Properties of Continuous Function

Chapter : Properties of Continuous Function

Paper-Analysis II (Real Analysis)

Lesson Developed: Rajinder Kaur

College/ Department: S.S.N. College, University of Delhi

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Chapter: Properties of Continuous Functions

1. Learning Outcomes

After studying this chapter you will learn that

- A continuous function on a closed and bounded interval is bounded and attains its maximum and minimum values.
- A continuous function on an interval takes all of the intermediate values between two of its values.
- An algorithm, known as Bisection method for finding solution of equations of the form $f(x)=0$, where f is a continuous function.
- The image of an interval under a continuous function is an interval.

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"Education is not the learning of the facts but training of the minds to think"

2. Introduction

This chapter is devoted to a study of some of the properties of continuous functions. Several of these properties such as Intermediate Value Theorem and Extreme Value Theorem will be familiar to the reader. The Intermediate value theorem provides one of the fantastic tools used in finding the roots of an equation $f(x)=0$, for a continuous function f. The Intermediate value theorem also assures that the range of a continuous function includes all of its intermediate values. In addition to including all intermediate values, the range of a continuous function also includes its extreme values, atleast when its domain is restricted to a closed and bounded interval. For this reason, this result is known as the Extreme Value Theorem. The proof of both the results depends on the completeness Axiom. Then there is a result about preservation of intervals which states that continuous function map intervals into intervals.

3. Some Definitions:

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Definition 1: A function $f: A \to \mathbb{R}$ is said to be **bounded above** on A if there exists a constant M such that $f(x) \le M \ \forall x \in A$. The number M is called an upper bound of f on A.

Definition 2: A function $f: A \to \mathbb{R}$ is said to be **bounded below** on A if there exists a constant m such that $f(x) \ge m \ \forall x \in A$. The number *m* is called a lower bound of f on A.

Definition 3: A function $f: A \to \mathbb{R}$ is said to be **bounded** on A if it is both bounded above and bounded below on A, that is, if there exist a constant $M^*> 0$ such that $|f(x)| \le M^*$ for all $x \in A$. The number M^* is called bound for *f* on *A*.

In other words, a function *f* is bounded above, bounded below, or bounded on a set A if its range $f(A) = \{f(x) \mid x \in A\}$ is bounded above, bounded below or bounded respectively.

A function that is not bounded is said to be **unbounded.**

Example 1:

(a) $f(x) = \sin x$ is bounded on R.

(b) $g(x) = \frac{1}{x}$ is bounded on [c, ∞) for any c > 0 but g is not bounded above on (0, ∞) as $\lim_{x \to 0^+} \frac{1}{x} = +\infty$.

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This example shows that continuous function need not be bounded. In the next theorem however, we show that continuous function on a certain type of interval are necessarily bounded.

I.Q.1.

Theorem 1: Let f be a continuous real valued function on a closed interval $[a, b]$. Then f is a bounded function.

Proof: We shall prove the result by contradiction.Suppose $f$is not bounded on $[a, b]$. Then for each positive integer n there exists $x_n \in$ $[a \; b]$ such that

$$| f(x_n) | > n \qquad (1)$$

The sequence $< x_n >$ is contained in the closed and bounded interval $[a, b]$ therefore it is bounded, and so, by BolzanoWeierstrass Theorem $< x_n >$ has a convergent subsequence, say $< x_{n_k} >$converging to a point of $[a,b]$ Suppose $\lim_{k \to \infty} x_{n_k} = c$. Since$f$ is continuous at 'c' therefore, we have $\lim_{k \to \infty} f(x_{n_k}) = f(c)$. As every convergent sequence is bounded, $\langle f(x_{n_k}) \rangle$ is bounded, but this contradicts (1). Therefore, $f$must be bounded.

Value Addition: Remark
The conclusion of the boundedness theorem fails if any of the

hypothesis is relaxed.

(a) The interval must be bounded. The function $f(x) = x^3$ on the closed interval [0 ∞)is continuous but not bounded.

