Lesson: Numerical Solutions of Algebraic Equations: Direct Methods

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2. Introduction:

Consider a system of n linear algebraic equations in n unknowns $x_1, x_2, \ldots, x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ . \quad . \quad \ldots \quad . \quad &. \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \tag{1}$$

where $a_{ij}\ (i = 1, 2, \ldots, n\ \&\ j = 1, 2, \ldots, n)$ are the known coefficients, $b_i\,(i = 1, 2, \ldots, n)$ are the known values and $x_i\,(i = 1, 2, \ldots, n)$ are the unknowns to be determined.

Above system of linear equations may be represented at the matrix equations as follow

$$AX = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & \ldots & . \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad and \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

The system of equations given above is said to be homogeneous if all the $b_i\,(i = 1, 2, \ldots, n)$ vanish otherwise it is called as non-homogeneous system of equations.

By finding a solution of a system of equations we mean to obtain the value of $x_1, x_2, \ldots, x_n$ such that they satisfy the given equations and a solution vector of system of equations (1) is a vector X whose components constitute a solution of (1)

There are two types of numerical methods to solve the above system of equations

(I) Direct Methods: direct methods such as Gauss Elimination method, in such methods the amount of computation to get a solution can be specified in advance.

(II) Indirect or Iterative Methods: Such as Gauss-Siedel Methods, in such methods we start from a (possibly crude) approximation and improve it stepwise by repeatedly performing the same cycle of composition with changing data.

3. Direct Methods to Solve the System of Linear Equations:

The necessary and sufficient condition for the existence of a solution of the system of equations

$$AX = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad and \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

is that

Rank [A] = Rank [A : b]

or we can say that rank of the coefficient matrix is the same as the rank of the augmented matrix.

Value Addition: Existence of a Solution of the Equation AX=b
(I) if $b_i = 0 \text{ and } \det A = 0$, then there exist infinite number of non-trivial solutions besides trivial solution X = 0. (II) If $b_i = 0 \text{ and } \det A \neq 0$, then the system has the only unique trivial solution X = 0. In this case Rank (A) = n (Number of variables). (III) If $b_i \neq 0 \text{ and } \det A \neq 0$, then the system has only unique solution and in this case Rank (A) = n (Numbers of variables). (IV) If $b_i \neq 0 \text{ and } \det A = 0$, then there exist infinite number of solutions provided the equations are consistent. In this case we have Rank (A) < n.

3.1. Method of Solution using Inverse of the Matrix:

Consider the system of linear equations

AX = b (1)

We know that A is an invertible matrix iff $\det A \neq 0$. Now if A is invertible matrix, then from equation (1), we have

$$A^{-1}AX = A^{-1}b$$

$$\Rightarrow \quad X = A^{-1}b$$

where

$$A^{-1} = \frac{1}{\det A}\text{Adj}\,A$$

$$\Rightarrow \quad X = \frac{\text{Adj}\,A}{\det A}b$$

Hence solution is determined.

3.2. Cramer's Rule:

Consider the system of linear equations

$$AX = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad and \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

then the j[th] component of the solution vector X is determined by

$$x_j = \frac{\det A_j}{\det A}$$

where $\det A_j$ is the determinant obtained by replacing j[th] column of det A by b, i.e.

$$A_j = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1(j-1)} & b_1 & a_{1(j+1)} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2(j-1)} & b_2 & a_{2(j+1)} \ldots a_{2n} \\ . & . \ \ldots\ . & . & . \ \ldots\ . \\ a_{n1} & a_{n2} \ldots a_{n(j-1)} & b_n & a_{n(j+1)} \ldots a_{nn} \end{bmatrix}.$$

Value Addition
Cramer's Rule is feasible only when n = 2, 3 or 4 only.

Example 1: Solve the systems of linear equations

$$3x+2y+2z=3$$
$$2x-3y-z=-3$$
$$x+2y+z=4$$

using inverse of the matrix method.

Solution: Given system of equations is

AX = b

where

$$A=\begin{bmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\quad and\quad b=\begin{bmatrix}3\\ -3\\ 4\end{bmatrix}$$

$$\Rightarrow\qquad \det A=|A|=\begin{vmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{vmatrix}=8$$

$$adj\ A=\begin{bmatrix}-1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}$$

$$\Rightarrow\qquad A^{-1}=\frac{1}{|A|}adj\ A=\frac{1}{8}\begin{bmatrix}-1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}$$

$$\Rightarrow\qquad X=A^{-1}b=\frac{1}{8}\begin{bmatrix}-1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}\begin{bmatrix}3\\ -3\\ 4\end{bmatrix}=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

Thus,

$$x=1,\ y=2\ \ and\ \ z=-1.$$

Example 2: Solve the systems of linear equations

$$x + 2y - z = 2$$
$$3x + 6y + z = 1$$
$$3x + 3y + 2z = 3$$

using Cramer's Rule.

