Rank of Matrices

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Table of Contents

1. Learning outcomes:

After studying this chapter you should be able to

- Understand the subspace of R^n.
- Define and determine the column space and null space of a matrix.
- Understand the basis for a subspace.
- Calculate the dimension of a subspace.
- Define and determine the rank of a matrix.

2. Introduction:

The rank of the matrix is the most important concept in Matrix Algebra. The rank is one of the fundamental pieces of data associated to a matrix. A matrix always represents a linear transformation between two vector spaces. With the help of the rank of the matrix we come to know several things about this linear transformation e.g. whether it is bijective, how it maps elements etc.

3. Subspace of R^n :

Let S be any subset of R^n. Then the set S is said to be the subspace of R^n if and only if it satisfies the following three properties :

(i) The zero vector is in S.

(ii) For each x, y $\in$ S, the sum x + y $\in$ S

(iii) For each x $\in$ S and to each scalar a, the vector ax $\in$ S.

Value Addition: Note
(1) A set S of R^n is called subspace of R^n if and only if it is closed under addition and scalar multiplication. (2) Three condition in the definition of subspace of R^n may be replaced by the equivalent one condition only i.e. A set S of R^n is called subspace of R^n if any only if to each x, y $\in$ S and for any scalars a, b $\in$ R, ax+by$\in$S. (3) R^n is also a subspace of itself because it satisfies the three properties required for a subspace. (4) The set consisting of only the zero vector in R^n is also a subspace of R^n, called the zero subspace of R^n.

Example 1: Consider a set S = span {v_1, v_2} where $v_1, v_2 \in R^n$. Show that S is a subspace of R^n.

Solution: To show that S is subspace of R^n. We will show that

(i) Zero vector is in S.

(ii) for all x,y $\in$ S $\Rightarrow$ x+y $\in$ S

(iii) for all x $\in$ S & for any scalar a, ax $\in$ S

(i) To show zero vector is in S.

we can write

$$\bar{0} = 0v_1 + 0v_2$$

Thus $\bar{0}$ vector is a linear combination of v_1 & v_2 therefore $\bar{0} \in S$.

(ii) Let x, y $\in$ S

Then x and y are the linear combination of v_1 & v_2 such that

$x = a_1v_1 + a_2v_2$, $a_1, a_2 \in R$ and $y = b_1v_1 + b_2v_2$, $b_1, b_2 \in R$.

Then $x + y = (a_1v_1 + a_2v_2) + (b_1v_1 + b_2v_2)$

$$= (a_1 + b_1)v_1 + (a_2 + b_2)v_2, \quad (a_1 + b_1), (a_2 + b_2) \in R$$

This show that x + y is a linear combination of v_1 and v_2. Hence x+y $\in$ H.

(iii) Let x $\in$ S and let a be any scalar, then

$$x = a_1v_1 + a_2v_2 , \quad a_1, a_2 \in R$$

Now $ax = a(a_1v_1 + a_2v_2)$

$$ax = (aa_1)v_1 + (aa_2)v_2, \quad aa_1, aa_2 \in R$$

This shows that ax is a linear combination of v_1 and v_2. Hence ax $\in$ s. Since S satisfies all the three condition. Therefore S is a subspace of R^n.

Value Addition: Note
Defining a set by S = span{v_1, v_2 ,. . ., v_n} means that every vector of the set S can be written as the linear combination of the vectors v_1, v_2,. . ., v_n. i.e. let S = span { v_1, v_2,. . ., v_n} and let u$\in$S, then there exists the scalars a_1, a_2, . . ., a_n such that $u = a_1v_1 + a_2v_2 + . . . + a_nv_n$.

3.1. Column space of a matrix:

The column space of a matrix A is denoted by Col A and defined as the set of all linear combinations of the columns of A.

Value Addition: Note
1. Consider a matrix A = [a_1 a_2 . . . a_n] where a_1, a_2,. . ., a_n are the columns in R^m. Then Col A = span [a_1, a_2,. . ., a_n] 2. The column space of an m $\times$ n matrix is a subspace of R^m. 3. To check whether a vector b is in the column space of a given matrix A, or not, we will find the solution of the system Ax = b. If the system is consistent i.e. the system has a solution then the vector b is in the column space of A.