(b) The interval must be closed. The function $g(x) = \frac{1}{x^2}$ on the semi-open interval (0, 1] is continuous but not bounded.

(c)The function must be continuous. Consider the function h defined on the closed and bounded interval [0 1] by

$$h(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 2 & x = 0 \end{cases}$$

hereh is unbounded on [0, 1] because h fails to be continuous at *x* = 0.

Definition 4: Let I be an interval,let $f : I \to \mathbb{R}$ and let c ∈ I.

(a) The function f has a **maximum value** at *c* if $f(x) \le f(c)$ for all x ∈I.

(b) The function f has a **minimum value** at*c* if $f(x) \ge f(c)$ for all x ∈I.

(c) The function f has an **extreme value** at *c* if it has either a maximum value or a minimum value at '*c*'.

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Value addition: If a function has a maximum (minimum) value, then the points where it is attained is not necessarily unique. Consider for example $f(x)=x^2$ on [–1, 1]. Here maximum value is attained at two points, x = 1 and –1 whereas minimum value is attained at single point *x* = 0.

I.Q.2.

I.Q.3.

4. Extreme Value Theorem:

Theorem 2: Suppose *f* is continuous on [a, b]. Then f attains its maximum and minimum values on [a, b], that is, there exists c_1, $c_2 \in$ [a, b] such that

$$f(c_1) \le f(x) \le f(c_2) \text{ for all } x \in [a \quad b]$$

Proof: Suppose f is continuous on [a , b] then by previous theorem f is bounded on [a, b]. Consider $f[a \;\; b]=\{f(x) \mid x \in [a \;\; b]\}$ then *f* [*a*, *b*] is a non-empty bounded subset of $\mathbb{R}$.

Let $\alpha = \inf([a,b])$ and let $\beta = \sup(f([a,b]))$

then $\alpha \le f(x) \le \beta$ for all $x \in [a \;\; b]$. We shall prove that there exists point c_1 and c_2 in [a b] such that $f(c_1)=\alpha$ and $f(c_2)=\beta$.

Because $\alpha = \inf\{f(x) \mid x \in [a \;\; b]\}$ for each n $\in \mathbb{N}$ there is a point $x_n \in$ [a b] such that

$$\alpha \le f(x_n) < \alpha + \frac{1}{n}.$$

The sequence $\langle x_n \rangle$ is in [a b] and therefore by BolzanoWeierstrass theorem, it has a convergent subsequence say $\langle x_{n_k} \rangle$ converging to a point c_1 in [a, b]. Since f is continuous at c_1 therefore

$$f(c_1) = \lim_{k \to \infty} f(x_{n_k}). \quad (1)$$

Since $\langle f(x_{n_k}) \rangle$ is a subsequence of $\langle f(x_n) \rangle$ therefore

$$\lim_{k \to \infty} \langle f(x_{n_k}) \rangle = \lim_{n \to \infty} f(x_n) = \alpha. \quad (2)$$

From (1) and (2), we get $f(c_1) = \alpha$

Thus *f* assumes its minimum at c_1. Similarly, we can show that f assumes its maximum on [a, b].

Value Addition: Note
The Extreme value Theorem has three hypotheses; first the function must be continuous, second the interval must be closed and third the interval must be bounded. If any one of these hypotheses is not satisfied then the conclusion of the Extreme Valuetheorem may not be valid.The following examples illustrate this point.

Example 2: The function f defined by $f(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0 & x=0\end{cases}$ does not have a maximum value on [0 1]. Here f fails to be continuous at x = 0.

Example 3: The function g defined by $g(x)=x^2$ does not have a maximum value on the interval [–1 2). Here the interval is not closed.

I.Q.4.

Value addition:
A continuous function on a set does not necessarily attain maximum or minimum values for eg. $f(x)=\frac{1}{x}$ has neither maximum value nor minimum value on (0, ∞). The same function has neither maximum value nor minimum value when it is restricted to (0, 1), while it has both maximum value and minimum value when it is restricted to [1, 2]. In addition, $f(x)=\frac{1}{x}$ has maximum value but no minimum value when restricted to [1 ∞), but no maximum and no minimum value when restricted to (1 ∞).