Solution: Given system of equations is

$$AX = b \tag{1}$$

where

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & 1 \\ 3 & 3 & 2 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; and \; b = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad \det A = |A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 1 \\ 3 & 3 & 2 \end{vmatrix} = 12$$

Also $A_1 = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 6 & 1 \\ 3 & 3 & 2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} \; and \; A_3 = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 1 \\ 3 & 3 & 3 \end{bmatrix}$

$$\Rightarrow \quad |A_1| = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 6 & 1 \\ 3 & 3 & 2 \end{vmatrix} = 35, \; |A_2| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 3 & 3 & 2 \end{vmatrix} = -13 \; and \; |A_3| = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 6 & 1 \\ 3 & 3 & 3 \end{vmatrix} = -15$$

Using Cramer's rule we have

$$\Rightarrow \quad x = \frac{|A_1|}{|A|} = \frac{35}{12}, \; y = \frac{|A_2|}{|A|} = \frac{-13}{12} \; and \; z = \frac{|A_3|}{|A|} = -\frac{15}{12}$$

Thus,

$$x = \frac{35}{12}, \; y = -\frac{13}{12} \; and \; z = -\frac{5}{4}.$$

I.Q. 1:

I.Q. 2:

4. Method of Factorization (Triangularization Method):

This method is also known as decomposition method. This method is based on the fact that a square matrix A can be factored into the product of a lower triangular matrix L and an upper triangular matrix U, if all the principal minors of A are non-singular, i.e. if

$$|a_{11}| \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

Thus, the matrix A can be expressed as

$$A = LU \qquad (1)$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ . & . & . & \dots & . \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{bmatrix} \quad and \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

Using the matrix multiplication rule to multiply the matrices L and U and comparing the elements of the resulting matrix with those of A we obtain

$$l_{i1}u_{1j} + l_{i2}u_{2j} + . . . + l_{in}u_{nj} = a_{ij} \quad (i = 1, 2, ..., n \ and \ j = 1, 2, ..., n)$$

where

$$l_{ij} = 0 \ if \ i < j \quad and \quad u_{ij} = 0 \ if \ i > j$$

this system of equations involves $n^2 + n$ unknowns. Thus there are n parameters family of solutions. To produce a unique solution it is convenient to choose either

$$u_{ii} = 1 \ or \ l_{ii} = 1 \quad i = 1, 2, ..., n$$

Now,

4.1. Doolittle's method:

If we take $l_{ii} = 1$, in the factorization method then the factorization method is called Doolittle's method.

Now if $l_{ii} = 1$, then we have

$$L = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ l_{21} & 1 & 0 & \dots & 0 \\ l_{31} & l_{32} & 1 & \dots & 0 \\ . & . & . & \dots & . \\ l_{n1} & l_{n2} & l_{n3} & \dots & 1 \end{bmatrix} \quad and \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

thus, for the system of equations

AX = b (2)

We have

LUX = b (3)

Putting UX = y in equation (3), we have

Ly = b (4)

On solving equation (4) by forward substitution, we find the vector y now solve the system of equations

UX = b

by backward substitution we get the values

$x_1, x_2, \dots, x_n$.

We have

UX = y

and Ly = b

$\Rightarrow \quad y = L^{-1}b \quad and \quad x = U^{-1}y$

Thus the inverse of A can also be determined as

$A^{-1} = U^{-1}L^{-1}$.

Example 3: Solve the system of equations

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

using factorization method (Doolittle's method).

Solution (Doolittle's method): We have system of equations

AX = b (1)

where

$$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\ and\ \ b=\begin{bmatrix}9\\ 6\\ 8\end{bmatrix}$$

Now let

A = L U (2)

where

$$L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\ and\ \ U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

Thus from equation (2) we have

L U = A

$$\Rightarrow\quad \begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$$

$$\Rightarrow\quad \begin{aligned}&u_{11}=2,\ u_{12}=3,\ u_{13}=1\\ &l_{21}u_{11}=1\Rightarrow\ l_{21}=\frac{1}{2}\ \ and\ \ l_{31}u_{11}=3\Rightarrow\ l_{31}=\frac{3}{2}\\ &l_{21}u_{12}+u_{22}=2\Rightarrow\ u_{22}=\frac{1}{2}\ \ and\ \ l_{21}u_{13}+u_{23}=3\Rightarrow\ u_{23}=\frac{5}{2}\\ &l_{31}u_{12}+l_{32}u_{22}=1\Rightarrow\ l_{32}=-7\ \ and\ \ l_{31}u_{13}+l_{32}u_{23}+u_{33}=2\Rightarrow\ u_{33}=18\end{aligned}$$

Thus, we have

$$L=\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ \frac{3}{2} & -7 & 1\end{bmatrix}\ and\ \ U=\begin{bmatrix}2 & 3 & 1\\ 0 & \frac{1}{2} & \frac{5}{2}\\ 0 & 0 & 18\end{bmatrix}$$

Now using equation (1) and (2) we have

LUX = b

Let

$$UX=Y \tag{3}$$

where $Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow$ LY = b

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} & y_1 = 9 \\ & \frac{1}{2}y_1 + y_2 = 6 \\ & \frac{3}{2}y_1 - 7y_2 + y_3 = 8 \end{aligned}$$

On solving using forward substitution we have

$$y_1 = 9,\ y_2 = \frac{3}{2}\ \ and\ \ y_3 = 5$$

Now using the equation (3) we have

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} & 2x + 3y + z = 9 \\ & \frac{1}{2}y + \frac{5}{2}z = \frac{3}{2} \\ & 18z = 5 \end{aligned}$$

On solving using backward substitution we have

$$x = \frac{35}{18},\ y = \frac{29}{18}\ \ and\ \ z = \frac{5}{18}$$

Thus, the solution of the given system of equations is

$$x = \frac{35}{18},\ y = \frac{29}{18}\ \ and\ \ z = \frac{5}{18}.$$

4.2. Crout's method:

If we take $u_{ii} = 1$, in the factorization method then the factorization method is called the Crout's method.