Example 2 : Let $A = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$. Determine whether b is in the column space of A.

Solution : For the solution of the equation Ax = b, we have the augmented matrix

$$[A \quad b] = \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 + \frac{1}{2}R_2 \; .$$

There is a free variable therefore, the system Ax = b has a non-zero solution i.e. the system is consistent, thus b is in the column space of A.

Example 3: Let $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -6 \\ 3 \end{bmatrix}$, $p = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$ and $A = [v_1 \; v_2 \; v_3]$, then

(i) How many vectors are in $\{v_1, v_2, v_3\}$

(ii) How many vectors are in Col A ?

Is p in Col A ? Why or why not ?

Solution: (i) There are three vectors in $\{v_1, v_2, v_3\}$.

(iii) We have

$$A = [v_1 \; v_2 \; v_3] = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix}$$

Augmented matrix

$$[A \quad p] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & -1 & 3 & -7 \end{bmatrix} \qquad R_3 \to R_3 + 2R_1$$

$$\sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 + \frac{1}{2}R_2 \,.$$

since the system Ax = p is consistent, therefore p is in the column A.

(ii) There are infinity many vectors is Col A, because the equation Ax = p has infinite solutions.

3.2. Null Space of a Matrix:

The null space of a matrix A is denoted by Nul A and defined as the set of all solutions to the homogenous equation Ax = 0.

Value Addition: Note
1. To check whether a given vector b is in Nul A, we will compute Ab to see whether Ab is the zero vector or not. If Ab is the zero vector, then b is in the null space of A otherwise not. 2. The null space of A is a subset of R^n, infect Nul A is the subspace of R^n.

Example 4 : Let $A = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$. Check whether b is in null space of A or not.

Solution : We have

$$Ab = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix} \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} 12 + 30 - 44 \\ -48 - 80 + 66 \\ 36 + 70 - 77 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}$$

since $Ab \neq 0$, therefore, b is not in the null space of A.

Theorem 1: The null space of an m×n matrix A is a subspace of R^n.

or

The set of all solution to a system Ax=0 of m homogeneous linear equations in n unknowns is a subspace of R^n.

Proof: To show that null space of A (Nul A) is subspace of R^n, we will show that it satisfies the three conditions of subspace.

(i) Zero vector is in null space of A

We know that

$$A\overline{0} = \overline{0}$$

Thus zero vector is in the null space of A.

(ii) Let u an v are two vectors, s.t. u, v∈Nul A

Then $Au = \overline{0}$ and $Av = \overline{0}$

Now, we have

$$A(u+v) = Au + Av \qquad \text{[By property of matrix multiplication]}$$

$$= \overline{0} + \overline{0}$$

$$= \overline{0}$$

Thus, u + v ∈ Nul A

(iii) Let u ∈ Nul A and let s be any scalar then $Au = \overline{0}$.

Now

$$A(su) = s(Au) = s.\overline{0} = \overline{0}$$

Thus, su∈ Nul A for all u∈Nul A and s∈R.

Thus Nul A is a subspace of R^n.

4. Basis for a Subspace:

A set B = {b_1, b_2, . . ., b_n} of vectors is a basis for a subspace S of R_n if it has the following two properties:

(i) B is linearly independent

(ii) B spans S, i.e. each vector of S can be written as a linear combination of vectors of B.

Value Addition: Note
1. **Definition :** A set B = {b_1, b_2, . . ., b_n} of vectors is a basis of S if every s∈S can be written uniquely as a linear combination of the basis vectors.
2. Let e_1 = [1], then the set {e_1} is a standard basis for R.
3. Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then the set {$e_1$, e_2} is the standard basis for R^2.
4. Similarly, in this way $e_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \quad ---------- \quad e_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$ Then the set {e_1, e_2, . . . , e_n} is called the standard basis for R^n.