Theorem 3 (Intermediate Value Theorem): Suppose f is a continuous real valued function on an interval [a, b], $f(a)\neq f(b)$

and k is any number between f(a) and f(b). Then there exists atleast one point c $\in$ (a, b) such that f(c) = k.

Proof: Case I $f(a) < k < f(b)$

Define the set $S = \{x \in [a\ b] \mid f(x) \le k\}$. Then S is both nonempty (as a$\in$ S) and bounded (as S $\subset$ [a, b]). By completeness axiom sup S exists.

Let c = sup S. We shall prove that f(c) = k.If *c* = *b*, then *f*(*c*) = *f*(*b*) >*k*. if c < b, then because c is an upper bound for S, for all x such that $c < x \le b$ we have $x \notin S$, so $f(x) > k$. Therefore, using the continuity of f at c, we get

$$f(c) = \lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) \ge k$$

i.e. $f(c) \ge k$ (1)

Similarly, because c is the least upper bound for S therefore for each n $\in$ there exist $x_n \in S \cap [a, c]$ such that

$c - \frac{1}{n} < x_n \le c$. By squeeze Theorem $\langle x_n \rangle \to c$ also $f(x_n) \le k$ as $x_n \in$ S $\forall$ n.Therefore by sequential definition of continuity it follows that

$$f(c) = \lim_{n\to\infty} f(x_n) \le k.$$

i.e. $f(c) \le k$ (2)

From (1) and (2) we get $f(c) = k$.

Case 2. $f(b) < k < f(a)$

Define $g(x) = -f(x)$. Then g is continuous on [*a, b*] and

$$-f(a) = g(a) < -k < g(b) = -f(b)$$

Thus, from above, there exists a point c in (a b) such that

$g(c) = -k$ that is $f(c) = k$.

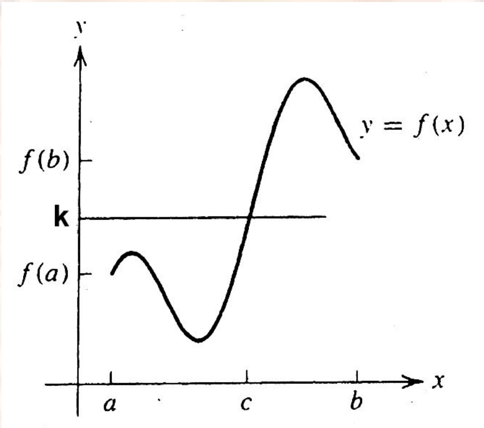

Figure 1: A continuous graph from (*a, f*(*a*)) to (*b*, *f*(*b*)) must cross the line *y* = *k*.

5. Geometrical Interpretation of Intermediate Value Theorem:

If $f(a) < k < f(b)$ as in figure above, then we assert that the graph of $y = f(x)$ cross the horizontal line *y* = *k* at least once as the graph is traced from (*a*, *f*(*a*)) to (*b*, *f*(*b*)).

Example 4: Show that every number in [–1, 1] is the tangent of some number in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Solution: Let c be any number in [–1, 1] since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $\tan\left(\frac{\pi}{4}\right) = 1$. By Intermediate Value Theorem there exist θ in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $\tan\theta = c$.

Corollary 1: A function f, which is continuous on a closed interval [a, b] assumes every value between its bounds (maximum and minimum values of f).

Proof: Since the function f is continuous on the closed interval [a, b] therefore it is bounded on [a, b] and attains its bounds on [a, b] by Extreme Value Theorem. Therefore there exists two numbers α, β in [a, b] such that

$$f(\alpha) = M \text{ and } f(\beta) = m$$

where M and m are respectively, the supremum and infimum of f. Since f is continuous on [a b] therefore it is continuous on $[\alpha \quad \beta]$ or $[\beta \quad \alpha]$ and consequently by **Intermediate Value Theorem** f assumes every value between $f(\alpha)$ and $f(\beta)$ that is *f* assumes every value between its bounds.