For the matrix A where

A = LU (1)

if $l_{ii} = 1$, then we have

$$L = \begin{bmatrix} l_{11} & 0 & 0 \ldots & 0 \\ l_{21} & l_{22} & 0 \ldots & 0 \\ l_{31} & l_{32} & l_{33} \ldots & 0 \\ . & . & . \ldots & . \\ l_{n1} & l_{n2} & l_{n3} \ldots & l_{nn} \end{bmatrix} \ and\ U = \begin{bmatrix} 1 & u_{12} & u_{13} \ldots & u_{1n} \\ 0 & 1 & u_{23} \ldots & u_{2n} \\ 0 & 0 & 1 \ \ldots & u_{3n} \\ . & . & . \ \ldots & . \\ 0 & 0 & 0 \ \ldots & 1 \end{bmatrix}$$

thus, for the system of equations

AX = b (2)

We have

LUX = b (3)

Putting UX = y in equation (3), we have

Ly = b (4)

On solving equation (4) by forward substitution, we find the vector y now solve the system of equations

UX = b

by backward substitution we get the values

$x_1, x_2, \ldots, x_n$.

We have

UX = y

and Ly = b

$\Rightarrow \quad y = L^{-1}b \ \ and \ \ x = U^{-1}y$

Thus the inverse of A can also be determined as

$$A^{-1} = U^{-1}L^{-1}.$$

Example 4: Solve the system of equations

$$\begin{aligned} x + y + z &= 1 \\ 4x + 3y - z &= 6 \\ 3x + 5y + 3z &= 4 \end{aligned}$$

using factorization method (Crout's method).

Solution (Crout's method): We have system of equations

AX = b (1)

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ and \ b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Now let

A = L U (2)

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \ and \ U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus from equation (2) we have

L U = A

$$\Rightarrow \quad \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} & l_{11} = 1, l_{21} = 4, l_{31} = 3 \\ & l_{11}u_{12} = 1 \Rightarrow u_{12} = 1 \ and \ l_{11}u_{13} = 1 \Rightarrow u_{13} = 1 \\ & l_{21}u_{12} + l_{22} = 3 \Rightarrow l_{22} = -1 \ and \ l_{21}u_{13} + l_{22}u_{23} = -1 \Rightarrow u_{23} = 5 \\ & l_{31}u_{12} + l_{32} = 5 \Rightarrow l_{32} = 2 \ and \ l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3 \Rightarrow l_{33} = -10 \end{aligned}$$

Thus, we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \quad and \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

Now using equation (1) and (2) we have

LUX = b

Let

UX=Y (3)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\Rightarrow$ LY = b

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} &y_1 = 1 \\ &4y_1 - y_2 = 6 \\ &3y_1 + 2y_2 - 10y_3 = 4 \end{aligned}$$

On solving using forward substitution we have

$$y_1 = 1,\ y_2 = -2 \ \ and \ \ y_3 = -\frac{1}{2}$$

Now using the equation (3) we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -\frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{aligned} &x + y + z = 1 \\ &y + 5z = -2 \\ &z = -\frac{1}{2} \end{aligned}$$

On solving using backward substitution we have

$$x = 1,\ y = \frac{1}{2}\ \ and\ \ z = -\frac{1}{2}$$

Thus, the solution of the given system of equations is

$$x = 1,\ y = \frac{1}{2}\ \ and\ \ z = -\frac{1}{2}.$$

I.Q. 3:

I.Q. 4:

4.3. Cholesky Method:

This method is also known as the square root method. If the coefficient matrix A is symmetric and positive definite, then the matrix A can be decomposed as

$$A = LL^T$$

where $L = \ell_{ij}$ ℓ_{ij} [illegible] $< j$

Thus, the system of equations

$$AX = b \qquad (1)$$

becomes

$$LL^T X = b \qquad (2)$$

Let $L^T X = Y$, then (2) becomes

$$LY = b \qquad (3)$$

Now system of equations in (3) can be solved by forward substitution and solution vector X is determined from

$$L^T X = Y \qquad (4)$$

by the backward substitutions.

The inverse of coefficient matrix A can also be obtained from

$$A^{-1} = (LL^T)^{-1} = (L^T)^{-1} L^{-1} = (L^{-1})^T L^{-1}.$$

Value Addition: Note
1. The matrix A can also be decomposed as $A = UU^T$ Where $U = u_{ij},\ \ u_{ij} = 0\ if\ i > j$.

2. If the coefficient matrix A is symmetric but not positive definite, then the Cholesky's method could still be applied, but then leads to a complex matrix L, so that it becomes impractical.

Example 5: Solve the system of equations

$$x+2y+3z=5$$
$$2x+8y+22z=6$$
$$3x+22y+82z=-10$$

using Cholesky method.