Example 5: Consider a matrix $A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix}$. Find a basis for Nul A and Col A.

Solution: (i) To find the basis for Nul A, we have Ax = 0

Augmented matrix

$$\begin{bmatrix} A & \bar{0} \end{bmatrix} = \begin{bmatrix} 4 & 5 & 9 & -2 & 0 \\ 6 & 5 & 1 & 12 & 0 \\ 3 & 4 & 8 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 6 & 5 & 1 & 12 & 0 \\ 3 & 4 & 8 & -3 & 0 \end{bmatrix} \qquad R_1 \to R_1 - 3R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 6 & 0 \\ 0 & 1 & 5 & -6 & 0 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 1 & 5 & -6 & 0 \end{bmatrix} \qquad R_2 \to (-1) R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - R_2$$

Thus, we have

$$x_1 + x_2 + x_3 + x_4 = 0$$

$$x_2 + 5x_3 - 6x_4 = 0$$

Taking x_3 and x_4 as free variables, we have

$$x_2 = -5x_3 - 6x_4$$

and $\quad x_1 = 4x_3 - 7x_4$

Thus, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4x_3 - 7x_4 \\ -5x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix} = x_3 u + x_4 v$$

Thus, {u, v} generates Nul A. Also u and v are linearly independent because

$$x_3 u + x_4 v = \overline{0}$$

has a unique solution if and only if $x_3 = 0$ and $x_4 = 0$.

Thus, {u, v} is a basis for Nul A.

(ii) For the basis of column space of A, we have.

$$A = \begin{bmatrix} 4 & 5 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

since the column a_1 and a_2 of A are linearly independent and all other columns can be written as linear combination of a_1 and a_2 therefore $\{a_1,a_2\}$ span Col A.

Thus the pivot columns of A i.e.

$a_1 = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}$, form the basis for Col A.

Value Addition: Note
The pivot columns of a matrix A form the basis for the column space of A.

Value Addition: Cautions
Always use the pivot columns of A itself for the basis of column space of A. the columns of an echelon form B are often not in the column space of A. For example, the columns in the echelon form of A, have zeros in their last entries and therefore cannot generate the columns of A.

5. Coordinate System:

Let the set B = $\{b_1, b_2, . . ., b_n\}$ is a basis for a subspace S. Then for each $s \in S$, the coordinates of s relative to the basis B is denoted by

$$[s]_B = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

is a column vector in R^n.

such that

$s = a_1b_1 + a_2b_2 + - - -+ a_nb_n.$

where $a_1, a_2, - - -, a_n$ are constants.

Then $[s]_B$ is called the coordinate vector of s (relative to B) or the B–coordinate vector of s.

Example 6 : Let $b_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$, $s = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ and B = $\{b_1, b_2\}$ then

(i) Show that B is a basis for S = span $\{b_1, b_2\}$.

(ii) Determine if s is in S, and if it is, find the coordinate vector of s relative to B.

Solution : (i) To show B is a basis for S = span $\{b_1, b_2\}$ B is linearly independent. We have

$$\begin{bmatrix} 1 & -2 \\ -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \qquad R_2 \to R_2 - 4R_1$$

thus, b_1 and b_2 are linearly independent. Hence B = $\{b_1, b_2\}$ is a basis for S = span $\{b_1, b_2\}$

(ii) To check whether s is in S or not. If s is in S then the equation

$a_1b_1 + a_2b_2 = s$

must be consistent, for this

$$a_1 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

Now using the augmented matrix, we have

$$\begin{bmatrix} 1 & -2 & -3 \\ -4 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & -1 & -5 \end{bmatrix} \qquad R_2 \to R_2 + 4R_1$$

$$\sim \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 5 \end{bmatrix} \qquad R_2 \to (-1) R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \end{bmatrix} \qquad R_1 \to R_1 + 2R_2$$

Since, the system is consistent therefore s is in S. and $a_1 = 7$ and $a_2 = 5$, therefore

$$[s]_B = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Value Addition: Note
If B = $\{b_1, b_2, - - -, b_n\}$ is a basis for S, then the mapping $s \to [s]_B$ is a one–to–one correspondence that makes S look and act the same as R^n [Even through the vectors in S themselves may have more than n entries].

