Discipline Courses-I
Semester-I
Paper: Calculus-I
Lesson: Properties of Conics
Lesson Developer: Harindri
College/Department: Department of Mathematics, Deshbandu College, University of Delhi

Properties of Conics

Table of Contents:

1. Learning Outcomes:

After you have read this chapter, you should be able to

- Define the eccentricity,
- Define and find the eccentricity of ellipse,
- Define and find the eccentricity of hyperbola,
- Identify the conic with the help on discriminant.

Properties of Conics

"I know not what I appear to the world, but to myself I seem to have been only like a boy playing on the sea–shore, and diverting myself in now and then finding a smoother pebble or a prettier shell, whilest the great ocean of truth lay all undiscovered before me." – *SIR ISAAC NEWTON*

"The method of fluxions [the calculus] is the general key by help whereof the modern mathematicians unlock the secrets of Geometry and consequently of Nature" – *BISHOP BERKELY*

1. Introduction:

Modern mathematics began with two great advances–analytic geometry and the calculus. Analytic geometry took definite form in the year 1637 while the calculus took the definite shape in 1666. Rene Descartes, the great French mathematician of the seventeenth century, solved the problem of describing the position of a point in a plane. His method was development of the older idea of latitude and longitude. In honour of Descartes, the system used for describing the position of a point in a plane is also known as the Cartesian system. When he introduced analytic geometry he was interested in it as an aid in the solution of equation. It soon developed, however, into a method of solving problem in plane and solid geometry algebraically, greatly influenced the study of both geometry and algebra. Analytic geometry is divided into two branches: plane analytic geometry, dealing with points, lines and curves which are restricted to a plane; and solid analytic geometry, dealing with points, lines, planes, curves and surfaces in the three–dimensional space but in this chapter we study plane analytic geometry. On the other hand, both Newton and Leibnitz, share the credit for inventing "Calculus" independently in the seventeenth century. The former used physical approach while the latter used geometrical approach. Further Newton used the term "rate of change" in his second law of motion. Thus the calculus, sometimes, may be defined as mathematics of motion and change.

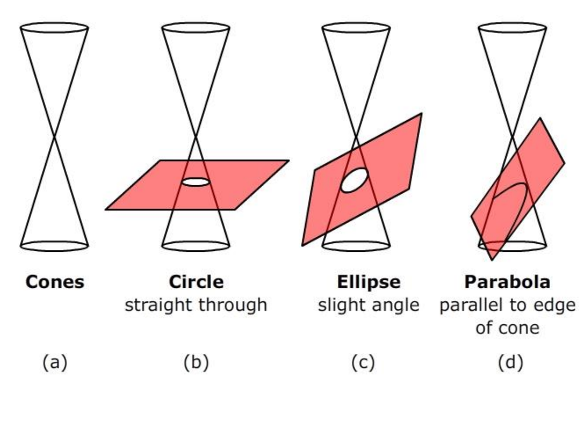
Conic section is a curve obtained as the intersection of a double–napped right circular conic with a plane. Conics are broadly classified as degenerate conic which includes a point, a line and a pair of intersecting lines or non–degenerate conic which includes circle, parabola, ellipse and hyperbola. The conics are the path travelled by planets, satellites and other bodies whose motion are driven by inverse square forces.

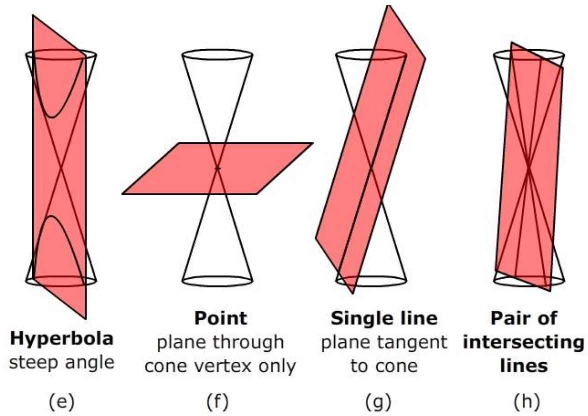
Properties of Conics

In this chapter we will discuss focus–directrix characterization of a conic section which introduces the concept of eccentricity and is defined for degenerate conics. The eccentricity of a conic section in a measure of how far it diviates from being circular. Rene Descartes applied his newly discovered analytic geometry to the study of conics which further reduces the geometrical problem of conics to problem in algebra.

It was also observed that every general equation of second degree represents a conic under certain conditions and for that discriminant test is also established. Conics has a wide range of application, for example, every partial differential equation is classified as parabolic, elliptic or hyperbolic. The behaviour and theory of these different types of partial differential equation is strikingly different – representative examples is that the Poisson equation is elliptic, the heat equation is parabolic and the wave equation is hyperbolic. Furthermore, conics have many practical application also. The parabola has various technical application. Familiar examples are the parabolic reflector, the parabolic arch and the parabolic suspension of cables. The property used in such cases in called the reflective property of the parabola. It is observed that sound waves follow elliptic paths and the reflective property of ellipse is used in constructing whispering galleries. Water pipes are sometimes designed with elliptical cross-section to allow for expansion when the water freezes. Also, hyperbolic paths arise in Einstein's theory of relativity and form the basis for the LORAN (Long range navigation) radio navigation system. The chapter ends with the statement and proof of the reflective properties of the parabola, ellipse and hyperbola.

We observe that a conic is just a section or a slice through a cone.





We notice that the conics such as circle, parabola, ellipse and hyperbola are defined only using a straight line called the directrix and a point (called the focus). Now measure the distance from the focus on the curve and the distance perpendicularly from the directrix to that point (see figure 2). We observe that these two distances will always be in some ratio. Based on these two distances we associate a number called eccentricity with each conic section.