Solution: Given system of equations is

AX = b (1)

where

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 8 & 22\\ 3 & 22 & 82\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\ and\ \ b=\begin{bmatrix}5\\ 6\\ -10\end{bmatrix}$$

Now let

$$A=LL^T \tag{2}$$

where

$$L=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$$

Thus from equation (2) we have

$$\Rightarrow \quad \begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ 2 & 8 & 22\\ 3 & 22 & 82\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned}&l_{11}^2=1\Rightarrow l_{11}=1,\\ &l_{11}l_{21}=2\Rightarrow l_{21}=2,\\ &l_{11}l_{31}=3\Rightarrow l_{31}=3,\\ &l_{21}^2+l_{22}^2=8\Rightarrow\ l_{22}=2,\\ &l_{31}l_{21}+l_{32}l_{22}=22\Rightarrow\ l_{32}=8,\\ &l_{31}^2+l_{32}^2+l_{33}^2=82\Rightarrow\ l_{33}=3.\end{aligned}$$

Thus, we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

Now using equation (1) and (2) we have

$$LL^T X = b$$

Let

$$L^T X = Y \qquad (3)$$

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\Rightarrow \quad LY = b$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} y_1 &= 5 \\ 2y_1 + 2y_2 &= 6 \\ 3y_1 + 8y_2 + 3y_3 &= -10 \end{aligned}$$

On solving using forward substitution we have

$$y_1 = 5,\ y_2 = -2 \ \ and \ \ y_3 = -3$$

Now using the equation (3) we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x + 2y + 3z &= 5 \\ 2y + 8z &= -2 \\ 3z &= -3 \end{aligned}$$

On solving using backward substitution we have

$$x = 2,\ y = 3 \ \ and \ \ z = -1$$

Thus, the solution of the given system of equations is

$x = 2,\ y = 3\ \ and\ \ z = -1$.

Example 6: Solve the system of equations

$$4x + 2y + 14z = 14$$
$$2x + 17y - 5z = -101$$
$$14x - 5y + 83z = 155$$

using Cholesky method.

Solution: Given system of equations is

$$AX = b \tag{1}$$

where

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}\ \ and\ \ b = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

Now let

$$A = LL^T \tag{2}$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

Thus from equation (2) we have

$$\Rightarrow \quad \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} & l_{11}^2 = 4 \Rightarrow l_{11} = 2, \\ & l_{11}l_{21} = 2 \Rightarrow l_{21} = 1, \\ & l_{11}l_{31} = 14 \Rightarrow l_{31} = 7, \\ & l_{21}^2 + l_{22}^2 = 17 \Rightarrow\ l_{22} = 4, \\ & l_{31}l_{21} + l_{32}l_{22} = -5 \Rightarrow\ l_{32} = -3, \\ & l_{31}^2 + l_{32}^2 + l_{33}^2 = 83 \Rightarrow\ l_{33} = 5. \end{aligned}$$

Thus, we have

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$$

Now using equation (1) and (2) we have

$$LL^T X = b$$

Let

$$L^T X = Y \qquad (3)$$

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\Rightarrow \quad LY = b$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -101 \\ 155 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2y_1 &= 14 \\ y_1 + 4y_2 &= -101 \\ 7y_1 - 3y_2 + 5y_3 &= 155 \end{aligned}$$

On solving using forward substitution we have

$$y_1 = 7,\ y_2 = -27 \ \ and \ \ y_3 = 5$$

Now using the equation (3) we have

$$\begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -27 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2x + y + 7z &= 7 \\ 4y - 3z &= -27 \\ 5z &= 5 \end{aligned}$$

On solving using backward substitution we have

$$x = 3,\ y = -6 \ \ and \ \ z = 1$$

Thus, the solution of the given system of equations is

$x = 3,\ y = -6\ \ and\ \ z = 1$.

Theorem 1: (Stability of the Cholesky Factorization): The Cholesky LL^T-factorization is numerically stable.

Proof: For the Cholesky method, we know that

$$A = LL^T \tag{1}$$

where $\quad L = \ell_{ij} \quad \ell_{ij} \quad\quad < j$

Thus,

$$a_{jj} = \ell_{j}^{2} \quad \ell_{j}^{2} \quad\quad \ell_{jj}^{2} \qquad \text{[using equation (1)]} \tag{2}$$

Hence for all ℓ_{jk}^{2} ($\ell_{jk} \quad\quad r\ k > j$). We obtain

$$\ell_{jk}^{2} \quad \ell_{j}^{2} \quad \ell_{j}^{2} \quad\quad \ell_{jj}^{2} \quad\quad {}_{jj}$$

Thus, ℓ_{jk}^{2} is bounded by an entry of A, which means stability against round-off.

I.Q. 5:

5. Positive Definite Matrix:

A matrix A is said to be positive definite matrix, if $X^* AX > 0$ for any vector $X \neq 0$ and $X^* = (\overline{X})^T$. Further, $X^* AX = 0$ if and only if X = 0.

Value Addition: Note
If a matrix A is Hermitian, strictly diagonal dominant matrix with positive real diagonal entries, then A is positive definite.

5.1. Properties of Positive Definite Matrices:

A positive definite matrix has the following properties:

(I) If A is non-singular and positive definite matrix then $B = A * A$ is Hermitian and positive definite.

(II) The eigenvalues of a positive definite matrix are all positive.

(III) All the leading minors of A are positive.

Value Addition: Note
1. A real matrix B is said to have 'property A' iff there exists a permutation matrix P such that $PBP^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} *and* A_{22} are diagonal matrices. 2. Inverse of a symmetric matrix is a symmetric matrix. 3. Inverse of a upper triangular matrix is a upper triangular matrix. 4. Inverse of a lower triangular matrix is a lower triangular matrix.

I.Q. 6:

6. Pivoting:

In the Gauss elimination process, sometimes it may happen that any one of the pivot element $a_{11}, a'_{22}, a'_{33}, \ldots, a'_{nn}$ vanishes or becomes very small compared to other elements in that column, then we attempt to rearrange the remaining rows so as to obtain a non-vanishing pivot or to avoid the multiplication by a large number. This process is called pivoting.