6. Dimension of a Subspace:

The dimension of a nonzero subspace S is denoted by dim S and is equal to the number of vectors in any basis for S.

For Example: The space R^2 has dimension 2, then every basis for R^2 consists of two vectors, similarly the space R^2 has dimension 3 and every basis for R^3 consists of three vectors. In general the space R^n has dimension n, and every basis of R^n consists n vectors.

Value Addition: Note
1. If a subspace S has a basis of n vectors, then every basis of S must consists of exactly n vectors.
2. A plane through $\bar{0}$ in R^3 is two dimensional and a line through $\bar{0}$ is one dimensional

7. Rank of a matrix :

Let A be an m x n matrix corresponding to a linear napping T form R^n to R^m relative to a basis $\{\alpha_1, \alpha_2, - - -, \alpha_n\}$ for R^n and basis $\{\beta_1, \beta_2, - - -, \beta_m\}$ for R^m . For j = 1, 2, - - -, n the j^{th} column of the matrix A is the m–tuple of scalars that represent $T(\alpha_j)$ relative to the β–basis. The rank of the matrix A or we can say that the rank of the linear transformation T is defined to be the dimension of the range space R(T), which is the space spanned by the vector $T(\alpha_1)$, $T(\alpha_2)$,- - -, $T(\alpha_n)$ (The column vectors of A).

$$A = \begin{bmatrix} a_{11} & a_{12} & --- & a_n \\ a_{21} & a_{22} & --- & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}$$

$$T(\alpha_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}, \quad T(\alpha_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m2} \end{bmatrix} ---, \quad T(\alpha_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix}$$

Therefore

Rank of T = Rank of A

= Number of linearly independent column vectors $\{T(\alpha_1), T(\alpha_2), - - -, T(\alpha_n)\}$ of A.

7.1. Methods of Finding Rank of a Matix:

There are mainly three methods to determine the rank of a matrix.

7.1.1. Minors Method :

Let A be any mxn matrix. Then it has square sub–matrices of different orders. The determinates of these square sub–matrices are called minors of A. To determine the rank of A we will follow these steps.

Step 1: Start with the highest order minor (or minors) of A. Let their order be r. If any one of them is non–zero then rank (A) = r.

Step 2: If all of them are zero, start with minors of next lower order (r–1), and so on until we get a non–zero minor. The order of that minor will be the rank of the matrix A.

Thus, the order of the highest non-zero minor is the rank of the matrix.

Value Addition: Note
(i) A matrix A is said to be of rank r if (a) It has at least one non–zero minor of order r. (b) All the minors of order higher than r are zero. (ii) If A is a null matrix, then rank (A) = 0 (iii) If A is an non–zero mxn matrix, then $1 \leq$ rank (A) $\leq$ minimum of m and n. (iv) If A is a non–singular nxn matrix, then rank (A) = n

7.1.2. Echelon Form Method:

Let A be a non–zero mxn matrix. In the Echelon form of the matrix, each of the first r elements of the leading diagonal is 1 and every element below this diagonal or r^{th} row is zero.

A non–zero matrix A can be reduced in Echelon form by following steps.

Step 1: The first non–zero element in a row should be unity.

Step 2: All the non–zero rows, if any, precede the zero rows.

Then the number of non–zero diagonal elements in the reduced echelon–form is the rank of the matrix.

In other words the number of non-zero rows in the echelon form of matrix is called the rank of the matrix.

Value Addition: Do you know?
1. Elementary operation (or Transformations) Any one of the following operations on a matrix is called an elementary operation or E–operation. (i) Interchange of two rows or two columns. (ii) Multiplication of (each element of) a row or column by a non–zero number k. (iii) Addition of k times the elements of a row (or column) to the corresponding elements of another row (or column), $k \neq 0$. 2. On operating elementary (row or column or both) operations on a matrix, the rank of the matrix does not change. 3. The matrix B obtained from a matrix A by operating one or more E–operation, is called the equivalent matrix to A.