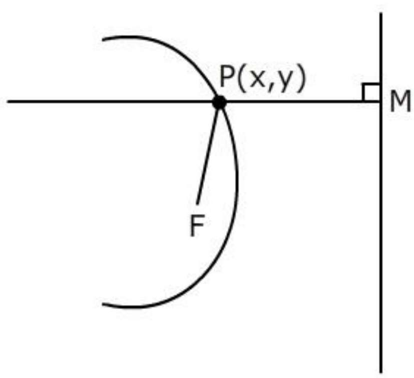


Figure 2

3. Eccentricity: Eccentricity, denoted by e, of a curve is defined as

$$e = \frac{\text{Distance from the focus to a point on the curve}}{\text{Perpendicular distance from the directrix to that point}}$$

$$= \frac{PF}{PM} \qquad \text{-----------------------} \qquad (1)$$

So a conic section may be re-defined as the locus of all points whose distance to the focus is equal to the eccentricity times the distance to the directrix.

3.1. Eccentricity of Ellipse: The eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is defined as $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$ ------------------ (2)

where a > b.

3.2. Eccentricity of Hyperbola: The eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is defined as

$$e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a} \qquad \text{------------------------------} \ (3)$$

Value Addition: Remarks

(a) We conclude, in case of a parabola, e = 1.

(b) Note that for an ellipse we have $0<e<1$ because the foci are closer together than the vertices.

(c) For a hyperbola, we conclude that e>1, because the foci are farther apart than the vertices. Further, if the eccentricity is smaller, The curve is more curved and if it is bigger, then the curve is less curved.

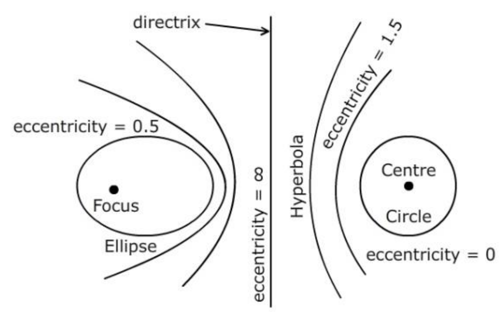


Figure 3

(d) We observe that a circle of radius *a* may be regarded as the limiting case of an ellipse whose major axis is 2a and whose eccentricity tends to zero. So the eccentricity shows how "un-circular" the curve is. The bigger the eccentricity, the less curved it is (see fig. 3).

(e) The latus rectum is a chord of the ellipse passing through the focus and perpendicular to the axis. Its length is $\frac{2b^2}{a}$ (where *a* is the semi major axis and *b* is the semi minor axis)

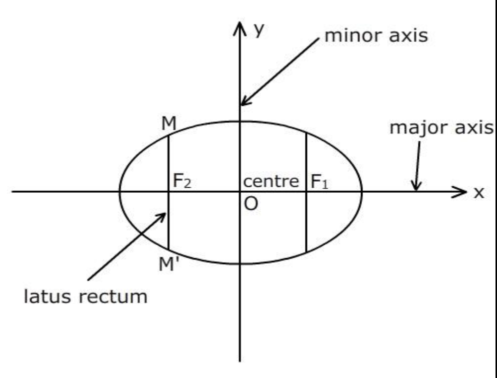


Figure 4

By definition, MF_2M' is the latus rectum.

Then $MF_2M' = 2MF_2$

Let MF_2 = y, thus the coordinates of M is (-ae, y)

Since M lies on the ellipse, therefore we have $\frac{(-ae)^2}{a^2} + \frac{y^2}{b^2} = 1$.

$i.e., \ y^2 = b^2\left(1-e^2\right) = \frac{b^4}{a^2}$ by using $1-e^2 = \frac{b^2}{a^2}$

$\textit{Therefore} \ \ y = \frac{b^2}{a}. \quad \textit{Hence latus rectum} = \frac{2b^2}{a}$

(f) The latus rectum is a chord of the hyperbola passing through the focus and perpendicular to the focal axis. Its length is $\frac{2b^2}{a^2}$ (for proof see remark (e)).

(g) We observe that the directrices of an ellipse are given by $x = \pm\frac{a}{e}$ which corresponds to the foci (±c, 0).

Example 1: Find the eccentricity, coordinates of the foci, and the length of the latus rectum of the ellipse $4x^2 + 9y^2 = 1$. Also find its directrices.

Solution: The equation of the ellipse can be re-written as

$$\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1.$$

This implies that $a^2 = \frac{1}{4}, \ b^2 = \frac{1}{9}$ (by comparing with equation (7)).

From equation (16),

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{1/4 - 1/9}}{1/2} = \frac{\sqrt{5}}{3}$$

Thus, coordinates of the foci are $(\pm\, ae, 0), i.e., \left(\pm\frac{\sqrt{5}}{6}, \ 0\right)$

Also, the length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{2 \times \frac{1}{9}}{\frac{1}{2}} = \dfrac{4}{9}$.

Hence, the equation of the directrices are

$$x = \pm a/e = \pm \frac{3}{2\sqrt{5}}$$

Example 2: Find the length of the axes and of the latus rectum of the ellipse $4x^2 + 3y^2 = 24$.

Solution. Rewriting the given equation, we have

$$\frac{x^2}{6} + \frac{y^2}{8} = 1$$

By comparing with equation (7), we obtain

$a^2 = 6$, $b^2 = 8$

Thus in this case the x-axis lies along the minor axis and y–axis along the major axis and therefore the lengths of the axes are $2\sqrt{8}$ and $2\sqrt{6}$.

Also, the length of the latus rectum $= \dfrac{2b^2}{a} = \dfrac{2 \times 6}{2\sqrt{2}} = 3\sqrt{2}$.

Example 3: Find the centre and eccentricity of the ellipse

$$2x^2 + 3y^2 - 4x + 5y + 4 = 0$$

Solution: Re-writing the given equation, we obtain

$$2\left(x^2 - 2x + 1\right) + 3\left(y^2 + \frac{5}{3}y + \frac{25}{36}\right) = 2 + \frac{25}{12} - 4 = \frac{1}{12}$$

So that,

$$\frac{(x-1)^2}{1/24} + \frac{\left(y + \frac{5}{6}\right)}{1/36} = 1$$

Let X = x – 1, Y = y + $\frac{5}{6}$. Then the given equation becomes

$$\frac{X^2}{1/24} + \frac{Y^2}{1/36} = 1$$

Thus, the given equation represents an ellipse whose centre is at $\left(1, \frac{-5}{6}\right)$ *and* $a^2 = \frac{1}{24}$; $b^2 = \frac{1}{36}$.