Definition 5: A function *f*defined on an interval I has I**ntermediate value property** on I if it satisfies the following condition: If *a* and *b* are distinct points in I and *v* is any number between *f*(*a*) and *f*(*b*) then there exists a point *c* between *a* and *b* such that *f*(*c*) = *v*.

A function with the intermediate value property is sometimes referred to as a Darboux function in honour of G. Darboux (1842-1917).

Example 5: Consider *f* : [0 2] $\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 4 - 2x & \text{if } 1 \le x \le 2 \end{cases}$$

Although the range of f is an interval and every horizontal line between *y* = 0 and y = 2 intersects the graph of f this function does not have intermediate value property on [0 2]. For if we take a = 0.5, b = 1 and v 1.5 there is no number c between a and b such that $f(c) = v$ (see figure)

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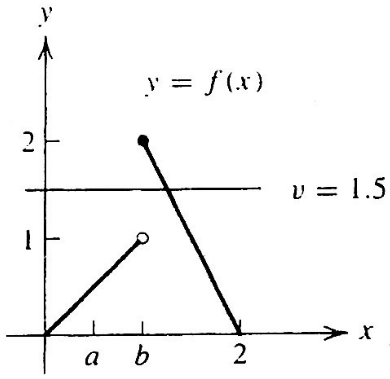

Figure 2: A function that does not have the intermediate value property on [0 2].

Value Addition: If f is continuous on an interval I, then by Intermediate Value Theorem f has the intermediate value property on I. The converse is not true. For example consider

$$g(x) = \begin{cases} \sin\frac{1}{x} & \text{for } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}.$$

Here g is not continuous at x = 0, but it does have the intermediate value property on [0   1].

Remark: Unlike continuity, the intermediate value property is not preserved under algebraic operations. For instance, the sum of two functions with the intermediate value property does not necessarily have the intermediate value property.

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The following result is a special case of Intermediate Value Theorem and is frequently used in finding roots of the equation $f(x)=0$ where *f* is a continuous, real valued function.

Corollary 2 (Location of Roots Theorem): Let $f:[a,\ b]\to\mathbb{R}$ be a continuous function. If $f(a)<0<f(b)$ or $f(a)>0>f(b)$ then there exists a number $c\in(a,b)$ such that $f(c)=0$.

Proof: We assume that $f(a)<0<f(b)$. We shall generate a sequence of intervals by successive bisections. Let $I_1=[a_1,\ b_1]$ where $a_1=a, b_1=b$, let p_1 be the midpoint, $p_1=\frac{1}{2}(a_1+b_1)$. If $f(p_1)=0$ we take $c=p_1$ and we are done. If $f(p_1)\neq 0$ then either $f(p_1)>0$ or $f(p_1)<0$. If $f(p_1)>0$ then we set $a_2=a_1, b_2=p_1$. If $f(p_1)<0$ then we set $a_2=p_1, b_2=b_1$. In either case $I_2=[a_2\ \ b_2]$. Now $I_2\subset I_1$ and $f(a_2)<0, f(b_2)>0$.

Continuing like this, suppose we obtain an interval I_K, with $f(a_k)<0, f(b_k)>0$ and se set $p_k=\frac{1}{2}(a_k+b_k)$. If $f(p_k)=0$. We take $c=p_k$ and we are done. If $f(p_k)>0$ we set $a_{k+1}=a_k, b_{k+1}=p_k$. If $f(p_k)<0$ we set $a_{k+1}=p_k$ and $b_{k+1}=b_k$. In either case $I_{k+1}=[a_{k+1}, b_{k+1}]$ with $I_{k+1}\subset I_k$ and $f(a_{k+1})<0\ f(b_{k+1})>0$. If the process terminates by locating a point p_n such that $f(p_n) = 0$ then we are done. If the process does not terminate we obtain a nested sequence of closed and bounded intervals $I_n = [a_n, b_n]$ such that

