Numerical Integration

Paper: Numerical Methods

Lesson :Numerical Integration

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1. Learning Outcomes:

After studying this chapter you will learn that

(1) How we use Interpolation techniques (methods) to approximate the value of the integral for the functions whose antiderivative can't be found,

(2) Lagrange's and Newton Interpolation formulae are used is Numerical Integration.

(3) Effect of choice of method and number of subpartitions of interval for the approximation of Integral is studied.

(4) Error analysis is done to see the accuracy of approximation.

(5) Comparison of the rate of convergence of different numerical formulas like Trapezoidal Rule and Simpson's Rule is explained.

2. Introduction:

The fundamental problem of numerical integration (which is also called numerical quadrature) is that given a function, 'f ' continuous on [a, b] approximate its integral $I(f)=\int_a^b f(x)dx$

Following cases arise when we try to find Integral of a function.

(i) If for the function $f(x)$ its antiderivative $F(x)$ exists such that $F'(x)=f(x)$,then with the help of Fundamental Theorem of Calculus we can find

$$I(f)=F(b)-F(a)$$

(ii) In some cases for the function $f(x)$, its antiderivative can't be expressed in terms of standard function (or simply can't be found)

For example, for $f(x)=e^{-x^2}$, $\int_a^b e^{-x^2}dx$ is not easy to find exactly.

(iii) Functions used in practice are often defined in terms of discrete data, not a formulae so methods of integration are not directly applicable.

Numerical quadrature is the process of finding or evaluating the value of a definite integral from a set of numerical values of the integrand. The problem of numerical integration is solved by first approximating the integrand by polynomial with the help of an interpolation formulae and than integrating this approximation between the smaller limits between which the entire interval is divided.

The areas are than added up to obtain the integral for the entire range.

Quadrature Formula generally is of the form

$$I(f) \approx I_n(f) = \sum_{i-0}^{n} w_i f(x_i)$$

$I_n(f)$ is Newton –Cotes Approximation Integral , x_i are known as the quadrature points or abscissas and the w_i are called the quadrature weights.

Value addition:
$I_n(f)$ is a linear operator just like $I(f)$ that is $I_n(f+g) = I_n(f) + I_n(g)$ $I_n(cf) = c\,I_n(f)$

I.Q.1

I.Q.2

3. General Newton – Cotes Formula:

In Newton Cotes Quadrature formula x_i are taken as equally spaced points from within the intervals [a, b] and the weights w_i are computed by fitting a function to the $f(x_i)$ data and integrating the resulting function exactly.

The basic procedure for developing Newton – Cotes quadrature rules is to first fix the abscissas $x_0, x_1, x_2, ..., x_n \in [a,b]$.

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Next interpolate the integrand, *f*, at the abscissas by the polynomial $P_n(x)$. Finally we integrate the interpolating polynomial and set

$I(f)$	$\approx$	$I_n(f)$	$\equiv$	$I(P_n)$
Real value of integral of original integrand		Newton Cotes formula		Real value of integral of interpolating polynomial.

Because we want the final quadrature rule to show a clear dependence on the data values $f(x_i)$, the Lagrange's form of interpolating polynomial will be used

$$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x) f(x_i)$$

$$= \sum_{i=0}^{n} \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} f(x_i)$$

$\therefore$ Newton – Cotes Quadrature formulae will take the form

$$I_n(f) = \int_a^b \sum_{i=0}^{n} L_{n,i}(x) f(x_i) dx$$

$$= \sum_{i=0}^{n} \left(\int_a^b L_{n,i}(x) dx \right) f(x_i)$$

$$= \sum_{i=0}^{n} w_i f(x_i) \quad \text{where} \quad w_i = \int_a^b L_{n,i}(x) dx$$

We have two forms of Newton – Cotes formulas which differs in their choice of the abscissas within the interval [*a*, *b*]

(i) Closed Newton – Cotes formulas which include the end points of the integration interval $x = a$ and $x = b$

Here for a given '*n*' we take $\Delta x = (b-a)/n$

and $x_i = a + i\Delta x \qquad i = 0,1,2,\ldots,n$

(ii) Open Newton – Cotes formulas which do not include the end points of the integration interval