Now, $e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{\frac{1}{24} - \frac{1}{36}}}{\frac{1}{\sqrt{24}}} = \frac{1}{\sqrt{3}}$ and hence $e = \frac{1}{\sqrt{3}}$.

Example 4: Find the equation of the hyperbola centred at the origin of the xy–plane that has a focus at (–2,0) and the line $x = -\frac{1}{2}$ as the corresponding directrix.

Solution: By definition, the focus is

(–c, 0) = (–2, 0)

so that c = 2.

Now equation of the directrix is given by

$$x = -\frac{a}{e} = -\frac{1}{2},\ i.e.,\ \mathrm{e} = 2\mathrm{a}.$$

$$Also,\quad e = \frac{c}{a} = \frac{2}{e/2} = \frac{4}{e},$$

i.e., $e^2 = 4$ and hence e = 2.

Let P(x, y) be any point on the hyperbola.

We know that PF = e.PM (by using equation (15))

$$i.e.,\quad \sqrt{(x+2)^2 + (y-0)^2} = 2\left|x + \frac{1}{2}\right|$$

$$i.e., \ (x+2)^2 + y^2 = 4\left(x+\frac{1}{2}\right)^2 \quad (squaring \ both \ sides)$$

$$i.e., \ -3x^2 + y^2 = -3$$

$$i.e., \ \frac{x^2}{1} - \frac{y^2}{3} = 1$$

which is the required equation of the hyperbola.

Example 5: Find the equation of the ellipse centred at the origin of the xy–plane that has focus at (0, ±3) and 0.5 as its eccentricity.

Solution: The focus F (0, ±c) is (0, ±3)

so that c = 3.

Now, $e = \frac{c}{a}$, i.e., $0.5 = \frac{3}{a}$ so that a = 6.

Also, $c^2 = a^2 - b^2$, i.e., $9 = 36 - b^2$ which gives $b^2 = 27$.

Thus, the equation of the ellipse is given by

$$\frac{x^2}{27} + \frac{y^2}{36} = 1 \ .$$

Example 6: Find the centre eccentricity, foci and direction of the hyperbola $9x^2 - 16y^2 = 144$.

Solution: The given hyperbola can be re-written as

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

This implies that $a^2 = 16$, $b^2 = 9$.

Then, $e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{16+9}}{4} = \frac{5}{4}$ (by using equation (3)).

Also, centre is at (0, 0).

Now, c = ae = $4 \times \frac{5}{4} = 5$

Thus, foci at (±5, 0).

Hence the equation of the directrices are given by

$$x = \pm\frac{a}{e} = \pm\frac{4}{5/4} = \pm\frac{16}{5}.$$

Example 7: A hyperbola of eccentricity 3/2 has one focus at (1, -3). the corresponding directrix is the line y = 2. Find an equation for the hyperbola.

Solution: Let P(x, y) be any point on locus of the hyperbola.

Then, by using equation (1), we have $PF_1^2 = e^2 PM_1^2$.

$$i.e.,\ \sqrt{(x-1)^2 + (y+3)^2} = \frac{3}{2}|y-2|$$

$$i.e.,\ (x-1)^2 + (y+3)^2 = \frac{9}{4}(y-2)^2 \quad (\textit{squaring both the sides})$$

$$i.e.,\ 4\left[(x-1)^2 + y^2 + 6y + 9\right] = 9\left(y^2 - 4y + 4\right)$$

$$i.e.,\ 4(x-1)^2 - 5\left(y^2 - 12y + 36 - 36\right) = 0$$

$$i.e.,\ 4(x-1)^2 - 5\left(y - 6\right)^2 + 36 \times 5 = 0$$

$$i.e.,\ \frac{(y-6)^2}{36} - \frac{(x-1)^2}{45} = 1$$

which is the required equation of the hyperbola.

Value Addition: Remember
Consider a second degree equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ --------------------------- (4) By using the various facts about conics we conclude that equation (4) represents (a) a circle if A=C≠ 0 (special cases: the graph is a point or there in no graph). (b) a parabola if equation (4) is quadratic (second degree) in one variable and linear in the other. (c) an ellipse if A and C are both positive and an imaginary ellipse if both A and C are negative. (special cases: circles, a single point or no graph

at all)

(d) a hyperbola if A and C have opposite signs (special cases: a pair of intersecting lines).

(e) a straight line if A=0 and C=0 and at last one of D and E is different from zero.

(f) One or two straight lines if L.H.S. of equation (4) can be factored into the product of two linear factors.

Value Addition: Note

Now the most general form of second degree equation is given by

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ ------------------------------- (5)

where A,B and C are not all zero.

(a) If B=0, then equation (5) reduces to equation (4).

(b) In equation (4) we notice that B=0 as the axes of the cones are parallel to the coordinate axes (counter clockwise rotation).

Theorem 1: Let the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be such that $B \neq 0$ and if and $x'y'$ – coordinate system is obtained by rotating the xy–axes through an angle α satisfying $\tan 2\alpha = \frac{B}{A-C}$, then, in $x'y'$ –coordinate, equation 5) will have the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0.$

Proof: We will rotate the coordinate axes through an angle α in order to eliminate the xy term, i.e., to find the condition on α so that B = 0.

