\Rightarrow \qquad x_3^{(1)} = \frac{1}{54}\left(110 - 3.15 - 3.54\right) = 1.91$$

Thus first approximation to the solution is

$$x_1^{(1)} = 3.15,\ x_2^{(1)} = 3.54 \ \ and \ \ x_3^{(1)} = 1.91$$

second approximation to x_1 *i.e.* $x_1^{(2)}$, using equation (3) we have

$$x_1^{(2)} = \frac{1}{27}\left(85 - 6x_2^{(1)} + x_3^{(1)}\right)$$

$$\Rightarrow \qquad x_1^{(1)} = \frac{1}{27}\left(85 - 6 \times 3.54 + 1.91\right) = 2.43$$

second approximation to x_2 *i.e.* $x_2^{(2)}$, using equation (4) we have

$$x_2^{(2)} = \frac{1}{15}\left(72 - 6x_1^{(2)} - 2x_3^{(1)}\right)$$

$$\Rightarrow \qquad x_2^{(2)} = \frac{1}{15}\left(72 - 6 \times 2.43 - 2 \times 1.91\right) = 3.57$$

second approximation to x_3 *i.e.* $x_3^{(2)}$, using equation (5) we have

$$x_3^{(2)} = \frac{1}{54}\left(110 - x_1^{(2)} - x_2^{(2)}\right)$$

$$\Rightarrow \qquad x_3^{(2)} = \frac{1}{54}\left(110 - 2.43 - 3.57\right) = 1.926$$

Thus second approximation to the solution is

$$x_1^{(2)} = 2.43,\ x_2^{(2)} = 3.57 \ \ and \ \ x_3^{(2)} = 1.926$$

Similarly third approximation to the solution is

$$x_1^{(3)} = 2.426,\ x_2^{(3)} = 3.572 \ \ and \ \ x_3^{(3)} = 1.926$$

Since the values of $x_1^{(2)},\ x_2^{(2)}\ and\ x_3^{(2)}$ are sufficiently close to $x_1^{(3)},\ x_2^{(3)}\ and\ x_3^{(3)}$ respectively. Hence the values $x_1^{(3)} = 2.426,\ x_2^{(3)} = 3.572\ and\ x_3^{(3)} = 1.926$ can be considered as the solution of the given system.

Example 10: Solve the system of equations

$$6x_1 + x_2 + x_3 = 20$$
$$x_1 + 4x_2 - x_3 = 6$$
$$x_1 - x_2 + 5x_3 = 7$$

by Gauss-Seidel iterative method.

Solution: Given system of equations is

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 20 \\ x_1 + 4x_2 - x_3 &= 6 \\ x_1 - x_2 + 5x_3 &= 7 \end{aligned} \qquad (1)$$

Rearranging the equations for the unknown with the largest coefficient in terms of the remaining unknowns,

$$\begin{aligned} x_1 &= \frac{1}{6}(20 - x_2 - x_3) \\ x_2 &= \frac{1}{4}(6 - x_1 + x_3) \\ x_3 &= \frac{1}{5}(7 - x_1 + x_2) \end{aligned} \qquad (2)$$

let $x_1^{(n)}, x_2^{(n)}$ and $x_3^{(n)}$ be the n^{th} approximations, then the $(n+1)^{th}$ approximations is given by

$$x_1^{(n+1)} = \frac{1}{6}\left(20 - x_2^{(n)} - x_3^{(n)}\right) \qquad (3)$$

$$x_2^{(n+1)} = \frac{1}{4}\left(6 - x_1^{(n+1)} + x_3^{(n)}\right) \qquad (4)$$

$$x_3^{(n+1)} = \frac{1}{5}\left(7 - x_1^{(n+1)} + x_2^{(n+1)}\right) \qquad (5)$$