There are two types of the pivoting

6.1. Partial Pivoting: Partial pivoting is done in the following steps:

Step 1: In the first stage of elimantion, we searched the first column for the largest element in magnitude and brought as the first pivot by interchanging the first equation with equation having the largest element in magnitude.

Step 2: In the second stage we searched the second column for the largest element in magnitude among the (n-1) elements leaving the first element and brought this element as the second pivot by interchanging the second equation with the equation having the largest element in magnitude.

This procedure is continued until we got the upper triangular matrix. In the partial pivoting, pivot is found by the algorithm choose j, the smallest integer for which

$$\left|a_{jk}^{(k)}\right| = \max\left|a_{ik}^{(k)}\right|; \quad k \le i \le n$$

and interchange rows k and j.

6.2. Complete Pivoting:

In the complete pivoting search the matrix A for the largest element in magnitude and bring it as the first pivot. In complete pivoting not only an interchange of equations requires but also an interchange of position of the variables requires.

In complete pivoting following algorithm is used to find the pivot choose l and m as the smallest integers for which

$$\left|a_{lm}^{(k)}\right| = \max\left|a_{ij}^{(k)}\right|, \quad k \le i,\ j \le n$$

and interchange rows k and l and columns k and m.

Value Addition: Note
If the matrix A is diagonally dominant or real, symmetric and positive definite then no pivoting is necessary.

7. Gauss Elimination Method:

From the previous methods, we have learnt that any system of linear algebraic equations can be solved by the use of determinants. But the method of solving the system of linear equations by determinants is not very practical, even with efficient methods for evaluating the determinants. Because if the order of the determinant is large, then the evaluation becomes tedious. Therefore to avoid these unnecessary computations, mathematicians have tried to develop simpler and less time consuming procedures and various methods for solving system of linear equations have been suggested. Gauss elimination method is one of the most important method to solve the system of linear equations .

Gauss elimination method for solving linear systems is a systematic process of elimination that reduces the system of linear equations to triangular form. In Gauss elimination method, we proceed with the following steps.

Step 1: Elimination of x_1 from the second, third, . . . , n^{th} equations

In the first step of Gauss elimination method we eliminate x_1 from the second, third, . . . , n^{th} equations by subtracting suitable multiple of first equation from second, third, . . . , n^{th} equations.

The first equation is called the pivot equation and the coefficient of x_1 in the first equation i.e., $a_{11} \neq 0$ is called the pivot. Thus first step gives the new system as follows.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}\,x_2 + \ldots + a'_{2n}\,x_n &= b'_2 \\
. \quad . \quad . \quad . \quad . \\
a'_{n2}\,x_2 + \ldots + a'_{nn}\,x_n &= b'_n
\end{aligned}
$$

Step 2: Elimination of x_2 from the third, . . . , n^{th} equation

In the second step of Gauss elimination method, we take the new second equation (which no longer contains x_1) as the pivot equation and use it to eliminate x_2 from the third, fourth, . . . , n^{th} equation.

In the third step we eliminate x_3 and in the fourth step we eliminate x_4 and so on. After (n-1) steps when the elimination is complete this process gives upper triangular system of the form

$$
\begin{aligned}
c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n &= d_1 \\
c_{22}x_2 + \ldots + c_{2n}x_n &= d_2 \\
. \quad . \quad . \quad . \quad . \\
c_{nn}x_n &= d_n
\end{aligned}
$$

Thus, the new system of equations is of upper triangular form that can be solved by the back substitution.

Value Addition: Note
In the Gauss elimination method the pivot equation remains unchanged also we may make the pivot as 1 before elimination at each step.

Example 7: Solve the system of equations

$$
\begin{aligned}
8x_2 + 2x_3 &= -7 \\
3x_1 + 5x_2 + 2x_3 &= 8 \\
6x_1 + 2x_2 + 8x_3 &= 26
\end{aligned}
$$

using Gauss elimination method.

Solution: Given system of equations is

$$8x_2 + 2x_3 = -7 \qquad (1)$$

$$3x_1 + 5x_2 + 2x_3 = 8 \qquad (2)$$

$$6x_1 + 2x_2 + 8x_3 = 26 \qquad (3)$$

Since the coefficient of x_1 in first equation is zero therefore we must rearrange the equations by interchanging first equation to third i.e.,

$$6x_1 + 2x_2 + 8x_3 = 26 \quad (4)$$

$$3x_1 + 5x_2 + 2x_3 = 8 \quad (5)$$

$$8x_2 + 2x_3 = -7 \quad (6)$$

Step 1: Elimination of x_1:

On subtracting $\frac{1}{2}$ times of equation (4) from equation (5) we have

$$6x_1 + 2x_2 + 8x_3 = 26 \quad (7)$$

$$4x_2 - 2x_3 = -5 \quad (8)$$

$$8x_2 + 2x_3 = -7 \quad (9)$$

Step 2: Elimination of x_2:

On subtracting 2 times of equation (8) from equation (9) we have

$$6x_1 + 2x_2 + 8x_3 = 26 \quad (10)$$

$$4x_2 - 2x_3 = -5 \quad (11)$$

$$6x_3 = 3 \quad (12)$$

On solving equation (10), (11) and (12) by back substitution we have

$$x_3 = \frac{1}{2},\ x_2 = -1 \ \ and \ \ x_1 = 4 \ .$$

Thus, the required solution is

$$x_1 = 4,\ x_2 = -1 \ \ and \ \ x_3 = \frac{1}{2}.$$

Example 8: Solve the system of equations

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1$$

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = 0$$

$$\frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 = 0$$

using Gauss elimination method.