$$f(a_n) < 0,\ f(b_n) > 0 \text{ for every } n \in \mathbb{N}.$$

Since the intervals are obtained by repeated bisection, the length of $I_n = b_n - a_n = \dfrac{b-a}{2^{n-1}}$. By Nested Interval Theorem, there exists a point $c \in I_n$ for all n $\in \mathbb{N}$. Since $a_n \le c \le b_n$ for all n $\in \mathbb{N}$ we have

$$0 \le c - a_n \le b_n - a_n = \frac{b-a}{2^{n-1}} \qquad (1)$$

and $$0 \le b_n - c \le b_n - a_n = \frac{b-a}{2^{n-1}} \qquad (2)$$

It follows that $\lim\limits_{n\to\infty} a_n = c = \lim\limits_{n\to\infty} b_n$. Since f is continuous at c we get

$$\lim_{n\to\infty}(f(a_n)) = f(c) = \lim_{n\to\infty}(f(b_n)).$$

Now $f(a_n) < 0$ for all n $\in \mathbb{N}$ gives that f(c) ≤ 0. Also $f(b_n) > 0$ for all n $\in \mathbb{N}$ gives that $f(c) \ge 0$. Therefore we must have $f(c) = 0$. Consequently c is a root of *f*.

The proof of location of roots Theorem provides an algorithm known as the Bisection Method, which is frequently used in Numerical Analysis.

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Example 6: Show that the equation $4\sin x = x$ has a positive solution.

Solution: Define a function g by $g(x) = 4\sin x - x$ for all x $\in \mathbb{R}$. Then g is continuous on $\mathbb{R}$. Now g(π/2) > 0 and $g(\pi) < 0$ therefore by above Theorem there exist a number $c \in (\pi/2, \pi)$ such that $g(c) = 0$. Hence, the equation $4\sin x = x$ has a positive solution.

Example 7: Show that any polynomial of odd degree must have atleast one real root.

Solution: Let $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ where n is odd $a_i \in \mathbb{R}$ and $a_n \neq 0$. Suppose $a_n > 0$. We can write

$$p(x) = x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n\right)$$

as x $\to -\infty$, $x^n \to -\infty$ and expression inside the bracket $\to a_n$ therefore $p(x) \to -\infty$. Similarly when x $\to +\infty$, $x^n \to +\infty$ we get $p(x) \to +\infty$. Thus, for any M > 0 there exist points x_1 and x_2 such that $p(x_1) < -M < 0 < M < p(x_2)$. By Intermediate Value Theorem there is a point c between x_1 and x_2 with p(c) = 0, that is there exists atleast one real root of $p(x)$.

I.Q.5.

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Definition 6: Let X be a nonempty set and $f: X \to X$ be a mapping. A point c $\in$ X is said to be **fixed point** of *f* if $f(c) = c$.

Example 8: $f: X \to X$ defined by $f(x) = x$ has infinitely many fixed points. In fact every point of X is a fixed point.

I.Q.6.

Example 9: Let $f: X \to X$ be defined by $f(x) = x + a$ where a $\neq$ 0. Then f has no fixed point.

Theorem 4 (Fixed Point Theorem): If $f:[a,b] \to [a,b]$ is continuous then there exists at least one point c $\in$[a, b] such that $f(c) = c$.

Proof: Here f is continuous and $f(x) \in [a\ b]$ for all $x \in [a,b]$. If $f(a) = a$ or $f(b) = b$ then we are done. So let us assume that $f(a) > a$ and $f(b) < b$.

Define $g(x) = f(x) - x \qquad \forall\, x \in [a\ b]$

Now $g(a) > 0,\ g(b) < 0$ and g is continuous on [a, b], therefore. By Intermediate Value Theorem there exists $c \in (a,b)$ such that $g(c) = 0$ this gives $f(c) = c$.