7.1.3. Normal form Method:

Let A be an mxn matrix. On operating a series of elementary (row or column or both) operations. This matrix can be transformed into one of the following forms called normal or canonical forms.

$$\begin{bmatrix} I_r & : & 0 \\ -- & -- & -- \\ 0 & : & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ -- \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & : & 0 \end{bmatrix}, \begin{bmatrix} I_r \end{bmatrix}$$

Where I_r is the unit matrix of order r.

Then the rank (A) = r.

This method is also called sweep out method or Pivotal method. In order to find the normal form or canonical form of a matrix, we use the following steps.

If the first element of the first row and I-column is zero then

Step 1: Interchange the rows (or columns) of the matrix A to obtain a non–zero element in I–row and I–column of given matrix.

Step 2: Make this non–zero element as I by dividing the first row by the number itself.

Step 3: Obtain zeros in the remainder of I–row and I–column by operating the elementary row and column operations.

Step 4: Repeat the above three steps starting with element in II row and II column.

Step 5: Continue this process down the main diagonal either until the end of diagonal is reached or all the remaining elements of matrix are zero.

7.2. Definition of Rank:

The rank of a matrix A, denoted by rank A, is the dimension of the column space of A.

Value Addition: Note
We have studied that pivot columns of A form the basis for column space of A (Col A). Therefore the number of pivot columns in A is equal to the rank of A.

Theorem 2: Colum rank and row rank of a mxn matrix A are equal and each is equal to the rank of the matrix A.

Proof: Consider a mxn matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1j} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2j} & --- & a_{2n} \\ . & & & & & \\ . & & & & & \\ a_{i1} & a_{i2} & --- & a_{ij} & --- & a_{in} \\ . & & & & & \\ . & & & & & \\ a_{m1} & a_{m2} & --- & aj & --- & an \end{bmatrix}$$

Suppose the matrix A is reduced to row–reduced echelon form by applying the elementary row operations. And let in the row–reduced echelon form the number of non zero rows are $r \leq m$.

Then these rows are linearly independent

Hence, the row rank of matrix A = r.

Now consider the column vectors of matrix A.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ a_{m2} \end{bmatrix}, ---, \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ a_{mn} \end{bmatrix}$$

These are n column vectors each of m components of i.e. of V_m a vector space of dimension m. Therefore, if n < m, Then these n vectors are linearly dependent in the row space of matrix A. Hence number of linearly independent vectors $r \leq m \leq n$.

If

$$r = \text{row rank of A} = \text{dimension of the row space of A}$$

Then $n > r$, and these n vectors are linearly dependent in r–dimensional vector space.

Therefore maximum number of linearly independent column vectors = r = row–rank of A .

Now, we have

row–rank of a matrix A = $r \leq m$ is equal to the number of non–zero rows reduced echelon form

Therefore, the row rank of matrix A = rank of A.

Hence Proved

Value Addition: Note
The maximum number of linearly independent row vectors of an m x n matrix A equals to the maximum number of linearly independent column vectors of A.

Theorem 3: For any m x n matrix A, rank (A) = rank (A*)

i.e. rank of a matrix A = rank of the transpose of matrix A

Proof : The rank of A* is the maximum number of linearly independent columns of A*. But the columns of A* are the rows of A.

Thus by the theorem "the maximum number of linearly independent row vectors of an mxn matrix A equal to the maximum number of linearly independent column vectors of A".

Thus, rank (A) = rank A*

Hence Proved

Example 7. Find the rank of the matrix

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Solution: (i) Given matrix is

A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}$ is a 2 x 4 matrix.

Therefore rank of (A) $\leq$ 2, the smaller of 2 and 4.

The second order minor $\begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} = 4 \neq 0$

Thus, Rank of A = 2.

(ii) Given matrix is

A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ is a 3 x 3 matrix

Therefore rank of (A) $\leq$ 3.