Here we take $\Delta x = (b-a)/(n+2)$

and then $x_i = a + (i+1)\Delta x \quad i = 0,1,2,\ldots,n.$

3.1. Some Closed Newton – Cotes Formulas:

3.1.1. Trapezoidal Rule:

In the closed Newton – Cotes formulae

We take $n = 1$

Then $\Delta x = b - a$ and $x_0 = a \quad x_1 = b$

$\therefore$ Lagrange's Polynomial associated with these points are

$$L_{1,0}(x) = \frac{b-x}{b-a}, \qquad L_{1,1}(x) = \frac{x-a}{b-a}$$

$\Rightarrow$ Quadrature weights are

$$w_0 = \int_a^b \frac{b-x}{(b-a)}dx \quad \text{and} \quad w_1 = \int_a^b \frac{x-a}{(b-a)}dx$$

Put $x = a + t\Delta x \quad \Rightarrow \quad dx = \Delta x\, dt$

when $x = a$, $\therefore \quad a = a + t\Delta x \quad \Rightarrow \quad t = 0 \quad (\because \Delta x = b - a)$

$x = b \quad \therefore \quad b = a + t\Delta x \quad \Rightarrow \quad (b-a) = t(b-a) \quad \Rightarrow \quad t = 1$

$$\therefore \quad w_0 = \Delta x \int_0^1 t\, dt = \frac{\Delta x}{2} = \frac{(b-a)}{2} = w_1$$

$\therefore$ Closed Newton – Cotes Quadrature formula for $n = 1$ is

$$I(f) \approx I_{1,\text{closed}}(f) = \frac{\Delta x}{2}\left[f(a)+f(b)\right] = \frac{(b-a)}{2}\left[f(a)+f(b)\right]$$

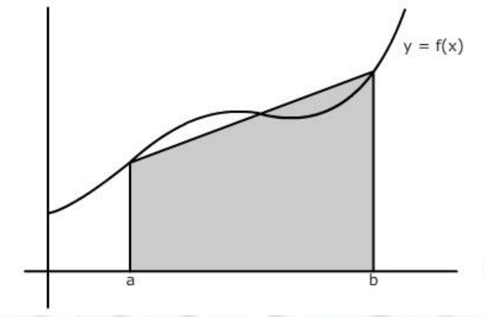

Figure 1: The Trapezoidal Rule

Geometrically, this quadrature rule approximate the value of the definite integral as the area of a trapezoid, so this rule is known as the trapezoidal rule.

3.1.2. Simpson's Rule:

When $n = 2$, the quadrature formulae produces a well known formulae that is, Simpson's Rule.

Here

$$\Delta x = \frac{(b-a)}{2}, \qquad x_0 = a, \qquad x_1 = a + \Delta x = (a+b)/2$$

$$x_2 = a + 2\Delta x = b$$

Weights are calculated as

$$w_0 = \int_a^b L_{2,0}(x)dx = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx$$

Put $\quad x = a + t\Delta x \Rightarrow dx = \Delta x dt$ where $\Delta x = \dfrac{(b-a)}{2}$

when $x = a \Rightarrow a = a + t\Delta x \Rightarrow t = 0$

when $\quad x = b, \Rightarrow \quad b = a + t\left(\dfrac{b-a}{2}\right) \Rightarrow (b-a) = t\left(\dfrac{b-a}{2}\right) \Rightarrow \dfrac{2(b-a)}{(b-a)} = t$

$\Rightarrow \; t = 2$

$$\therefore \qquad w_0 = \int_0^2 \frac{\left(\cancel{x_0} + t\Delta x - \cancel{x_0} - \Delta x\right)\left(\cancel{x_0} + t\Delta x - \cancel{x_0} - 2\Delta x\right)}{\left(\cancel{x_0} - \cancel{x_0} - \Delta x\right)\left(\cancel{x_0} - \cancel{x_0} - 2\Delta x\right)} \Delta x \, dt$$

$$= \int_0^2 \frac{(t-1)(t-2)(\Delta x)^3}{2\cancel{(\Delta x)^2}} \qquad = \frac{\Delta x}{2} \int_0^2 \left[t^2 - 3t + 2\right] dt$$

$$= \frac{\Delta x}{2} \left[\frac{t^3}{3} - \frac{3t^2}{2} + 2t\right]_0^2$$

$$= \frac{\Delta x}{2}\left[\frac{8}{3} - 6 + 4\right] = \frac{\Delta x}{\cancel{2}} \times \frac{\cancel{2}}{3} = \frac{\Delta x}{3}$$

Similarly

$$w_1 = \int_a^b L_{2,1}(x)\,dx = -\Delta x \int_0^2 t(t-2)\,dt = \frac{4}{3}\Delta x$$

$$w_2 = \int_a^b L_{2,2}(x)\,dx = \frac{\Delta x}{2} \int_0^2 t(t-1)\,dt = \frac{\Delta x}{3}$$

$$\therefore \qquad I(f) \approx I_{2,\text{closed}}(f) = \frac{\Delta x}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

which is known as Simpson's Rule.