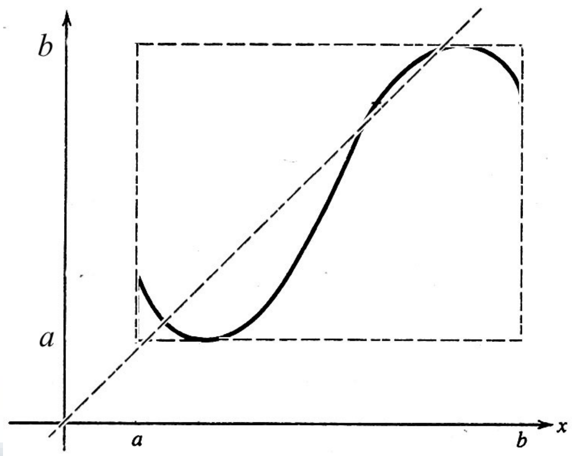

Figure 3

Geometrically, this theorem asserts that when f is a continuous function such that range of f is contained in domain of f then graph of f crosses the line $y = x$ atleast once.

Theorem 5: Let I be a closed and bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) = \{f(x) \mid x \in I\}$ is a closed and bounded interval.

Proof: Let $m = \inf f(I)$ and $M = \sup f(I)$. Then by Extreme Value Theorem *m* and *M* belong to $f(I)$ and $f(I) \subseteq [m \quad M]$.

(1)

Suppose $k \in [m \quad M]$ then from the corollary 8there exists a point $c \in I$ such that $f(c) = k$ that is $k \in f(I)$ this gives

$[m,M]\subseteq f(I)$ (2)

From (1) and (2), we get

$f(I)=[m \quad M]$.Hence proved.

Value addition: The above theorem asserts that, if $f:[a \quad b]\to\mathbb{R}$ is continuous then $f[a,b]=[m \quad M]$and $f[a,b]\neq[f(a) \quad f(b)]$.

Theorem 6 (Preservation of Intervals): Let I be an interval. Let $f:I\to\mathbb{R}$ be continuous on I then the set $f(I)$ is an interval.

Proof: Let $\alpha,\beta\in f(I)$with $\alpha<\beta$. Then there exist points a, b $\in$ I such that $\alpha=f(a)$and $\beta=f(b)$. By Intermediate Value Theorem if $k\in(\alpha,\beta)$ then there exist a point $c\in I$ such that $k=f(c)\in f(I)$. This gives $[\alpha,\beta]\subseteq f(I)$. Therefore it follows by definition that $f(I)$ is an interval.

Here it is important to note that the intervals I and $f(I)$ are not necessarily of the same type. For example if $g(x)=\sin x$ and $I=(-\pi \quad \pi)$ then$g(I)=[-1 \quad 1]$. The converse of above theorem is not true. The range of a function where domain is an interval can be an interval even if the function is not continuous on the domain. For example consider $f(x)=x-[x]$ on $[0 \quad 2]$.

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Definition 7: Let I be an interval, let $f : I \to \mathbb{R}$

(a) The function *f* is **increasing** on I if $f(x) \le f(y)$ for all *x*, *y*$\in I$ that satisfy *x*<*y*.

(b) The function *f* is **strictly increasing** on I if $f(x) < f(y)$ for all *x*, *y*$\in I$ that satisfy *x*<*y*.

(c) The function *f* is **decreasing** on I if $f(x) \ge f(y)$ for all *x*, *y*$\in I$ that satisfy *x*<*y*.

(d) The function *f* is **strictly decreasing** on I if $f(x) > f(y)$ for all *x*, *y*$\in I$ that satisfy *x*<*y*.

(e) The function *f* is **monotone** on I if it is either increasing or decreasing on I .

(f) The function *f* is **strictly monotone** on I if it is either strictly increasing or strictly decreasing on I.

Value Addition: If $f : I \to \mathbb{R}$ is increasing on I then *g* = –*f* is decreasing on *I*.

I.Q.7,

I.Q.8

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Theorem 7: Let I be an interval and let $f: I \to \mathbb{R}$ be a monotone function. If $f(I)$ is an interval then f is continuous on I.