Starting with the initial approximations

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0 \ \ and \ \ x_3^{(0)} = 0$$

first approximation to x_1 *i.e.* $x_1^{(1)}$, using equation (3) we have

$$x_1^{(1)} = \frac{1}{6}\left(20 - x_2^{(0)} - x_3^{(0)}\right)$$

$$\Rightarrow \qquad x_1^{(1)} = \frac{1}{6}(20 - 0 - 0) = 3.3333$$

first approximation to x_2 *i.e.* $x_2^{(1)}$, using equation (4) we have

$$x_2^{(1)} = \frac{1}{4}\left(6 - x_1^{(1)} + x_3^{(0)}\right)$$

$\Rightarrow \qquad x_2^{(1)} = \frac{1}{4}(6 - 3.3333 + 0) = 0.6667$

first approximation to x_3 *i.e.* $x_3^{(1)}$, using equation (5) we have

$$x_3^{(1)} = \frac{1}{5}\left(7 - x_1^{(1)} + x_2^{(1)}\right)$$

$\Rightarrow \qquad x_3^{(1)} = \frac{1}{5}(7 - 3.3333 + 0.6667) = 0.8666$

Thus first approximation to the solution is

$$x_1^{(1)} = 3.3333, \; x_2^{(1)} = 0.6667 \quad and \quad x_3^{(1)} = 0.8666$$

second approximation to x_1 *i.e.* $x_1^{(2)}$, using equation (3) we have

$$x_1^{(2)} = \frac{1}{6}\left(20 - x_2^{(1)} - x_3^{(1)}\right)$$

$\Rightarrow \qquad x_1^{(2)} = \frac{1}{6}(20 - 0.6667 - 0.8666) = 3.0777$

second approximation to x_2 *i.e.* $x_2^{(2)}$, using equation (4) we have

$$x_2^{(2)} = \frac{1}{4}\left(6 - x_1^{(2)} + x_3^{(1)}\right)$$

$\Rightarrow \qquad x_2^{(2)} = \frac{1}{4}(6 - 3.0777 + 0.8666) = 0.9472$

second approximation to x_3 *i.e.* $x_3^{(2)}$, using equation (5) we have

$$x_3^{(2)} = \frac{1}{5}\left(7 - x_1^{(2)} + x_2^{(2)}\right)$$

$\Rightarrow \qquad x_3^{(2)} = \frac{1}{5}(7 - 3.0777 + 0.9472) = 0.9739$

Thus second approximation to the solution is

$$x_1^{(2)} = 3.0777, \; x_2^{(2)} = 0.9472 \quad and \quad x_3^{(2)} = 0.9739$$

third approximation to x_1 *i.e.* $x_1^{(2)}$, using equation (3) we have

$$x_1^{(3)} = \frac{1}{6}\left(20 - x_2^{(2)} - x_3^{(2)}\right)$$

$\Rightarrow \qquad x_1^{(3)} = \frac{1}{6}\left(20 - 0.9472 - 0.9739\right) = 3.0131$

third approximation to x_2 *i.e.* $x_2^{(2)}$, using equation (4) we have

$$x_2^{(3)} = \frac{1}{4}\left(6 - x_1^{(3)} + x_3^{(2)}\right)$$

$\Rightarrow \qquad x_2^{(3)} = \frac{1}{4}\left(6 - 3.0131 + 0.9739\right) = 0.9902$

third approximation to x_3 *i.e.* $x_3^{(2)}$, using equation (5) we have

$$x_3^{(3)} = \frac{1}{5}\left(7 - x_1^{(3)} + x_2^{(3)}\right)$$

$\Rightarrow \qquad x_3^{(3)} = \frac{1}{5}\left(7 - 3.0131 + 0.9902\right) = 0.9954$