Solution: Given system of equations is

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1 \quad (1)$$

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{4}x_3 = 0 \quad (2)$$

$$\frac{1}{3}x_1 + \frac{1}{4}x_2 + \frac{1}{5}x_3 = 0 \quad (3)$$

Step 1: Elimination of x_1:

On subtracting $\frac{1}{2}$ times of equation (1) from equation (2) and $\frac{1}{3}$ times of equation (1) from equation (3) we have

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1 \quad (4)$$

$$\frac{1}{12}x_2 + \frac{1}{12}x_3 = -\frac{1}{2} \quad (5)$$

$$\frac{1}{12}x_2 + \frac{4}{45}x_3 = -\frac{1}{3} \quad (6)$$

Step 2: Elimination of x_2:

On subtracting equation (5) from equation (6) we have

$$x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1 \quad (7)$$

$$\frac{1}{12}x_2 + \frac{1}{12}x_3 = -\frac{1}{2} \quad (8)$$

$$-\frac{1}{180}x_3 = -\frac{1}{6} \quad (9)$$

On solving equation (7), (8) and (9) by back substitution we have

$$x_3 = 30,\ x_2 = -36 \ \ and \ \ x_1 = 9.$$

Thus, the required solution is

$$x_1 = 9,\ x_2 = -36 \ \ and \ \ x_3 = 30.$$

Example 9: Solve the system of equations

$$\begin{aligned}
&10x_1 - 7x_2 + 3x_3 + 5x_4 = 6 \\
&-6x_1 + 8x_2 - x_3 - 4x_4 = 5 \\
&3x_1 + x_2 + 4x_3 + 11x_4 = 2 \\
&5x_1 - 9x_2 - 2x_3 + 4x_4 = 7
\end{aligned}$$

using Gauss elimination method.

Solution: Given system of equations can be written as

$$x_1 - 0.7x_2 + 0.3x_3 + 0.5x_4 = 0.6 \quad (1)$$

$$-6x_1 + 8x_2 - x_3 - 4x_4 = 5 \quad (2)$$

$$3x_1 + x_2 + 4x_3 + 11x_4 = 2 \quad (3)$$

$$5x_1 - 9x_2 - 2x_3 + 4x_4 = 7 \quad (4)$$

Step 1: Elimination of x_1:

On subtracting (-6) times of equation (1) from equation (2), 3 times of equation (1) from equation (3) and 5 times of equation (1) from eqution (4) we have

$$x_1 - 0.7x_2 + 0.3x_3 + 0.5x_4 = 0.6 \quad (5)$$

$$3.8x_2 + 0.8x_3 - x_4 = 8.6 \quad (6)$$

$$3.1x_2 + 3.1x_3 + 9.5x_4 = 0.2 \quad (7)$$

$$-5.5x_2 - 3.5x_3 + 1.5x_4 = 4 \quad (8)$$

Step 2: Elimination of x_2:

In the above equations (6), (7) and (8) coefficient of x_2 is maximum (numerically) in equation (8) therefore interchanging the equation (6) and (8) After that x_2 is eliminated from equations (7) and (8) we have

$$x_1 - 0.7x_2 + 0.3x_3 + 0.5x_4 = 0.6 \tag{9}$$

$$x_2 + 0.6363x_3 - 0.27275x_4 = -0.72727 \tag{10}$$

$$-1.61818x_3 + 0.03636x_4 = 11.36364 \tag{11}$$

$$1.12727x_3 + 10.34545x_4 = 2.45455 \tag{12}$$

Step 3: Elimination of x_3:

On eliminating x_3 from equation (12) we have

$$x_1 - 0.7x_2 + 0.3x_3 + 0.5x_4 = 0.6 \tag{13}$$

$$x_2 + 0.6363x_3 - 0.27275x_4 = -0.72727 \tag{14}$$

$$x_3 - 0.02247x_4 = -7.02247 \tag{15}$$

$$10.3607947x_4 = 10.37079 \tag{16}$$

On solving equation (13), (14), (15) and (16) by back substitution we have

$$x_4 = 1,\ x_3 = -7,\ x_2 = 4 \ \ and \ \ x_1 = 5.$$

Thus, the required solution is

$$x_1 = 5,\ x_2 = 4,\ x_3 = -7 \ \ and \ \ x_4 = 1.$$

Example 10: Solve the system of equations

$$10x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

using Gauss elimination method.

Solution: Since the given system is diagonally dominant therefore no pivoting is necessary. Thus we have

$$10x_1 - x_2 + 2x_3 = 4 \tag{1}$$

$$x_1 + 10x_2 - x_3 = 3 \tag{2}$$

$$2x_1 + 3x_2 + 20x_3 = 7 \tag{3}$$

Step 1: Elimination of x_1:

On eliminating x_1 from equations (2) and (3) we have

$$10x_1 - x_2 + 2x_3 = 4 \quad (4)$$

$$\frac{101}{10}x_2 - \frac{12}{10}x_3 = \frac{26}{10} \quad (5)$$

$$\frac{32}{10}x_2 + \frac{196}{10}x_3 = \frac{62}{10} \quad (6)$$

Step 2: Elimination of x_2:

On eliminating x_2 from equation (6) we have

$$10x_1 - x_2 + 2x_3 = 4 \quad (7)$$

$$\frac{101}{10}x_2 - \frac{12}{10}x_3 = \frac{26}{10} \quad (8)$$

$$\frac{20180}{1010}x_3 = \frac{5430}{1010} \quad (9)$$

On solving equation (7), (8) and (9) by back substitution we have

$$x_3 = 0.269,\ x_2 = 0.289 \ \ and \ \ x_1 = 0.375.$$

Thus, the required solution is

$$x_1 = 0.375,\ x_2 = 0.289 \ \ and \ \ x_3 = 0.269.$$

I.Q. 7:

I.Q. 8:

8. Gauss-Jordan Elimination Method:

M. Jordan in 1920 introduced another variant of the Gauss elimination method. In Gauss-Jordan method the coefficient matrix is reduced to a diagonal form rather than a triangular form in the Gauss elimination and we have the solution without further computations. Generally, this method is not used for the solution of a system of equations, because the reduction from the Gauss triangular to diagonal form requires more operations than back substitution does. Therefore this method is disadvantageous for solving system of equations. However it gives a simple method for finding the inverse of a given matrix by operating on the unit matrix I in the same way as the Gauss-Jordan method reducing A to I.

Example 11: Solve the system of equations

$$x_1 + 2x_2 + x_3 = 8$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$4x_1 + 3x_2 + 2x_3 = 16$$

using Gauss elimination method.

Solution: Given system of equations is

$$x_1 + 2x_2 + x_3 = 8 \quad (1)$$

$$2x_1 + 3x_2 + 4x_3 = 20 \quad (2)$$

$$4x_1 + 3x_2 + 2x_3 = 16 \quad (3)$$

Step 1: Elimination of x_1:

On eliminating x_1 from equations (2) and (3) we have

$$x_1 + 2x_2 + x_3 = 8 \quad (4)$$

$$-x_2 + 2x_3 = 4 \quad (5)$$

$$-5x_2 - 2x_3 = -16 \quad (6)$$

Step 2: Elimination of x_2:

On eliminating x_2 from equations (4) and (6) we have

$$x_1 + 0.x_2 + 5x_3 = 16 \quad (7)$$

$$-x_2 + 2x_3 = -36 \quad (8)$$

$$-12x_3 = -36 \quad (9)$$

Step 2: Elimination of x_3:

On eliminating x_3 from equations (7) and (8) we have

$$x_1 = 1 \quad (7)$$

$$-x_2 = -2 \quad (8)$$

$$12x_3 = 36 \quad (9)$$

This gives

$x_1 = 1,\ x_2 = 2\ \ and\ \ x_3 = 3$.

Example 12: Find the inverse of the coefficient matrix of the given system of equations

$$x_1 + x_2 + x_3 = 1$$
$$4x_1 + 3x_2 - 1x_3 = 6$$
$$3x_1 + 5x_2 + 3x_3 = 4$$

using Gauss elimination method with partial pivoting and hence solve the system of the equations..

Solution: Given system of equations is

$$AX = b \qquad (1)$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix},\quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\quad and\quad b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Using the augmented matrix [A | I], we have

$$[A\,|\,I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad [R_1 \leftrightarrow R_2]$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_1 \to \frac{1}{4}R_1$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right] \qquad R_2 \leftrightarrow R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right] \qquad R_2 \to \frac{4}{11}R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{14}{11} & 0 & \frac{5}{11} & -\frac{3}{11} \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & -\frac{1}{11} \end{array}\right] \qquad \begin{array}{l} R_1 \to R_1 - \frac{3}{4}R_2 \\ R_3 \to R_3 - \frac{1}{4}R_2 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{14}{11} & 0 & \frac{5}{11} & -\frac{3}{11} \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \qquad R_3 \to \frac{11}{10}R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \qquad \begin{array}{l} R_1 \to R_1 + \frac{14}{11}R_3 \\ R_2 \to R_2 - \frac{15}{11}R_3 \end{array}$$

Thus the inverse of the coefficient matrix A is

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Therefore the solution of the system of equation (1) is

$$X = A^{-1}b$$

$$\Rightarrow \qquad X = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Thus,

$$x_1 = 1, \; x_2 = \frac{1}{2} \; \textit{and} \; x_3 = -\frac{1}{2}.$$

I.Q. 9:

9. Error Analysis for Direct Methods:

The quality of a numerical method is judged in terms of:

Amount of storage

Amount of time ($\equiv$ Number of operations)

Effect of round-off error.

9.1. Operational Count for Gauss Elimination:

The number of divisions and multiplications involved in solving the system of equations is usually called the operational count for that method. For Gauss elimination, the operation count for a system of equations is as follows:

Elimination of x_1: For eliminating x_1, the factor $\frac{a_{21}}{a_{11}}$ is computed once. There are (n-1) multiplications in the (n-1) terms on the left side and 1 multiplication on the right side.