3.1.3. For *n* = 3 we have Simpson's three – eighth rule:

$$I(f) = \frac{(b-a)}{8}\left[f(a) + 3f(a + \Delta x) + 3f(a + 2\Delta x) + f(b)\right]$$

where $\Delta x = \dfrac{(b-a)}{3}$

3.1.4. For *n* = 4 we have Boole's Rule:

$$I(f)=\frac{(b-a)}{90}\left[7f(a)+32f(a+\Delta x)+12f(a+2\Delta x)+32f(a+3\Delta x)+7f(b)\right]$$

where $\Delta x=(b-a)/4$

3.1.5. For *n* = 6 we have Weddle's Rule:

$$I(f)=\frac{(b-a)}{6}\Big[5f(a)+18f(a+\Delta x)+27f(a+2\Delta x)+24f(a+3\Delta x)$$

$$+\frac{123}{10}f(a+4\Delta x)+\frac{33}{10}f(a+5\Delta x)+\frac{41}{140}f(a+6\Delta x)\Big]$$

3.2. Some Open Newton – Cotes Formulas:

3.2.1. Mid – Point Rule:

The simplest open Newton – Cotes formulae corresponds to $n = 0$

$$\Rightarrow \qquad \Delta x=\frac{(b-a)}{(n+2)}=\frac{(b-a)}{0+2}=(b-a)/2$$

and the only abscissa is $x_0=(a+b)/2$

The quadrature weight is

$$w_0=\int_a^b L_{0,0}(x)dx=\int_a^b dx=b-a$$

$\therefore$ Open Newton – Cotes Quadrature formulae is

$$I(f)\approx I_{0,\text{open}}(f)=(b-a)\times f\left(\frac{a+b}{2}\right) \qquad (1)$$

The formulae given by (1) is known as Mid – Point Rule

3.2.2. For *n* = 1, $\Delta x=\frac{(b-a)}{(n+2)} = \frac{(b-a)}{3}$

and the abscissa are $x_0=a+\Delta x$

$$x_1 = a + 2\Delta x$$

Quadrature weights are

$$w_0 = \int_a^b L_{1,0}(x) = \int_a^b \frac{x - x_1}{x_0 - x_1} dx = \int_a^b \frac{x - (a + 2\Delta x)}{(a + \Delta x - a - 2\Delta x)}$$

$$= \int_a^b \frac{a + 2\Delta x - x}{\Delta x} \qquad \text{Put} \quad x = a + t\Delta x, \quad dx = \Delta x\, dt$$

$$= \Delta x \int_0^3 (2 - t)\, dt = \Delta x \left[2t - \frac{t^2}{2} \right]_0^3$$

$$= \Delta x \left[6 - \frac{9}{2} \right] = \frac{3\Delta x}{2}$$

when $\quad x = a, \quad t = 0$

$$x = b = t = \frac{b - a}{\Delta x} = \frac{(b - a)}{\left(\frac{b - a}{3} \right)} = 3$$

$\therefore \quad w_0 = \dfrac{3\Delta x}{2}$

Similarly

$$w_1 = \int_a^b L_{1,1}(x) dx = \int_a^b \frac{(x - x_0)}{x_1 - x_0} dx = \int_a^b \frac{x - (a + \Delta x)}{(\not{a} + 2\Delta x - \not{a} - \Delta x)} dx$$

$$= \int_a^b \frac{x - (a + \Delta x)}{\Delta x} dx = \frac{3\Delta x}{2}$$

Here also we put $x = a + t\Delta x$

we have

$$\Delta x = \frac{(b - a)}{3}$$

∴ For $n = 1$ Open Newton Cotes formulae is

$$I(f) \approx I_{0,\text{open}}(f) = \frac{(b-a)}{2}\left[f(a+\Delta x) + f(a+2\Delta x) \right]$$

I.Q.3

I.Q.4

3.3. Error Analysis :

Approximation of Integrals involve some error.