Thus third approximation to the solution is

$$x_1^{(3)} = 3.0131,\; x_2^{(3)} = 0.9902 \;\; and \;\; x_3^{(3)} = 0.9954$$

Fourth approximation to x_1 *i.e.* $x_1^{(2)}$, using equation (3) we have

$$x_1^{(4)} = \frac{1}{6}\left(20 - x_2^{(3)} - x_3^{(3)}\right)$$

$\Rightarrow \qquad x_1^{(4)} = \frac{1}{6}\left(20 - 0.9902 - 0.9954\right) = 3.0024$

fourth approximation to x_2 *i.e.* $x_2^{(2)}$, using equation (4) we have

$$x_2^{(4)} = \frac{1}{4}\left(6 - x_1^{(4)} + x_3^{(3)}\right)$$

$\Rightarrow \qquad x_2^{(4)} = \frac{1}{4}\left(6 - 3.0024 + 0.9954\right) = 0.9982$

fourth approximation to x_3 *i.e.* $x_3^{(2)}$, using equation (5) we have

$$x_3^{(4)} = \frac{1}{5}\left(7 - x_1^{(4)} + x_2^{(4)}\right)$$

$\Rightarrow \qquad x_3^{(4)} = \frac{1}{5}\left(7 - 3.0024 + 0.9982\right) = 0.9991$

Thus fourth approximation to the solution is

$$x_1^{(4)} = 3.0024,\ x_2^{(4)} = 0.9982 \ \ and \ \ x_3^{(4)} = 0.9991$$

Since the values of $x_1^{(4)},\ x_2^{(4)}\ and\ x_3^{(4)}$ are sufficiently close to $x_1^{(3)},\ x_2^{(3)}\ and\ x_3^{(3)}$ respectively. Hence the values $x_1^{(4)} = 3.0024 \approx 3,\ x_2^{(3)} = 0.9982 \approx 1\ and\ x_3^{(3)} = 0.9991 \approx 1$ can be considered as the solution of the given system.

7.3. Successive Over Relaxation (SOR) Method:

This method is a generalization of the Gauss-Seidel method. SOR method is generally used when the coefficient matrix of the system of equations is symmetric and has property A.

Now, let us define an auxiliary vector $\widehat{X}$ as

$$\widehat{X}^{(n+1)} = -DLX^{(n+1)} - DUX^{(n)} + D^{-1}b \qquad (1)$$

Let the final solution from Gauss-Seidel method is now written as

$$X^{(n+1)} = X^{(n)} + w\left(\widehat{X}^{(n+1)} - X^{(n)}\right)$$

$$\Rightarrow \qquad X^{(n+1)} = (1-w)X^{(n)} + w\widehat{X}^{(n+1)} \qquad (2)$$

replacing the values from (1) in (2) we have

$$X^{(n+1)} = (D+wL)^{-1}\left[(1-w)D - wU\right]X^{(n)} + w(D+wL)^{-1}b$$

$$\Rightarrow \qquad X^{(n+1)} = HX^{(n)} + C,\ \ n = 0, 1,\ 2, \ldots \qquad (3)$$

where $H = (D+wL)^{-1}\left[(1-w)D - wU\right]$ and $C = w(D+wL)^{-1}b$.

Equation (3) can also be written as

$$X^{(n+1)} = X^{(n)} - (D+wL)^{-1}\left[(D+wL) - (1-w)D + wU\right]X^{(n)} + w(D+wL)^{-1}b$$

$$\Rightarrow \qquad X^{(n+1)} = X^{(n)} + w(D+wL)^{-1}r^{(n)}$$

where $r^{(n)} = b - AX^{(n)}$ is the residual.

This can also be written as

$$X^{(n+1)} - X^{(n)} = w(D+wL)^{-1}r^{(n)}$$

$$\Rightarrow \qquad (D+wL)\left(X^{(n+1)} - X^{(n)}\right) = wr^{(n)} \qquad (4)$$

This equation describes the SOR method in its error format.

When w = 1, then equation (4) reduces to the Gauss-Seidel method. The parameter w is called the Relaxation Parameter and $X^{(n+1)}$ is a weighted mean of $\hat{X}^{(n+1)}$ and $X^{(k)}$.

7.3.1. Over Relaxation and Under Relaxation Method:

If w > 1, then the SOR method is called Over Relaxation method and if w<1, then SOR method is called Under Relaxation method.

7.3.2. Optimal Relaxation Parameter for the SOR Method:

The optimal relaxation parameter for the SOR method is denoted by w_{opt} and defined as

$$w_{opt} = \frac{2}{\mu^2}\left[1-\sqrt{1-\mu^2}\right] = \frac{2}{\left[1+\sqrt{1-\mu^2}\right]}$$

where μ is the largest Eigen value in modulus of the Jacobi iteration matrix.