From fig. 5, we have

$x = ON = OP\cos(\theta+\alpha)$

$= OP\cos\theta\cos\alpha - OP\sin\theta\sin\alpha$

Also, $y = PN = OP\sin(\theta+\alpha)$

$= OP\sin\theta\cos\alpha + OP\cos\theta\sin\alpha$

But $OP\cos\theta = ON' = x'$ and $OP\sin\theta = N'P = Y'$

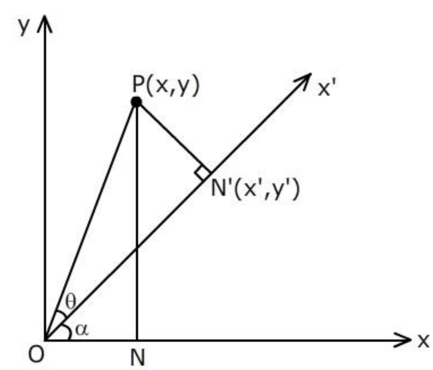


Figure 5

Thus the equations for rotating coordinates axes are given by

$$\left.\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha \\ y &= x'\sin\alpha + y'\cos\alpha \end{aligned}\right\} \quad \text{---------------} \quad (6)$$

Substituting x and y from the equation (6) in the equation (5) we obtain

$A' x'^2 + B' x' y' + C' y'^2 + D' x' + E' y' + F' = 0$ ------------------ (7)

where

$$\left.\begin{aligned}
A' &= A\cos^2\alpha + B\sin\alpha\,\cos\alpha + \sin^2\alpha \\
B' &= B\cos^2 2\alpha + (C - A)\sin 2\alpha \\
C' &= A\sin^2\alpha - B\sin\alpha\,\cos\alpha + C\cos^2\alpha \\
D' &= D\cos\alpha + E\sin\alpha \\
E' &= -D\sin\alpha + E\cos\alpha \\
F' &= F
\end{aligned}\right\} \text{------------------} \quad (8)$$

Now choose α in such a way that the coefficient of $x'y'$, i.e, B' in equation (7) may vanish.

Thus ,

$B\cos 2\alpha + (C-A)\sin 2\alpha = 0$ (by using equation(8))

i.e, $\tan 2\alpha = \dfrac{B}{A-C}$ or $\cot 2\alpha = \dfrac{A-C}{B}$ --------------------------- (9)

which is the required condition on α so that the equation (7) becomes $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ which completes the proof.

Value Addition: Remarks
For any rotation of axes, we have (i) $B^2 - 4AC = B'^2 - 4A'C'$ (ii) $A + C = A' + C'$ This shows that the above expressions does not alter under the rotation of axes. The next result shows that without rotating the coordinate axis (i.e., without eliminating the xy–term from equation (7)), it is possible to classify what type of conic section the equation (7) represents.

Theorem 2: Let $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be any second-degree equation.

(a) It $B^2 - 4AC < 0$, then the equation represents an ellipse, a circle, a point or else has no graph

(b) It $B^2 - 4AC > 0$, then the equation represents a hyperbola or a pair of interesting lines.

(c) It $B^2 - 4AC = 0$, then the equation represents a parabola, a line, a pair of parallel lines or else has no graph.

Proof: By using theorem1, we obtain

$$A'x'^2 + B'xy' + C'y'^2 + D'x' + E'y' + F' = 0 \qquad -------------- \ (7)$$

where $A', B', C', D', E', and\ F'$ are given by equation (8) and choose α is given by equation (9).

Then equation (7) reduces to

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \qquad ------------------------------- \ (10)$$

For any rotation of axes, we have

$$B^2 - 4AC = B'^2 - 4A'C' = -4A'C' \qquad \sin ce\ B' = 0.$$

$$i.e.,\ B'^2 - 4A'C' = -4A'C' \ ------------------------------ (11)$$

(a) If $B^2 - 4AC < 0$, then by equation (11), we have $A'C' > 0$. We first assume that $A' > 0\ and\ C' > 0$. Now we divide equation (10) by $A'C'$ so that it can be written in the from

$$\frac{1}{C'}\left(x'^2 + \frac{D'}{A'}x'\right) + \frac{1}{A'}\left(y'^2 + \frac{E'}{C'}y'\right) = \frac{-F'}{A'C'}$$

$$i.e., \quad \frac{1}{C'}\left(x' + \frac{D'}{2A'}\right)^2 + \frac{1}{A'}\left(y' + \frac{E'}{2C'}\right)^2 = \frac{D'^2}{4A'^2C'} + \frac{E'^2}{4A'C'^2} - \frac{F'}{A'C'}$$

$$i.e., \quad \frac{\left(x' + \frac{D'}{2A'}\right)^2}{\left(\sqrt{C'}\right)^2} + \frac{\left(y' + \frac{E'}{2C'}\right)^2}{\left(\sqrt{A'}\right)^2} = \Delta$$

where $\Delta = \frac{1}{4A'^2C'^2}\left(C'D'^2 + A'E'^2 - 4A'C'F'\right)$

$$i.e., \quad \frac{(x' - h)^2}{\left(\sqrt{C'}\right)^2} + \frac{(y' - k)^2}{\left(\sqrt{A'}\right)^2} = \Delta \ ------------------------ (12)$$

where $h = \frac{-D'}{2A'}$, $k = \frac{-E'}{2C'}$.

There are three cases to discuss.

(i) Let Δ >0.

Then from the equation (12), we conclude that the graph is either a circle if $A' = C'$ or an ellipse if $A' \neq C'$

(ii) Let Δ = 0.

Then, the equation (12) is satisfied only by $x' = h$ and $y' = k$ which implies that the graph in this case is the single point (h, k)

(iii) Let Δ < 0.

In this case there is no graph, since the left side of equation (12) is nonnegative for all x' and y'.

Similar arguments holds true for the case when

$A' < 0$ and $C' < 0$ so that $A'C' > 0$.

(b) If $B^2 - 4AC > 0$, then by equation (11) we have $A'C' < 0$ which implies that either $A' > 0$ and $C' < 0$ or $A' < 0$ and $C' > 0$.

We assume that $A' > 0$ and $C' < 0$.

Again there are three cases to discuss.

(i) Let Δ > 0.

Then from equation (12) we deduce that the graph is a hyperbola of the form

$$\frac{(y'-k)^2}{a^2} - \frac{(x'-h)^2}{b^2} = 1$$

where $a = \sqrt{\Delta A'}$ and $b = \sqrt{\Delta C'}$.

(ii) Let Δ < 0.

Then the equation (12) represents a graph of the hyperbola of the from

$$\frac{(x'-h)^2}{a^2} - \frac{(y'-k)^2}{b^2} = 1$$

where $a = \sqrt{\Delta C'}$ and $b = \sqrt{\Delta A'}$

(iii) Let Δ = 0.