Now we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 - R_2$$

There are only two non-zero rows in the echelon form of the matrix therefore the rank of (A) = 2.

Example 8: Reduce the matrix to its normal form and hence find the rank of A.

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

Solution: Given Matrix is

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

To reducing the matrix into normal form, we have

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 5 & -3 \end{bmatrix} \qquad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 5 & -3 \end{bmatrix} \qquad R_2 \to \frac{1}{4} R_2$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 4\\0 & 0 & 1 & 0\\0 & 0 & 0 & -4\\0 & 0 & 0 & -3\end{bmatrix}\qquad \begin{matrix}R_3 \to R_3 - 5R_2\\ R_4 \to R_4 - 5R_2\end{matrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 4\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & -3\end{bmatrix}\qquad R_3 \to \frac{(-1)}{4} R_3$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 4\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}\qquad R_4 \to R_4 + 3R_3$$

$$\sim\begin{bmatrix}1 & 2 & 0 & 4\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}\qquad R_1 \to R_1 + R_2$$

$$\sim\begin{bmatrix}1 & 2 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}\qquad R_1 \to R_1 - 4R_3$$

$$\sim\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}\qquad C_2 \leftrightarrow C_3$$

$$\sim\begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}\qquad C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_4 \to C_4 - 2C_1$$

Thus, we have

$$\sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & \vdots & O \\ \cdots & \cdots & \cdots \\ O & \vdots & O \end{bmatrix}$$

which is the normal form and Hence rank of (A) = 3.

Example 9: Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}.$$

and also determine the dimensions of Col A and Nul A.

Solution : Reducing the matrix A into echelon form, we have

$$A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & -3 & -6 & -7 & -5 \\ 0 & 4 & 8 & 11 & 10 \end{bmatrix} \qquad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix} \qquad \begin{matrix} R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix} \qquad R_3 \to \frac{1}{5} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to R_4 + 5R_3$$

Since, the matrix has 3 pivot columns or in other words we can have the matrix has three non–zero row in the echelon form, therefore rank A = 3. Thus, dimension Col A = 3.

Since there are two free variables in Ax = 0 because two of the five columns of A are not pivot columns. Thus, dimension Nul A = 2.

Value Addition: Note
1. The non pivot columns correspond to the free variables in Ax = 0. 2. Dim Col A = Number of pivot columns in echelon form = Rank A 3. Dim Nul A = Number of non–pivot columns in Echelon form 4. If a matrix A has n columns, then dim Col A + dim Nul A = n or Rank A + dim Nul A = n

Example 10: Prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if and only if the rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than three.

Solution: The condition is necessary

Let the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. Now, we have to prove that the rank of matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than three.

Since the points are collinear, therefore, the area of the triangle formed by these points is zero. Hence

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ The rank of matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.

Hence the condition is necessary.

Value Addition: Note
If the determinant of nxn matrix is zero than the rank of the matrix is always less than n.

The condition is sufficient

Let the rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3 then we have to prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear. Since rank of matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3. Therefore

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Hence the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

Exercise:

1. Find the rank of the matrices

(i) $\begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

2. Determine the rank of the matrices by reducing them to the normal form.

(ii) $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$

3. Find the rank of the following matrices using elementary transformations:

(i) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -3 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 3 & 4 & 1 & -2 \end{bmatrix}$

Rank of Matrices

4. Find the rank of the following matrices by reducing it to normal form (or canonical form):

(i) $\begin{bmatrix} 1 & 3 & 2 & 5 & 1 \\ 2 & 2 & -1 & 6 & 3 \\ 1 & 1 & 2 & 3 & -1 \\ 0 & 2 & 5 & 2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 8 & 11 & 14 & 7 \end{bmatrix}$

5. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$; a, b, c being all real.

6. Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$ and $p = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$. Is p in Nul A? Is p in Col A? Justify your answer.