The error term associated with a quadrature rule provides two information's.

(i) The error term indicates precisely how the error depends on the length of the integration interval.

(ii) The error term allows us to determine the degree of precision, which characterizes the class of polynomials for which quadrature formulae produces exact result.

Definition: The degree of accuracy (precision) of a quadrature rule $I_n(f)$ is the positive integer m s.t.

$I(P) = I_n(P)$ for every polynomial P of degree $\leq m$

$I(P) \neq I_n(P)$ for some polynomial P of degree $m+1$

Based on this definition, a quadrature rule that integrates every constant polynomial, every linear polynomial and every quadratic polynomial exactly but fails to integrate at least one cubic polynomial exactly would be said to have degree of precision equal to 2.

3.4. Rate of Convergence:

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Let f be a function defined on the interval (a, b) that contains x = 0 and suppose that $\lim_{x\to 0} f(x) = L$. If there exists a function $g(x)$ for which $\lim_{x\to 0} g(x) = 0$ and a positive constant K such that

$$| f(x) - L | \le K | g(x) |$$

for all sufficiently small values of x, then $f(x)$ is said to converge to L with rate of convergence $O(g(x))$.Generally $g(x)$ is of the form x^a, we replace x by h in our discussion.

A powerful tool for deriving error terms associated with quadrature formulas is the following theorem.

4. Weighted Mean – Value Theorem for Integrals:

Statement : If f is continuous on [a, b], g is integrable on [a, b] and $g(x)$ does not change sign on [a, b], then there exists a number $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

As a first example to show the error associated with a Newton – Cotes formulae, let's consider the trapezoidal rule

Using Interpolation

$$f(x) = P_1(x) + f[a,b,x](x-a)(x-b) \qquad (1)$$

where $P_1(x)$ is the unique linear polynomial that interpolates the integrand at $x = a$ and $x = b$

Integrating both sides of (1)

$$I(f) = I_{1,\text{closed}}(f) + \int_a^b f[a,b,x](x-a)(x-b)dx \qquad (2)$$

We observe that $(x-a)(x-b) \le 0 \;\forall\; x \in [a,b]$

$\therefore$ Applying weighted mean value theorem for Integrals in equation (2)

$$I(f) - I_{1,\text{closed}}(f) = f[a,b,\xi]\int_a^b (x-a)(x-b)dx$$

$$= -\frac{(b-a)^3}{6} f[a,b,\xi]$$

$$= -\frac{(b-a)^3}{12} f''(\xi) \quad \text{where } a < \xi < b$$

Since second derivative of every constant and every linear polynomial is zero.

⇒ Trapezoidal rule integrates a constant and linear polynomial exactly.

⇒ Degree of precision of trapezoidal rule is 1.

Value Addition : Note
When the error term for a quadrature rule involves the n-th derivative of the integrand, the rule has degree of precision (n – 1).

I.Q.5

Example 1: Verification of Trapezoidal Rule Degree of Precision

The following table demonstrates explicitly that the trapezoidal rule integrates 1 and x exactly, but fails to integrate x^2 exactly; hence, the degree of precision is 1.

$f(x)$	$\int_a^b f(x)dx$	$h[f(a) + f(a+h)]/2$
1	$b-a$	$b-a$
x	$(b^2 - a^2)/2$	$(b^2 - a^2)/2$
x^2	$(b^3 - a^3)/3$	$(b^3 - a^3 + ba^2 - ab^2)/2$

Example 2: Verification of Simpson's Rule Degree of Precision

$f(x)$	$\int_a^b f(x)dx$	$h[f(a) + 4f(a+h) + f(a+2h)]/3$

1	$b-a$	$b-a$
x	$(b^2-a^2)/2$	$(b^2-a^2)/2$
x^2	$(b^3-a^3)/3$	$(b^3-a^3)/3$
x^3	$(b^4-a^4)/4$	$(b^4-a^4)/4$
x^4	$(b^5-a^5)/5$	$(5a^5-b^4a+2b^3a^2-2b^2a^3+ba^4-5a^5)/24$

Clearly this example shows degree of precision of Simpson's rule is 3.