Then equation (12) reduces to

$$A'(x'-h)^2 + C'(y'-k)^2 = 0.$$

which is separable into linear factors because A' and C' have opposite signs.

Hence, equation (12) in this case represents two straight lines intersecting in the point (h, k), i.e,

$$\left(\frac{-D'}{2A'}, \frac{-E'}{2C'}\right)$$

Likewise, the case for which $A' < 0$ and $C' > 0$ can be discussed.

(c) If $B^2 - 4AC = 0$. then by equation (11) we obtain

$$A'C' = 0.$$

which implies that either $A' = 0$ or $C' = 0$ or both are zero.

(i) Let $A' = 0$. Then equation (10) reduces to

$$C'y'^2 + D'x' + E'y' + F' = 0.$$

$$i.e., \quad y'^2 + \frac{D'}{C'}x' + \frac{E'}{C'}y' + \frac{F'}{C'} = 0$$

which represents a graph of parabola provided $D' \neq 0$ axis parallel to the x–axis

If $D' = 0$, then equation (10) represents a pair of parallel lines or has no graph.

(ii) Let $C' = 0$. Then equation (10) becomes

$$A'x'^2 + D'x' + E'y' + F' = 0$$

$$i.e., \quad x'^2 + \frac{D'}{A'}x' + \frac{E'}{A'}y' + \frac{F'}{A'} = 0$$

which represents a graph of the parabola provided $E' \neq 0$ having its axis parallel to the y–axis.

If $E' = 0$, then equation (12) represents a pair of parallel lines or has no graph.

(iii) Let $A' = 0$ and $C' = 0$.

Then equation (10) represents a line if either $D' \neq 0$ or $E' \neq 0$.

4. Discriminant: The term $B^2 - 4AC$ in the theorem 2, is called the discriminant of the equation (5).

Value Addition: Note
Sometimes theorem 2, is also known as discriminant test.

Example 8: Identify the graph of the conic

$$x^2 + 2\sqrt{3}\,xy + 3y^2 + 2\sqrt{3}\,x - 2y = 0. \qquad \text{............................(13)}$$

Solution. We have $A = 1,\ B = 2\sqrt{3},\ C = 3,\ D = 2\sqrt{3},\ E = -2,\ F = 0$

so that $B^2 - 4AC = \left(2\sqrt{3}\right)^2 - 4 \times 1 \times 3 = 12 - 12 = 0$

Thus, the given equation represents a parabola, a line, a pair of parallel lines or has no graph.

Now, $\tan 2\alpha = \dfrac{B}{A-C} = \dfrac{2\sqrt{3}}{1-3} = -\sqrt{3}$.

$$\text{Then} \quad 2\alpha = -\frac{\pi}{3} \quad \text{so that} \quad \alpha = -\frac{\pi}{6}.$$

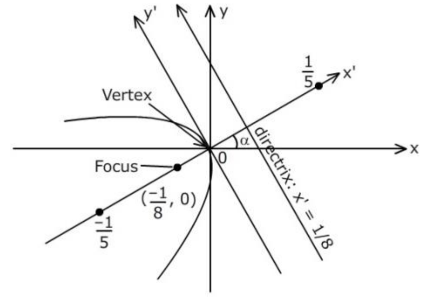


Figure 6

Further $x = x'\cos\left(\frac{-\pi}{6}\right) - y'\sin\left(\frac{-\pi}{6}\right)$

$$= \frac{\sqrt{3}x'}{2} + \frac{y'}{2} \; and \; y = x'\sin\left(\frac{-\pi}{6}\right) + y'\cos\left(\frac{-\pi}{6}\right) = \frac{-x'}{2} + \frac{y'\sqrt{3}}{2}$$

On substituting x' and y' in the equation (13), we obtain

$$\left(\frac{\sqrt{3}\,x'}{2} + \frac{y'}{2}\right) + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}x' + \frac{y'}{2}\right)\left(\frac{-x'}{2} + \frac{y'\sqrt{3}}{2}\right) + 3\left(\frac{-x'}{2} + \frac{y'\sqrt{3}}{2}\right)^2$$

$$+2\sqrt{3}\left(\frac{\sqrt{3}}{2}x' + \frac{y'}{2}\right) - 2\left(\frac{-x'}{2} + \frac{y'\sqrt{3}}{2}\right) = 0$$

i.e., $\frac{1}{4}\left(3x'^2 + y'^2 + 2\sqrt{3}\,x'y'\right) + \frac{2\sqrt{3}}{4}\left(-\sqrt{3}x'^2 + 3x'y' - x'y' + \sqrt{3}y'^2\right)$

$$+\frac{3}{4}\left(x'^2 + 3y'^2 - 2\sqrt{3}\,x'y'\right) + \left(3x' + \sqrt{3}y'\right) - x' - y'\sqrt{3} = 0$$

i.e., $\frac{3}{4}x'^2 - \frac{6}{4}x'^2 + \frac{3}{4}x'^2 + \frac{y'^2}{4} + \frac{6}{4}y'^2 + \frac{9}{4}y'^2 + \frac{\sqrt{3}}{2}x'y' + \sqrt{3}x'y' - \frac{3\sqrt{3}}{2}x'y' + 2x' = 0$

i.e., $\frac{16}{4}y'^2 + 2x' = 0$

i.e., $4y'^2 = -2\,x'$, *i.e.,* $y'^2 = -\frac{1}{2}x' = 4\left(-\frac{1}{8}\right)x'$

so that $a = -\frac{1}{8}$.

Thus the given equation represents a parabola.

Example 9: Trace the conic

$3x^2 + 2xy + 3y^2 = 19$ --------------------------------- (14)

Solution. By comparing equation (14) with equation (5) we obtain A = 3, B = 2, C = 3 and F = -19

Consider $B^2 - 4AC = (2)^2 - 4 \times 3 \times 3 = 4 - 36 = -32 < 0$.