Example 11: Consider the system of equations

$$\begin{bmatrix} 1 & a \\ 2a & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad a = \frac{\sqrt{2}}{2}, \ a \ real$$

for a = 0.25 determine the optimal relaxation factor, if the system is to be solved with relaxation method.

Solution: Given system of equation is

$$\begin{bmatrix} 1 & a \\ 2a & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad a = \frac{\sqrt{2}}{2}, \ a \ real$$

using the Jacobi-iterative method we have

$$x_1 = -ax_2 + b_1$$
$$x_2 = -2ax_1 + b_2$$

Thus the $X^{(n+1)th}$ iteration of the Jacobi iteration method we have

$$X^{(n+1)} = \begin{bmatrix} 0 & -a \\ -2a & 0 \end{bmatrix} X^{(n)} + b$$

for the Eigen values of the matrix $\begin{bmatrix} 0 & -a \\ -2a & 0 \end{bmatrix}$, we have

$$\begin{vmatrix} -\lambda & -a \\ -2a & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^2 - 2a^2 = 0$

$\Rightarrow \quad \lambda = \pm\sqrt{2}a$

Thus largest Eigen value in modulus is $\mu = \sqrt{2}a$.

Thus, Optimal relaxation factor is

$$w_{opt} = \frac{2}{\mu^2}\left[1-\sqrt{1-\mu^2}\right] = \frac{2}{\left[1+\sqrt{1-\mu^2}\right]} = \frac{2}{\left[1+\sqrt{1-2a^2}\right]}$$

for a = 0.25, we have

$$w_{opt} = \frac{2}{\left[1+\sqrt{1-2(0.25)^2}\right]} \approx 1.034.$$

Example 12: Consider the system of equations

$$\begin{bmatrix} 1 & -k \\ -k & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

for k = 0.5 determine the optimal relaxation factor, if the system is to be solved with relaxation method.

Solution: Given system of equation is

$$\begin{bmatrix} 1 & -k \\ -k & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

using the Jacobi-iterative method we have

$$x_1 = kx_2 + b_1$$
$$x_2 = kx_1 + b_2$$

Thus the $X^{(n+1)th}$ iteration of the Jacobi iteration method we have

$$X^{(n+1)} = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} X^{(n)} + b$$

for the Eigen values of the matrix $\begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$, we have

$$\begin{vmatrix} -\lambda & k \\ k & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \qquad \lambda^2 - k^2 = 0$

$\Rightarrow \qquad \lambda = \pm k$

Thus largest eigen value in modulus is $\mu = k$.

Thus, Optimal relaxation factor is

$$w_{opt} = \frac{2}{\mu^2}\left[1-\sqrt{1-\mu^2}\right] = \frac{2}{\left[1+\sqrt{1-\mu^2}\right]} = \frac{2}{\left[1+\sqrt{1-k^2}\right]}$$

for k = 0.5, we have

$$w_{opt} = \frac{2}{\left[1+\sqrt{1-(0.5)^2}\right]} \approx 1.0718.$$

8. Convergence Analysis of Iterative Methods:

To discuss the convergence of the iteration method for solving system of equations $AX = b$, we study the behaviour of the difference between the exact solution X and an approximation $X^{(k)}$. An iterative method is said to converge for an initial solution $X^{(0)}$ if the corresponding iterative sequence $X^{(0)}, X^{(1)}, X^{(2)}, \ldots$ converges to a solution of the given system. Convergence of an iterative method depends on the relation between $X^{(n)}$ *and* $X^{(n+1)}$.

Theorem 2: Let A be a square matrix. Then

$$\lim_{n\to\infty} A^n = 0$$

if $\|A\| < 1$ or iff $\rho(A) < 1$.