5. Composite Newton – Cotes Quadrature Method.

Methods of Interpolation suggests us that since we are dealing with polynomial interpolation at equally spaced points, the sequence of approximation by increasing n (number of f nodal points) may not converge to value of exact integral.

An alternative approach for improving the accuracy of an approximation is to subdivide the integration interval [a, b] into pieces and than apply Newton – Cotes formulae on each sub – interval. Numerical integration performed in this manner is referred to as composite Newton – Cotes integration formula

6. Composite Trapezoidal Rule:

We know that

$$I(f)=I_{1,\text{closed}}(f)+\text{error}$$

$$=\frac{(b-a)}{2}\left[f(a)+f(b)\right]-\frac{(b-a)^3}{12}f''(\xi) \qquad (1)$$

If the integration interval [a, b] is split into n subintervals by defining $h=(b-a)/n$ and $x_j=a+jh;\ 0\le j\le n$, and then the trapezoidal rule is applied on each subinterval $\left[x_{j-1},x_j\right]$.

We get

$$I(f)=\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} f(x)dx$$

$$= \sum_{j=1}^{n} \frac{(x_j - x_{j-1})}{2} \left[f(x_{j-1}) + f(x_j) \right] - \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^3}{12} f''(\xi_j)$$

$$= \frac{h}{2} \left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n) \right] - \frac{h^3}{12} \sum_{j=1}^{n} f''(\xi_j)$$

Composite trapezoidal rule Error

$(\because h = x_j - x_{j-1})$ for each j and $x_{j-1} < \xi_j < x_j$ (2)

6.1. Error Analysis:

Suppose f has two continuous derivatives then the Extreme Value Theorem guarantees that there exist two constants $c, d \in [a,b]$ such that

$$f''(c) = \min_{a \le x \le b} f''(x),$$

$$f''(d) = \max_{a \le x \le b} f''(x)$$

$\Rightarrow$ for each j

$$f''(c) \le f''(\xi_j) \le f''(d)$$

Summing over each subinterval $\left[x_{j-1}, x_j \right]$

We find that

$$n\, f''(c) \le \sum_{j=1}^{n} f''(\xi_j) \le n\, f''(d)$$

$$\Rightarrow \quad f''(c) \le \frac{1}{n} \sum_{j=1}^{n} f''(\xi_j) \le f''(d) \qquad (3)$$

$\therefore$ Using Intermediate Value Theorem on relation (1) there exists $\xi \in [a,b]$ such that $f''(\xi) = \frac{1}{n}\sum_{j=1}^{n} f''(\xi_j)$ (4)

$\therefore$ Error for the composite traperiodal rule can be written using relation (2) and (4) as

$$-\frac{h^3}{12}\sum_{j=1}^{n} f''(\xi_j) = -\frac{nh^3}{12} f''(\xi) = -\frac{(b-a)h^2}{12} f''(\xi) \quad [\because nh = (b-a)]$$

Hence

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right] - \frac{(b-a)h^2}{12} f''(\xi)$$

Note 1: Extreme Value Theorem (Statement)

If $f:[a,b] \to R$ is continuous on [a, b], then there exist points c and d in [a, b] such that

$f(c) \le f(x) \le f(d)$ for all $x \in [a,b]$

that is

$f(c) = \min_{a \le x \le b}\{f(x)\}, \quad f(d) = \max_{a \le x \le b}\{f(x)\}$

Note 2: Intermediate Value Theorem (Statement)

Suppose that f : [a, b] $\to$ R is continuous on [a, b]. If v is a number between $f(a)$ and $f(b)$, then there is a point $c \in (a, b)$ such that $f(c) = v$.

Note 3: If the integrand has two continuous derivatives the composite trapezoidal rule has rate of convergence $0(h^2)$.

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Example 3: Numerical Verification of Rate of Convergence.

Consider the integral

$$I(f) = \int_0^{\pi} \sin x \, dx \qquad \text{(Exact value of integral = 2)}$$

The composite trapezoidal rule approximation $T_h(f)$ to $I(f)$ computed using a subinterval size of *h* (*h* changing).