Hence the curve (14) is an ellipse, a point, a circle or else has no graph

Now, $\tan 2\alpha = \dfrac{B}{A-C} = \dfrac{2}{3-3} = \infty$

Then, $\alpha = \dfrac{\pi}{4}.$

The rotation equation are given by

$$x = x'\cos\frac{2\pi}{4} - y'\sin\frac{2\pi}{4}\ ;\ y = x'\sin\frac{2\pi}{4} + y'\cos\frac{x\pi}{4}.$$

$$i.e.,\ x = \frac{x' - y'}{\sqrt{2}};\quad y' = \sin\frac{x' + y'}{\sqrt{2}}\ \text{-----------------}\ (15)$$

Substituting equation (15) in the equation (14), we get

$$3\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x' - y'}{\sqrt{2}}\right)^2\left(\frac{x' - y'}{\sqrt{2}}\right) + 3\left(\frac{x' + y'}{\sqrt{2}}\right)^2 = 19$$

which on simplification gives

$$8x'^2 + 4y'^2 = 38$$

$$\frac{x'^2}{19/4} + \frac{y'^2}{19/2} = 1$$

which is the required equation of ellipse.

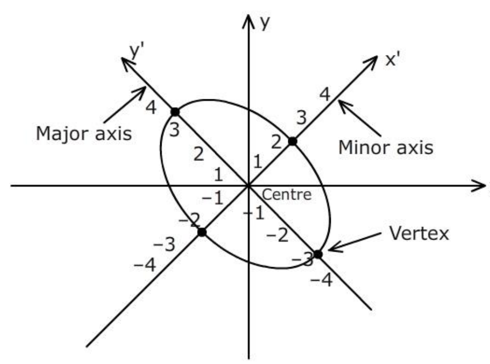


Figure 7

Example 10: Identify the conic and also trace its graph.

$$x^2 + xy + y^2 - 1 = 0 \quad \text{-----------------------}\ (16)$$

Solution: A =1, B = 1, C = 1, D = E = 0, F = -1

Now, $B^2 - 4AC = 1 - 4 = -3 < 0$

Therefore (16) represents an ellipse.

Also, $\tan 2\theta = \dfrac{B}{A-C} = \dfrac{1}{1-1} = \infty \quad i.e.,\quad 2\theta = \pi/2 \quad i.e.,\quad \theta = \pi/4$

Then,

$$\left.\begin{aligned} x &= x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \\ y &= x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}} \end{aligned}\right\} \text{------------------------} (17)$$

On substituting equation (17) in equation (16), we obtain

$$\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)^2 + \left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right)\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + \left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right)^2 - 1 = 0$$

$$i.e.,\ \frac{1}{2}\left(x'^2 - 2x'y' + y'^2\right) + \frac{1}{2}\left(x'^2 + x'y' - x'y' - y'^2\right) + \frac{1}{2}\left(x'^2 + 2x'y' + y'^2\right) - 1 = 0$$

$$i.e.,\ \left(\frac{x'^2}{2} + \frac{x'^2}{2} + \frac{x'^2}{2}\right) + \left(\frac{y'^2}{2} - \frac{y'^2}{2} + \frac{y'^2}{2}\right) = 1$$

$$ie.,\ \frac{x'^2}{\left(\sqrt{2/3}\right)^2} + \frac{y'^2}{\left(\sqrt{2}\right)^2} = 1$$ which represent an ellipse.

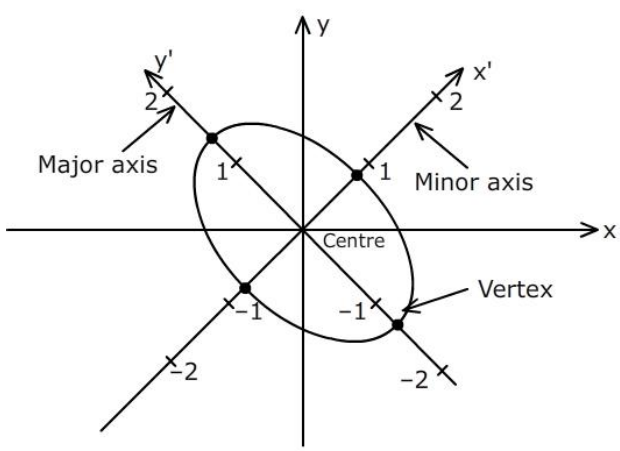


Figure 8

Example 11: Find the sine and cosine of an angle through which the co-ordinate axes can be rotated to eliminate the cross product term from the equation. Also identify the conic.

$$4x^2 - 4xy + y^2 - 8\sqrt{5x} - 16\sqrt{5y} = 0 \qquad \text{--------------------}(18)$$

Solution: Here $A = 4,\ B = -4,\ C = 1,\ D = -8\sqrt{5},\ E = -16\sqrt{5},\ F = 0$.

Consider B^2 –4AC = (-4)^2 – 4 × 4 × 1 = 16 – 16 = 0

Therefore, equation (18) represents a parabola.

Now, $\tan 2\alpha = \frac{B}{A-C} = \frac{-4}{4-1} = \frac{-4}{3}$ (by using theorem 2.13)

then $\cos 2\alpha = \frac{-3}{5}$ so that $\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}}$

$$\cos\alpha = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5-3}{5}}{2}} = \sqrt{\frac{\frac{2}{5}}{2}} = \frac{1}{\sqrt{5}}$$

and $\sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}}$

$$\sin\alpha = \sqrt{\frac{1+\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5+3}{5}}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \frac{2}{\sqrt{5}}.$$

Also, $x = \frac{x'}{\sqrt{5}} - \frac{2y'}{\sqrt{5}}, \quad y = \frac{2x'}{\sqrt{5}} + \frac{y'}{\sqrt{5}}.$

Therefore the given equation becomes

$$\frac{4}{5}\left(x'^2 - 4x'y' + 4y'^2\right) - \frac{4}{5}\left(2x'^2 + x'y' - 2y'x' - 2y'^2\right) + \frac{1}{5}\left(4x'^2 - y'^2 + 4x'y'^2\right)$$

$$-8\sqrt{5} = \left(\frac{x'-2y'}{\sqrt{5}}\right) - 16\sqrt{5}\left(\frac{2x'+2y'}{\sqrt{5}}\right) = 0.$$

i.e., $\left(\frac{4}{5}x'^2 - \frac{8}{5}x'^2 + \frac{4}{5}x'^2\right) + \left(\frac{-16}{5}x'y' + \frac{12}{5}x'y' + \frac{4}{5}x'y'\right) + \frac{4}{5}\left(4y'^2 + 2y'^2 + \frac{y'^2}{4}\right)$

$$-8x' + 16y' - 32x' - 16y' = 0$$

i.e., $y'^2 = \frac{32}{5}x'$ which represents a parabola.