Proof: Let us suppose that

$$\|A\| < 1,$$

then we have

$$\|A^n\| \le \|A\|^n$$

and $\left\|\lim_{n\to\infty} A^n\right\| \le \lim_{n\to\infty}\left\|A^n\right\| = 0.$

For simplicity, assume that all the eigenvalues of A are distinct. Then, there exists a similarity transformation P, such that

$$A = P^{-1}DP$$

where D is the diagonal matrix having the eigenvalues of A on the diagonal. Therefore

$$A^n = P^{-1}D^nP$$

where

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & \dots & 0 \\ & \lambda_2^n & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_m^n \end{bmatrix}$$

Thus, $\lim_{n\to\infty} A^n = 0$, if and only if the eigenvalues satisfy $|\lambda_i| < 1$, that is $\rho(A) < 1$.

Theorem 3: The infinite series

$$I + A + A^2 + \dots$$

converges if and only if $\lim_{n\to\infty} A^n = 0$. The series converges to $(I - A)^{-1}$.

Proof: (Necessary Condition) We know that

$\lim_{n\to\infty} A^n = 0$, then $\rho(A) < 1$,

Hence $|I - A| \ne 0$ *and* $(I - A)^{-1}$ exists.

Consider the identity

$$\left(I + A + A^2 + \dots + A^{n-1}\right)(I - A) = I - A^n$$

multiplying $(I - A)^{-1}$ to the left, we have

$$\left(I + A + A^2 + \dots + A^{n-1}\right) = \left(I - A^n\right)(I - A)^{-1}$$

As $n \to \infty$, we get

$$I + A + A^2 + \dots = (I - A)^{-1}.$$

Theorem 4: The iteration method for the solution of system of equations converges to the exact solution for an initial vector $X^{(0)}$, if the norm of iteration matrix is less than unity i.e., $\|H\| < 1$.

Proof: For an iterative method the (n+1)th approximation is given by

$$X^{(n+1)} = HX^{(n)} + C \tag{1}$$

where H is called the iteration matrix.

Without loss of generality may assume that the initial vector is

$$X^{(0)} = 0.$$

We have

$$X^{(1)} = C$$

$$X^{(2)} = HX^{(1)} + C = HC + C = (H+I)C$$

$$X^{(3)} = HX^{(2)} + C = H(H+I)C + C = (H^2 + H + I)C$$

$$X^{(4)} = HX^{(3)} + C = H(H^2 + H + I)C + C = (H^3 + H^2 + H + I)C$$

.

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$$X^{(n+1)} = HX^{(n)} + C = (H^n + H^{n-1} + \ldots + H + I)C$$

$$\Rightarrow \quad \lim_{n\to\infty} X^{(n+1)} = \lim_{n\to\infty}(H^n + H^{n-1} + \ldots + H + I)C$$

$$= (I - H)^{-1} C$$

if $\|H\| < 1$ *or iff* $\rho(H) < 1$.

Hence proved.

Theorem 5: If the coefficient matrix A of the system of equations $AX = b$ is a strictly diagonally dominant matrix, then any of the iteration method converges for any initial starting vector.

Proof: We know that the $(n+1)^{th}$ iteration of the system of equations $AX = b$ by the iteration method is given by

$$X^{(n+1)} = HX^{(n)} + C \quad (1)$$

where H is called the iteration matrix.

(I) Jacobi Iteration Method:

For the Jacobi iteration method, the iteration matrix H is given by

$$H = -D^{-1}(L+U)$$

$$\Rightarrow \quad H = -D^{-1}(A-D) = (\mathrm{I} - D^{-1}A)$$

We know that the iteration method converge if

$$\|H\| < 1$$

$$\Rightarrow \quad \|\mathrm{I} - D^{-1}A\| < 1 \quad (2)$$

Using absolute row sum norm, we have from (2)

$\frac{1}{|a_{ii}|}\sum_{\substack{j=1 \\ j\neq 1}}^{n} |a_{ij}| < 1 \;\; for \;\; all \;\; i$ Which is true, since the matrix A is strictly diagonally dominant.**(I)Gauss-Seidel Iteration Method:**