We observe that the sequence is converging with rate of convergence $O(h^2)$

n	h	$T_h(f)$	$\ell_h = \lvert I(f) - T_h(f) \rvert$	ℓ_{2h}/ℓ_h
1	π	0.000000	2.0000000	
2	π/2	1.5707963	0.4292036	4.659792
4	π/4	1.8961188	0.1038811	4.131681
8	π/8	1.9742316	0.0257683	4.0.31337
16	π/16	1.9935703	0.0064296	4.007741
32	π/32	1.9983933	0.0016066	4.001929
64	π/64	1.9995983	0.0004016	4.000482
128	π/128	1.9998996	0.0001004	4.000120

7. Composite Simpson's Rule:

Since the basic Simpson's rule divides the interval [a, b] into two pieces

∴ For Simpson's composite rule, we divide the interval [a, b] into even no. of subinterval

$\therefore$ Let $n = 2m$, define $h = \frac{b-a}{n} = \frac{b-a}{2m}$

$$x_i = a + ih \quad (0 \le i \le 2m)$$

and apply Simpson's rule m times once over each subinterval $[x_{2j-2}\ x_{2j}],\ j = 1, 2, \ldots, m$

$$\therefore \quad I(f) = \sum_{j=1}^{m} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx$$

$$= \sum_{j=1}^{m} \frac{(x_{2j} - x_{2j-2})}{6}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right] - \sum_{j=1}^{m} \frac{(x_{2j} - x_{2j-2})^5}{2880} f^4(\xi_j)$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{m} f(x_{2j-1}) + 2\sum_{j=1}^{m-1} f(x_{2m})\right] - \frac{h^3}{90}\sum_{j=1}^{m} f^{(4)}(\xi_j)$$

$(\because x_{2j} - x_{2j-2} = 2h)$

Provided f has four continuous derivatives.

Similar to composite Trapezoidal Rule

We can find a number $\xi \in [a,b]$ such that $f^4(\xi) = \frac{1}{m}\sum_{j=1}^{m} f^{(4)}(\xi_j)$

$\therefore$ Error Term is $\frac{-h^5 m}{90} f^{(4)}(\xi) = \frac{-(b-a)h^{(4)}(\xi)}{180}$

where $hm = (b-a)/2$

Hence the composite Simpson's rule is

$$I(f) = \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{m} f(x_{2j-1}) + 2\sum_{j=1}^{m-1} f(x_{2j}) + f(x_{2m})\right] - \frac{(b-a)h^4}{180} f^{(4)}(\xi)$$

Note: The composite Simpson's Rule has rate of convergence $O(h^4)$.

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I.Q.6.

Example 4: Numerical Verification of Rate of Convergence of Composite Simpson's Rule

Reconsider the integral $I(f) = \int_0^{\pi} \sin x dx = 2$.

We calculate a sequence of $S_h(f)$ values where $S_h(f)$ is the composite Simpson's rule approximation to I(f) for several values of h (h = size of subinterval) observe that we double the number of subintervals n and subinterval size is reduced by a factor of two

n	h	$S_h(f)$	$e_h = \lvert I(f) - S_h(f) \rvert$	e_{2h}/e_h
2	π/2	2.09439510239	0.09439510239	
4	π/4	2.00455975498	0.00455975498	20.701792
8	π/8	2.00026916995	0.00026916995	16.940059
16	π/16	2.00001659105	0.00001659105	16.223806
32	π/32	2.00000103337	0.00000103337	16.055292
64	π/64	2.00000006453	0.00000006453	16.013782
128	π/128	2.00000000403	0.00000000403	16.003442

The ratio e_{2h}/e_h shows that rate of convergence is $O(h^4)$ that is the error ratio in the last column is approaching $16 = \left(\frac{2h}{h}\right)^4$.

I.Q.7

Example 5: Evaluate $I = \int_0^1 \frac{1}{1+x} dx$ correct to three decimal places.

Solution: We solve this example by both the Trapezoidal and Simpson’s rules with Δx = 0.5, 0.25, 0.125

$$f(x) = \frac{1}{1+x}$$

(i) $\Delta x = 0.5$ the values of x and $f(x)$ are

x	0	0.5	1.0
$f(x)$	1.0000	0.6667	0.5

(a) Trapezoidal rule give

$$I = \frac{1}{4}\left[1.0000 + 2(0.6667) + 0.5\right] = 0.7084$$

(b) Simpson's rule gives

$$I = \frac{1}{6}\left[1.0000 + 4(0.6667) + 0.5\right] = 0.6945$$

(ii) $\Delta = 0.25$ the tabulated values of x and $f(x)$ are

x	0	0.25	0.50	0.75	1.0
$f(x)$	1.0000	0.8000	0.6667	0.5714	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{8}\left[1.0 + 2(0.8000 + 0.6667 + 0.5714) + 0.5\right] = 0.6970$$