Now we will discuss the reflective properties of parabola, ellipse & hyperbola which a useful in different types of applications.

Theorem 3: The tangent line at a point P on a Parabola makes equal angles with the line through P parallel to the axis of symmetry and the line through P and the focus.

Proof: Let us consider a point P(at^2, 2at) on the parabola. Further, suppose that PT is the tangent to the parabola

$$y^2 = 4ax \quad \text{--} \quad (19)$$

and F (a, 0) is the focus in the figure 9.

Also, L is the directrix and PM is the line parallel to the axis of symmetry

We first find the equation of the tangent PT to the given parabola. For this, consider any two point P(at^2, 2at) and Q(x_1, y_1) on the given parabola.

Then equation of the chord PQ in given by

$$y - 2at = \frac{2at - y_1}{at^2 - x_1}\left(x - at^2\right) \quad \text{-------------------} \quad (20)$$

Since Q lies on (19), therefore $y_1^2 = 4ax_1$ ----------------(21)

By substituting (21) in (20), we obtain

$$y - 2at = \frac{2at - y_1}{at^2 - \frac{y_1^2}{4a}}\left(x - at^2\right)$$

$$i.e., \quad y - 2at = \frac{4a}{2at + y_1}\left(x - at^2\right)$$

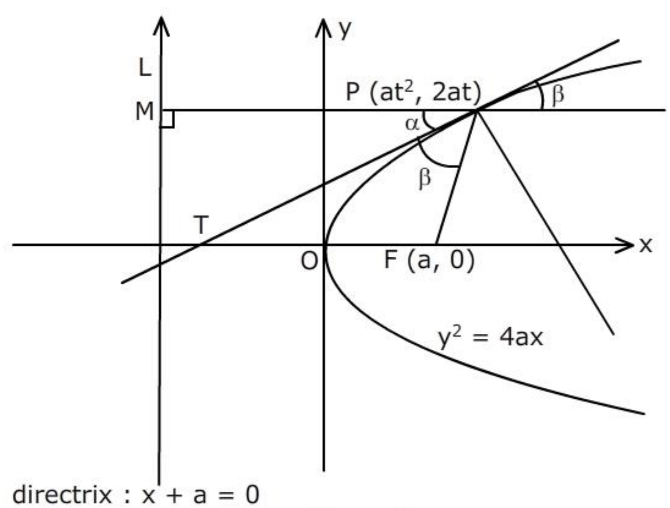


Figure 9

Now, the chord PQ will be a tangent to the parabola (19) if Q coincides with P, i.e., if $at^2 = x_1$ and $2at = y_1$.

Then, $$y - 2at = \frac{4a}{2at + 2at}\left(x - at^2\right)$$

$$i.e., \quad ty = x + at^2$$

which is the required equation of the tangent PT to the given parabola.

Then slope of PT is $\frac{1}{t}$

The slopes of PM and FP are 0 and $\frac{2at}{at^2 - a}$ respectively

Let ∠TPM = α, ∠FPT = β.

$$Then \tan\alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{1}{t} - 0}{1 + \frac{1}{t}.0} = \frac{1}{t}$$

$$and \tan\beta = \frac{\frac{2at}{at^2 - a} - \frac{1}{t}}{1 + \frac{2at}{at^2 - a}.\frac{1}{t}} = \frac{1}{t}$$

Thus, tanα = tanβ, i.e α=β and hence the proof is complete.

Theorem 4: A line tangent to an ellipse at a point P makes equal angles with the lines joining P to the foci.

Proof: Let the equation of an ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Take a point P(x_1, y_1) on it. Suppose that the tangent and normal at P meet the x–axis (in this case the major axis) in the points M and N respectively.

By using the similar arguments as used in the theorem 3, the equation of the tangent PT to the given ellipse is as follows.

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

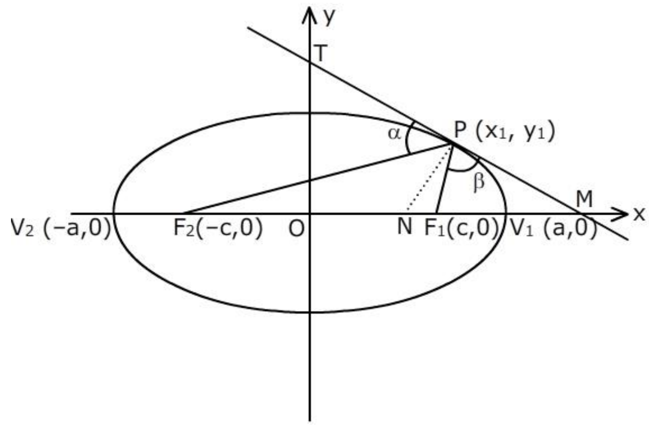


Figure 10

Now, slope of the tangent PT

$$= \frac{\frac{-x_1}{a^2}}{\frac{y_1}{b^2}} = -\frac{b^2 x_1}{a^2 y_1}$$

so that the slope of the normal PN = $\frac{a^2 y_1}{b^2 x_1}$.

Thus, equation of the normal PN at (x_1, y_1) is given by

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

i.e.,
$$\frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}$$

Therefore, normal cuts the x–axis (put y = 0) when $x = x_1\left(1 - \frac{b^2}{a^2}\right) = e^2 x_1$.