For the Gauss-Seidel iteration method, the iteration matrix H is given by

$$H = -(D+L)^{-1}U$$
$$= -(D+L)^{-1}[A-(D+L)]$$

$$\Rightarrow \quad H = [\mathrm{I} - (D+L)^{-1}A]$$

We know that the iteration method converge if

$$\|H\| < 1$$

$$\Rightarrow \quad \|\mathrm{I} - (D+L)^{-1}A\| < 1 \quad (2)$$

We know that the iteration method will be convergent if

$$\rho\left[\mathrm{I} - (D+L)^{-1}A\right] < 1$$

Let λ be an eigenvalue of $\mathrm{I} - (D+L)^{-1}A$. then

$$\left(\mathrm{I} - (D+L)^{-1}A\right)X = \lambda X$$

or $$(D+L)X - AX = \lambda(D+L)X$$

or $$-\sum_{j=i+1}^{n} a_{ij}x_j = \lambda\sum_{j=1}^{i} a_{ij}x_j,\ 1 \le i \le n$$

$$\Rightarrow \quad \lambda a_{ii}x_j = -\sum_{j=i+1}^{n} a_{ij}x_j - \lambda\sum_{j=1}^{i-1} a_{ij}x_j$$

$$\Rightarrow \quad \left|\lambda a_{ii}x_j\right| = \left|-\sum_{j=i+1}^{n} a_{ij}x_j - \lambda\sum_{j=1}^{i-1} a_{ij}x_j\right|$$

$$\le \left|\sum_{j=i+1}^{n} a_{ij}x_j\right| + \left|\lambda\sum_{j=1}^{i-1} a_{ij}x_j\right|$$

$$\le \sum_{j=i+1}^{n} \left|a_{ij}x_j\right| + |\lambda|\sum_{j=1}^{i-1} \left|a_{ij}x_j\right|$$

$$\Rightarrow \quad \left|\lambda a_{ii}x_j\right| \le \sum_{j=i+1}^{n} \left|a_{ij}\right|\left|x_j\right| + |\lambda|\sum_{j=1}^{i-1} \left|a_{ij}\right|\left|x_j\right| \qquad (3)$$

Since X is an eigenvector, $X \ne 0$. Without loss of generality, we assume that

$$\|X\|_\infty = 1$$

Choose an index i such that

$$|X_i| = 1 \text{ and } |X_j| \le 1 \text{ for all } j \ne i.$$

From equation (3), we have

$$|\lambda||a_{ii}| \le \sum_{j=i+1}^{n} |a_{ij}| + |\lambda|\sum_{j=1}^{i-1} |a_{ij}|$$

$$\Rightarrow \quad |\lambda|\left[|a_{ii}| - \sum_{j=1}^{i-1}|a_{ij}|\right] \le \sum_{j=i+1}^{n} |a_{ij}|$$

$$\Rightarrow \quad |\lambda| \le \frac{\sum_{j=i+1}^{n} |a_{ij}|}{\left[|a_{ii}| - \sum_{j=1}^{i-1}|a_{ij}|\right]} < 1$$

Which is true, since the matrix A is strictly diagonally dominant.

Exercise:

1. Solve the following system of equations using Jacobi Iteration method:

(I)
$$4x+5y=7$$
$$12x+14y=18$$

(II)
$$5x+9y+2z=24$$
$$9x+4y+z=25$$
$$2x+y+z=11$$

(III)
$$4x+6y+8z=0$$
$$6x+34y+52z=-160$$
$$8x+52y+129z=-452$$

2. Solve the following system of equations using Gauss-Seidel iteration method

(I)
$$-4x-y=1$$
$$-x+4y-z=0$$
$$-y+4z=0$$

(II)
$$2x+2y+3z=1$$
$$4x+2y+3z=2$$
$$x+y+z=3$$

(III)
$$4x_1+x_2+x_3=4$$
$$x_1+4x_2-2x_3=4$$
$$3x_1+2x_2-4x_3=6$$

(IV)
$$x_1+x_2-x_3=2$$
$$2x_1+3x_2+5x_3=-3$$
$$3x_1+2x_2-3x_3=6$$

4. For the following system of equations

(I)
$$2x+y+3z=1$$
$$4x-y+5z=-7$$
$$-3x+2y+4z=-3$$

(II)
$$\begin{aligned} 4x_1+2x_2+4x_3&=10\\ 2x_1+2x_2+3x_3+2x_4&=18\\ 4x_1+2x_2+6x_3+3x_4&=30\\ 2x_2+3x_3+9x_4&=61 \end{aligned}$$