Hence the number of multiplications/divisions required for eliminating x_1 is

$$(1 + n - 1 + 1 = n + 1),$$

Since x_1 is eliminated from (n-1) equations. Therefore, the total number of multiplications/divisions required to eliminated x_1 from (n-1) equations is

$$(n-1)(n+1)=(n-1)(\mathrm{n}+2-1),$$

Elimination of x_2: For eliminating x_2, the total number of multiplications/divisions required to eliminate x_2 from (n-2) equations is

$$(n-2)n=(n-2)(\mathrm{n}+2-2),$$

Elimination of x_3: For eliminating x_3, the total number of multiplications/divisions required to eliminate x_3 from (n-3) equations is

$$(n-3)(n-1)=(n-3)(\mathrm{n}+2-3),$$

Elimination of x_k: For eliminating x_k, the total number of multiplications/divisions required to eliminate x_k from (n-k) equations is

$$=(n-k)(\mathrm{n}+2-k),$$

Elimination of $x_{(n-1)}$: For eliminating $x_{(n-1)}$, the total number of multiplications/divisions required to eliminate $x_{(n-1)}$ from (n-k) equations is

$$=\left[\mathrm{n}-(n-1)\right]\left[\mathrm{n}+2-(n-1)\right]=1.3.$$

Thus, the total number of operations required to eliminate x_1, x_2, x_3, . . ., x_{n-1} are as follows

$$\sum_{k=1}^{n-1}(n-k)(n+2-k)=\sum_{k=1}^{n-1}\left[(n-k)^2+(n-k)\right]$$

$$=\sum_{k=1}^{n-1}\left[n^2+k^2-2nk+2n-2k\right]$$

$$=n^2(n-1)+\frac{(n-1)\mathrm{n}(2\mathrm{n}-2+1)}{6}-$$

$$2n\frac{(n-1)\mathrm{n}}{2}+2n(n-1)-2\frac{(n-1)\mathrm{n}}{2}$$

$$\Rightarrow \quad \sum_{k=1}^{n-1}(n-k)(n+2-k)\approx\frac{n^3}{3}$$

Thus, the total number of multiplications and divisions required in Gaussian elimination method is $\frac{n^3}{3}$.

Note 1: Similarly, it can be shown that the Gauss-Jordan Method requires $\frac{n^3}{2}$ arithmetic operations. Hence, Gauss elimination method is preferred to Gauss-Jordan method to solve the large system of equations.

Note 2: In L-U decompositions method total number of operations count is $\frac{n^3}{3}$ same as in Gauss elimination method.

Note 3: In Cholesky method total number of operations count is $\frac{n^3}{6}$.

I.Q. 10:

Exercise:

1. Solve the following system of equations using LU-factorization method:

(I)
$$\begin{aligned} 4x+5y&=7 \\ 12x+14y&=18 \end{aligned}$$

(II)
$$\begin{aligned} 5x+9y+2z&=24 \\ 9x+4y+z&=25 \\ 2x+y+z&=11 \end{aligned}$$

(III)
$$\begin{aligned} 4x+6y+8z&=0 \\ 6x+34y+52z&=-160 \\ 8x+52y+129z&=-452 \end{aligned}$$

2. Solve the following system of equations using Cholesky method

(I)
$$\begin{aligned} -4x-y&=1 \\ -x+4y-z&=0 \\ -y+4z&=0 \end{aligned}$$

(II)
$$\begin{aligned} 4x_1-x_2&=1 \\ -x_1+4x_2-x_3&=0 \\ -x_2+4x_3-x_4&=0 \\ -x_3+4x_4&=0 \end{aligned}$$

3. Solve the following system of equations using Gauss elimination method

(I)
$$\begin{aligned} 2x+2y+3z&=1\\ 4x+2y+3z&=2\\ x+y+z&=3 \end{aligned}$$

(II)
$$\begin{aligned} 2x_1+x_2+x_3+2x_4&=2\\ 4x_1+2x_3+x_4&=3\\ 3x_1+2x_2+2x_3&=-1\\ x_1+3x_2+2x_3&=-4 \end{aligned}$$

(III)
$$\begin{aligned} 4x_1+x_2+x_3&=4\\ x_1+4x_2-2x_3&=4\\ 3x_1+2x_2-4x_3&=6 \end{aligned}$$

(IV)
$$\begin{aligned} x_1+x_2-x_3&=2\\ 2x_1+3x_2+5x_3&=-3\\ 3x_1+2x_2-3x_3&=6 \end{aligned}$$

4. Solve the following system of equations using Gauss-Jordan method:

(I)
$$\begin{aligned} 2x+y+3z&=1\\ 4x-y+5z&=-7\\ -3x+2y+4z&=-3 \end{aligned}$$

(II)
$$\begin{aligned} 2x_1+x_2-4x_3+x_4&=4\\ -4x_1+3x_2+5x_3-2x_4&=-1\\ x_1-x_2+x_3-x_4&=-1\\ x_1+3x_2-3x_3+2x_4&=-1 \end{aligned}$$

(III)
$$\begin{aligned} 4x_1+2x_2+4x_3&=10\\ 2x_1+2x_2+3x_3+2x_4&=18\\ 4x_1+2x_2+6x_3+3x_4&=30\\ 2x_2+3x_3+9x_4&=61 \end{aligned}$$

(IV)
$$\begin{aligned} 10x_1+2x_2+x_3&=59\\ x_1+8x_2+2x_3&=-4\\ 7x_1-x_2+20x_3&=5 \end{aligned}.$$

Summary:

In this lesson we have emphasized on the followings

- Direct methods to solve the system of linear equations
- Inverse of the matrix method
- Cramer's Rule
- Method of Factorization (Triangularization Method)
- Doolittle's method
- Crout's method
- Cholesky Method
- Positive Definite Matrix
- Gauss Elimination Method
- Pivoting
- Gauss-Jordan Elimination Method
- Error Analysis for Direct Methods
- Operational Count for Gauss Elimination

References:

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