This implies that ON = e^2x_1

Thus $F_2N = F_2O + ON = ea + e^2x_1 = e\ (a + ex_1) = eF_2P$

and $F_1N = F_1O – ON = ea - e^2x_1 = e\ (a – ex_1) = eF_1P$

Hence, $\frac{F_2P}{F_1P} = \frac{F_2N}{F_1N}$

which concludes that PN is the bisector of the angle F_2PF_1.

Now, $\angle F_2PT = \frac{\pi}{2} - \angle F_2PN = \frac{\pi}{2} - \angle NPF_1 = \angle F_1PM$

Thus, $\angle F_2PT = \angle F_1PM$, *i.e.,* $\alpha = \beta$

which completes the proof.

Theorem 5: A line tangent to a hyperbola at a point P makes equal angles with the lines joining P to the foci.

Proof: Let the equation of the hyperbola be represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

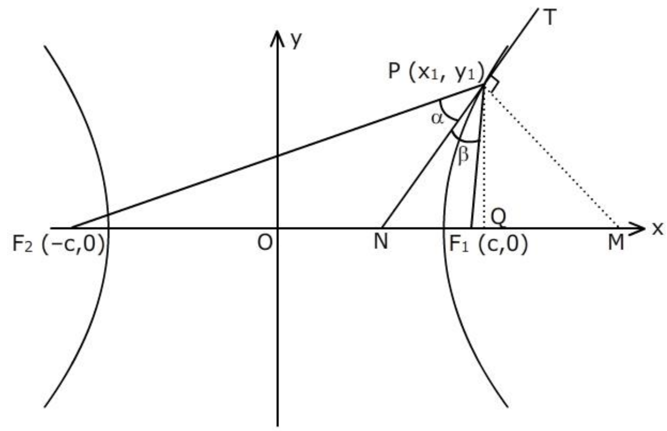


Figure 11

Consider any point $P(x_1, y_1)$ on the hyperbola and let the tangent PT at P meets the x–axis (i.e., focal axis) at the point N.

Then, the equation of the normal at P to the hyperbola is

$$\frac{x - x_1}{x_1/a^2} = -\frac{y - y_1}{y_1/b^2}$$

Now it meets the focal axis (i.e., y = 0) at the point M

$$x = \frac{x_1\left(a^2 + b^2\right)}{a^2} = e^2 x_1$$

Thus, $OM = e^2 OQ$

so that $F_1M = OM - OF_1, = e^2OQ - ea.$
$= e(ex_1 - a)$

$= eF_1P.$

By using the same arguments, we obtain

$$F_2M = eF_2P$$

This implies that $\frac{F_1M}{F_2M} = \frac{F_1P}{F_2P}$.

i.e., PM is the bisector (external) of the $\angle F_2PF_1$. Since the tangent is normal to the bisector, therefore it bisects the $\angle F_2PF_1$ internally which completes the proof of the theorem.

Properties of Conics

Exercise:

1. Find the eccentricity and equations of the directrices of the ellipse

$$x^2 + 9y^2 + 10x + 16 = 0.$$

2. The line $x - \frac{17}{4} = 0$ is one of the directrices of an ellipse, a vertex is at (3, 1), and the centre is at (–2, 1). Find the equation of the ellipse. Also trace the graph.

3. Find the equation of the hyperbola containing the point (5, 2) and having the asymptote $x - 2y = 0$. Also find the eccentricity.

4. Find the equation of the hyperbola having a focus at the point (–5, 0) and vertex at the point (3, 0).

5. Find the locus of a point which moves so that its distance from the point (10, 0) is five–halves of its distance from the line $x - \frac{8}{5} = 0$.

In each of the exercises 6–10, obtain a transformed equation of the given curve free from a term x'y' and then identify the conic. Also trace the graph.

6. $4x^2 + 24xy + 11y^2 - 48x - 44y + 24 = 0$

7. $10x^2 + 6xy + 2y^2 + 14x + 4y - 23 = 0$

8. $x^2 + xy + y^2 - 2x - 4y - 7 = 0$

9. $16x^2 - 24xy + 9y^2 + 130x + 90y = 0$

10. $5x^2 - 4xy + 2y^2 - 9x - 2 = 0$

Solutions:

1. $e = \frac{2\sqrt{2}}{3};\ x = \frac{-5 \pm 9\sqrt{2}}{4}$

2. $9x^2 + 25y^2 + 36x - 50y - 164 = 0$

3. $x^2 - 4y^2 = 9\ ;\ e = \frac{\sqrt{5}}{2}$

4. $16x^2 - 9y^2 = 144$

5. $21x^2 - 4y^2 = 336$

6. $20x'^2 - 5y'^2 - 64x' + 12y' + 24 = 0,\ \textit{hyperbola}$

7. $55x'^2 + 5y'^2 + 23\sqrt{10}x' - \sqrt{10}y' - 115 = 0,\ \textit{ellipse}$

8. $3x'^2 + y'^2 - 6\sqrt{2}x' - 2\sqrt{2}y' - 14 = 0,\ \textit{ellipse}$

9. $y'^2 - 6x' - 2y' = 0,\ \textit{parabola}$

10. $5x'^2 + 30y'^2 - 9\sqrt{5}x' + 18\sqrt{5}y' - 10 = 0,\ \textit{ellipse}$

Summary:

In this chapter, we have emphasized on the followings

- definition of the eccentricity,
- how to find the eccentricity of ellipse,
- how to find the eccentricity of hyperbola,
- how to identify the conic with the help on discriminant.

References:

1. Anton, H., Bivens I., and Davis, S., "Calculus", Wiley India, 2012

2. Cell, J.W., "Analytic geometry", Toppon company Ltd., Tokyo, Japan, 1960.

3. Fuller, G., "Analytic geometry", Addision–Wesley Publishing company. Inc., 1956.

4. Holmes, C.T., "Calculus and Analytic geometry", Mc graw–Hill Book Company, Inc. 1950.

5. Taylor, A.F., and Halberg Jr. Charles, J.A. "Calculus with analytic geometry" Prentice Hall International Inc, Englewood cliffs, N.J., 1969.

6. Thomas, Jr. G.B. and Finney, R.L. "Calculus and Analytic Geometry", Pearson Education Inc., 2012.