(III)
$$\begin{aligned} 10x_1+2x_2+x_3&=59\\ x_1+8x_2+2x_3&=-4\\ 7x_1-x_2+20x_3&=5 \end{aligned}.$$

(a) Show that the jacobi iteration method converges hence find its rate of convergence.

(b) Obtain the Jacobi iteration method in matrix form.

(c) Starting with $X^{(0)}=0$, iterate three times.

5. For the following system of equations

(I)
$$\begin{aligned} -3x+y&=-2\\ 2x-3y+z&=0\\ 2y-3z&=-1 \end{aligned}$$

(II)
$$\begin{aligned} 5x_1+x_2-2x_3&=2\\ 3x_1+4x_2-x_3&=-2\\ x_1-3x_2+5x_3&=10 \end{aligned}.$$

(a) Show that the Gauss-Seidel iteration method converges hence find its rate of convergence.

(b) Obtain the Gauss-Seidel iteration method in matrix form.

(c) Starting with $X^{(0)}=0$, iterate three times.

6. For the following system of equations

(I)
$$\begin{aligned} 4x+2z&=4\\ 5y+2z&=-3\\ 5x+4y+10z&=2 \end{aligned}$$

(II)
$$\begin{aligned} 3x_1+2x_2&=5\\ 2x_1+3x_2-x_3&=4\\ -x_2+2x_3&=1 \end{aligned}.$$

(III)
$$\begin{aligned} 3x_1 - x_2 &= 3 \\ -x_1 + 4x_2 - x_3 &= 2 \\ -x_2 + 4x_3 &= 3 \end{aligned}$$

(a) Determine the convergence factor for the Jacobi and Gauss-Seidel methods.

(b) Find the optimal relaxation parameter w_{opt} for the SOR iteration method.

7. For the following system of equations

(I)
$$\begin{aligned} 2x + 3y + z &= -1 \\ 3x + 2y + 2z &= 1 \\ x + 2y + 2z &= 6 \end{aligned}$$

(II)
$$\begin{aligned} x_1 + x_2 - x_3 &= -1 \\ 2x_1 + 3x_2 + 5x_3 &= -6 . \\ 3x_1 + 2x_2 - 3x_3 &= 4 \end{aligned}$$

(III)
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= -1 \\ 2x_1 + x_2 + 2x_3 &= 5 \\ 4x_1 + 2x_2 + x_3 &= 3 \end{aligned}$$

Show that both (i) Jacobi method and (ii) Gauss-Seidel iteration method diverges diverge for solving the system of equations.

8. Find the necessary and sufficient conditions on k, so that the (i) Jacobi method, (ii) Gauss-Seidel method converges for solving the system of equations $AX = b$ where

$$A = \begin{bmatrix} 1 & 0 & k \\ 2 & 1 & 3 \\ k & 0 & 1 \end{bmatrix}$$ and b is arbitrary.

Summary:

In this lesson we have emphasized on the followings:

- Vector Norms
- Matrix Norms
- Euclidean Norm or Frobenius Norm of a Matrix
- Maximum Norm of a Matrix
- Hilbert Norm or Spectral Norm of a Matrix
- ILL-Conditioned Linear Systems
- Condition Number of a Matrix

- Residual
- Method to Solve the ILL-Conditioned Systems
- Solutions of Linear Systems: Iterative Methods
- Jacobi-Iterative Method or Gauss-Jacobi Iterative Method
- Gauss-Seidel Iterative Method
- Successive Over Relaxation (SOR) Method
- Convergence Analysis of Iterative Methods

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