Proof: We assume that the interval I is open and *f* is increasing on I we shall prove the contrapositive statement. Suppose f is not continuous on I. Let c $\in$ I be a point at which *f* is not continuous.

Let $\alpha = \sup\{f(x) \mid x \in I, x < c\}$

$\beta = \inf\{f(x) \mid x \in I, x > c\}$

then $\alpha < \beta$ (as f is an increasing function). Let u, v $\in$ I be such that $u < c < v$. Let $y \in (\alpha, \beta) \setminus \{f(0)\}$ then y lies between *f*(*u*) and *f*(*v*) but y is not in the range of *f*. This shows that *f*(*I*) is not an interval.

Theorem 8 (Monotone Theorem): Let f be a one-one, continuous function on *I* = [*a* *b*]. Then f is strictly increasing or strictly decreasing on I.

Proof: Let f be one to one and continuous on I then $f(a) \neq f(b)$.

Suppose $f(a) < f(b)$. If f is not strictly increasing on I, then there exist

p, q$\in$*I* with *p*<*q* and $f(p) \geq f(q)$. Since f is one to one therefore $f(p) \neq f(q)$ and we must have $f(p) > f(q)$. There are two possibilities.

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Case 1: $f(p) > f(b)$. Choose $k \in (f(b), f(p))$. By Intermediate Value Theorem there exists $c \in (a \;\; p)$ with $f(c) = k$ and $d \in (p, b)$ with $f(d) = k$. But as $c \neq d$ therefore this contradicts the fact that f is one to one on I.

Case 2. $f(p) < f(b)$. Then $f(q) < f(b)$. Choose $k \in (f(p), f(b))$ obviously $k \in (f(q), f(b))$. By Intermediate Value Theorem there exist $u \in (a, q)$ with $f(u) = k$ and $v \in (q, b)$ with $f(v) = k$. Again as $u \neq v$, this contradicts the fact that f is one to one on I. Therefore it follows that f is strictly increasing on *I*.

When $f(a) > f(b)$ we apply the above argument to the function –*f*. Then –*f* is strictly increasing on *I*that is*f* is strictly decreasing on I.

Exercise

1. Show that the function defined on [0, 1] by $f(x) = \begin{cases} 2x+1 & x \in (0 \;\; 1] \\ 0 & x = 0 \end{cases}$ does not satisfy the conclusion of Intermediate value Theorem. Why?

2. Prove that $\cos x = x$ for some $x \in \left(0 \;\; \frac{\pi}{2}\right)$.

3. Show that $x.2^x = 1$ for some *x* in (0 1).

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4. Suppose f is continuous on [a b] and $f(x) \in Q \ \forall x \in [a \ \ b]$ prove that f is a constant function on [a b].

5. Show that every number in [–1 1] is the sine of some number in $\left[-\frac{\pi}{2} \ \ \frac{\pi}{2}\right]$.

6. Show that $2^x = 3x$ for some $x \in (0 \ \ 1)$.

7. Show that for $f(x) = (x+1)(x-1)(x-2)$ on [–2 4] there is at least one point $c \in [-2 \ 4]$ for which $f(c) = 8$.

8. Find a real root of the equation

$$f(x) = x^3 - x - 1 = 0$$

[Hint: $f(1) < 0, f(2) > 0$. Hence root lies between 1 and 2 repeat the process to arrive at the desired result]

9. Give example of a function which is continuous and bounded on (–1, 1] but does not have a maximum value on (–1, 1].

10. Suppose f, g : [a, b] $\to \mathbb{R}$ are continuous, $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for at least one $c \in (a,b)$

[Hint: Define $h(x) = f(x) - g(x)$. Then *h* is continuous on [*a, b*], *h*(*a*) < 0, *h*(*b*) > 0 therefore there exist at least one $c \in (a,b)$ such that $h(c) = 0$]

Properties of Continuous Function

Summary

In this chapter we have discussed

- Continuous functions on Intervals
- Properties of continuous functions
- Fixed point theorem.
- Bisection method for finding solution of equation of the form $f(x)=0$, where *f* is a continuous function.

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