(b) Simpson's rule gives

$$I = \frac{1}{2}\left[1.0 + 4(0.8000 + 0.5714) + 2(0.6667) + 0.5\right] = 0.6932$$

(iii) Finally we take $\Delta x = 0.125$

The tabulated values of x and $f(x)$ are

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1

$f(x)$	1	0.8889	0.8000	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

(a) Trapezoidal rule gives

$$I = \frac{1}{16}\Big[1.0 + 2(0.8889 + 0.8000 + 0.7273 + 0.6667 + 0.6154$$

$$+0.5714 + 0.5333) + 0.5\Big]$$

$$= 0.6941$$

(b) Simpson's rule gives

$$I = \frac{1}{24}\Big[1.0 + 4\left(0.8889 + 0.7273 + 0.6154 + 0.5333\right)$$

$$+2\left(0.8000 + 0.6667 + 0.5714\right) + 0.5\Big]$$

$$= 0.6932$$

The exact value of Integral $I = \log_e(2) = 0.693147$

$\therefore$ Approximate value can be taken = 0.693

This example demonstrates that in general Simpson's rule yields more accurate results than the Trapezoidal rule.

Numerical Integration

Exercise:

Q.1 Approximate the value of each of the following integrals using (i) Trapezoidal Rule (ii) Simpson's Rule

(a) $\int_1^2 \frac{1}{x}dx$ (b) $\int_0^1 e^{-x}dx$

(c) $\int_0^1 \frac{1}{1+x^2}dx$ (d) $\int_0^1 \tan(x)dx$

Q.2 (a) Determine values for the coefficient A_0, A_1, and A_2 so that the quadrature formulae

$$I(f)=\int_{-1}^{1} f(x)dx = A_0 f\left(-\frac{1}{3}\right)+A_1 f\left(\frac{1}{3}\right)+A_2 f(1)$$

has degree of precision at least 2.

(b) Once the values of A_0, A_1, and A_2 have been computed determine the overall degree of precision for the quadrature rule.

Q.3 (a) Derive the closed Newton – Cotes formula with $n = 3$

$$I(f)\approx I_{3,\text{closed}}(f)=\frac{(b-a)}{8}\left[f(a)+3f(a+\Delta x)+3f(a+2\Delta x)+f(b)\right]$$

(b) Verify that this formulae has degree of precision equal to 3.

(c) Derive the error term associated with this quadrature rule.

Q.4 Verify for the following that the composite traperiodal rule has rate of convergence $O(h^2)$, composite Simpson's rule has rate of convergence $O(h^4)$ by approximating the value of the indicated definite integral

(a) $\int_1^2 \frac{1}{x}dx$ (b) $\int_0^1 e^{-x}dx$ (c) $\int_0^1 \tan^{-x}(x)dx$

Q.5 Derive the composite mid-point rule with error

$$\int_a^b f(x)dx = 2h\sum_{j=1}^{n} f(x_j) + \frac{(b-a)h^2}{6} f''(\xi)$$

where $h = (b-a)/2h,\ x_j = a + (2j-1)h$ and $\xi \in [a,b]$.

Q.6 Approximate the value of the indicated definite integral using

(a) Composite trapezoidal rule

(b) Composite midpoint rule

(c) Composite Simpson's rule

(i) $\int_0^1 e^{-2x}dx$ (ii) $\int_1^2 \frac{\sin x}{x}dx$ (iii) $\int_0^1 \sqrt{1+x^2}dx$

Numerical Integration

Summary:

Numerical quadrature (Integration) is a procedure that helps us in finding the approximate integral value of a function whose exact antiderivative is either not possible or very difficult to find.First we approximate the integrand by a polynomial(polynomial interpolation) and then integrating that polynomial between the limits. The general formulae used is Newton-Cotes Quadrature rule then we derived some closed formulaes like Trapezoidal rule and Simpsons rule and open formulaes like Mid-Point method.

During the approximation of integral we are faced with some error as different formulaes have different level of accuracy. Estimation of error and the bounds on the error is shown for all the Quadrature formulaes.

Numerical Integration

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