7. Given $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find a vector in Nul A and a vector in Col A.

8. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -7 \\ 9 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ -8 \\ 6 \\ 5 \end{bmatrix}$, and $u = \begin{bmatrix} 4 \\ 10 \\ -7 \\ -5 \end{bmatrix}$. Determine if u is in the subspace of $\mathbb{R}^4$ generated by $\{v_1, v_2, v_3\}$.

9. Let $v_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$, $u = \begin{bmatrix} 8 \\ 2 \\ -9 \end{bmatrix}$. Determine if u is in the subspace of $\mathbb{R}^3$ generated by v_1 and v_2.

10. Let $v_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$, $p = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$, and $A = [v_1\ v_2\ v_3]$.

    a. How many vectors are in $[v_1, v_2, v_3]$?

b. How may vectors are in Col A?

c. Is p in Col A? Why or Why not?

11. Determine which of the following sets are bases for $\mathbb{R}^2$ or $\mathbb{R}^3$. Justify each answer.

(i) $\begin{bmatrix}-4\\6\end{bmatrix}, \begin{bmatrix}2\\-3\end{bmatrix}$ (ii) $\begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}-5\\-1\\2\end{bmatrix}, \begin{bmatrix}7\\0\\-5\end{bmatrix}$

(iii) $\begin{bmatrix}1\\-6\\-7\end{bmatrix}, \begin{bmatrix}3\\-4\\7\end{bmatrix}, \begin{bmatrix}-2\\7\\5\end{bmatrix}, \begin{bmatrix}0\\8\\9\end{bmatrix}$ (iv) $\begin{bmatrix}3\\-8\\1\end{bmatrix}, \begin{bmatrix}6\\2\\-5\end{bmatrix}$

12. Find a basis for column space and a basis for null space for the following matrices

(i) $\begin{bmatrix}-3 & 9 & -2 & -7\\ 2 & -6 & 4 & 8\\ 3 & -9 & -2 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}3 & -1 & 7 & 3 & 9\\ -2 & 2 & -2 & 7 & 5\\ -5 & 9 & 3 & 3 & 4\\ -2 & 6 & 6 & 3 & 7\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 4 & 8 & -3 & -7\\ -1 & 2 & 7 & 3 & 4\\ -2 & 2 & 9 & 5 & 5\\ 3 & 6 & 9 & -5 & -2\end{bmatrix}$

13. The vector s is in a subspace S with a basis B = (b_1 , b_2). Find the B–coordinate vector of s.

(i) $b_1 = \begin{bmatrix}1\\5\\-3\end{bmatrix},\ b_2 = \begin{bmatrix}-3\\-7\\5\end{bmatrix},\ s = \begin{bmatrix}4\\10\\-7\end{bmatrix}$ (ii) $b_1 = \begin{bmatrix}1\\-3\end{bmatrix},\ b_2 = \begin{bmatrix}-3\\5\end{bmatrix},\ s = \begin{bmatrix}-7\\5\end{bmatrix}$

(iii) $b_1 = \begin{bmatrix}1\\-4\end{bmatrix}, b_2 = \begin{bmatrix}-2\\7\end{bmatrix},\ s = \begin{bmatrix}-3\\7\end{bmatrix}$ (iv) $b_1 = \begin{bmatrix}-3\\1\\-4\end{bmatrix},\ b_2 = \begin{bmatrix}7\\5\\-6\end{bmatrix},\ s = \begin{bmatrix}11\\0\\7\end{bmatrix}$

14. Find the column space of A and null space of A, and then determine the dimensions of these subspaces.

(i) $\begin{bmatrix} 1 & 2 & -4 & 3 & 3 \\ 5 & 10 & -9 & -7 & 8 \\ 4 & 8 & -9 & -2 & 7 \\ -2 & -4 & 5 & 0 & -6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}$

15. Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

(i) $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 9 \\ -5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$

Summary:

In this lesson we have emphasize on the followings

- Subspace of R^n.
- Column space and null space of a matrix.
- Basis for a subspace.
- Dimension of a subspace.
- Rank of a matrix.
- Methods to determine the rank of matrix

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