**Discipline Courses-I**
**Semester-I**
**Paper: Calculus-I**
**Lesson: Reduction Formulae**

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## 1. Learning Outcomes

After reading this chapter, you should be able to understand how to derive reduction formulae for the following functions

- $\int \sin^n x\,dx$,
- $\int \cos^n x\,dx$,
- $\int \tan^n x\,dx$,
- $\int \sec^n x\,dx$,
- $\int \sin^n x \cos^m x\,dx$,
- $\int (\log x)^n\,dx$,
- $\int_0^{\pi/2} \sin^n x \cos^m x\,dx$ *etc.*

and how to apply them to find the value of the integrals.

## 2: Introduction:

In this lesson we will study a different method of integration to evaluate the integrals. Integration by parts is one of the effective and most useful methods to evaluate the integrals having an expression involving products or logarithm or inverse trigonometric functions, which cannot be solved directly using standard integral methods. We need to integrate powers of some trigonometric functions and their products in many problems in engineering and physics. In this chapter we will discuss the techniques to derive these kinds of integrals and derive some standard forms using this technique. Using the method of integration by parts, we will try to reduce a given integral in terms of another simple integral.

## 3: Reduction Formulae:

A standard technique to solve the complex problems is to reduce the given problem into similar problem but with lesser complexity. Sometimes integrand depends not only on the independent variables but also on the value n, called the parameter.
For example in order to evaluate

$$\int cos^n x\, dx$$

we may convert it into a relation which requires the solution of

$$\int cos^{n-1} x \cos x\, dx$$

and will get

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2} x\, dx \,,$$

Thus if we apply this formula repeatedly, the exponent can be reduced to 0 or 1, accordingly n is even or odd.
We can also use this technique to evaluate the integrals of the type

$$\int \sin^m x\, cos^n x\, dx$$

by reducing the power of m and n to small values like 0, 1, 2… etc. Since this process connects the given integral with another of the same type with reduced power, the relation obtained is called reduction formula.

| **Value Addition: Definition** |
|---|
| **Reduction Formulae:** A formula of the form $\int g(x,n)dx = h(x) + \int f(x,k)dx$ , where k < n, is called the reduction formulae. |

**Example 1:** Show that the integration of $\int x^n e^{ax} dx$ depends upon *x* and also on *n*, which is exponent of *x*.

**Solution:** Let $I_n = \int x^n e^{ax} dx$, on integrating by parts, we have

$$I_n = \int x^n e^{ax} dx = x^n \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx \Rightarrow I_n = x^n \frac{e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

Thus, the integration of $\int x^n e^{ax} dx$ depends upon *x* and also on *n*.

**4: Reduction Formulae for Trigonometric Functions:**

**4.1: Reduction formula for $\int \sin^n x\, dx$, *n* being a positive integer greater than unity**

Let

$$I_n = \int sin^n x\, dx = \int sin^{n-1} x\ sin\, x\, dx \qquad (n > 1)$$

(we can rewrite *sin* $^n$ *x* as *sin* $^{n-1}$ *x sin x*)

Using method of integration by parts, we get

$$I_n = sin^{n-1} x \int sin\, x\, dx - \left(\int\left(\frac{d\left(\sin^{n-1} x\right)}{dx}\int sin\, x\, dx\right)dx\right)$$

$$= -sin^{n-1} x \cos x + (n-1)\int sin^{n-2} x \cos x \cos x\, dx$$

$$= -sin^{n-1} x \cos x + (n-1)\int sin^{n-2} x\left(1 - \sin^2 x\right)dx$$

$$= -sin^{n-1} x \cos x + (n-1)\int sin^{n-2} x\ dx - (n-1)\int sin^n\ x\ dx$$

$$= -sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$\therefore \qquad I_n = \frac{-sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2} \qquad \dots (1)$$

which is valid for n ≥ 2.

**Example 1:** Integrate $(i)\int \sin^6 x\,dx \quad (ii)\int \sin^5 x\,dx$

**Solution:** *(i)* Let $I_6 = \int \sin^6 x\,dx$,

by equation (1), we have $I_n = \frac{-sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$

Putting n = 6, we get

$$I_6 = \frac{-sin^5 x \cos x}{6} + \frac{5}{6} I_4$$

Repeating the same for n = 4 and 2, we have

$$I_6 = \frac{-sin^5 x \cos x}{6} + \frac{5}{6}\left(\frac{-sin^3 x \cos x}{4} + \frac{3}{4} I_2\right),$$

$$I_6 = \frac{-sin^5 x \cos x}{6} + \frac{5}{6}\left(\frac{-sin^3 x \cos x}{4} + \frac{3}{4}\left(\frac{-sin\ x \cos x}{2} + \frac{1}{2}\int \sin^0 x\,dx\right)\right)$$

$$I_6 = \frac{-sin^5 x \cos x}{6} - \frac{5}{24} sin^3 x \cos x - \frac{5}{16} sin\ x \cos x + \frac{5}{16} x$$

*(ii)* Let

$$I_5 = \int \sin^5 x\,dx$$

proceeding the same as in part (i), we have

$$I_5 = \frac{-sin^4 x \cos x}{5} - \frac{4}{15} sin^2 x \cos x - \frac{8}{15} \cos x$$

| **Value Addition: Note** |
|---|
| Derivation of reduction formula is valid for all values of n except n = 1. For negative integer n = -m, $I_{-m} = \int \sin^{-m} x\,dx = \int \operatorname{cosec}^m x\,dx,$ which will be derived in next section. |

## 4.2: Reduction formula for $\int \operatorname{cosec}^n x\,dx$ n being a positive integer greater than unity

Let

$$I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\,dx$$

By using method of integration by parts, we get

$$I_n = \text{cosec}^{n-2} x \int \text{cosec}^2 x\,dx - \left( \int \left( \frac{d\left(\text{cosec}^{n-2} x\right)}{dx} \int \text{cosec}^2 x\,dx \right) dx \right)$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) \int \text{cosec}^{n-3} x \,\text{cosec}\, x \cot x \cot x\,dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) \int \text{cosec}^{n-2} x \left(\text{cosec}^2 x - 1\right) dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) \int \text{cosec}^{n} x\, dx + (n-2) \int \text{cosec}^{n-2} x\, dx$$

$$= -\text{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = -\text{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$$

$$\therefore \qquad I_n = \frac{-\text{cosec}^{n-2} x \cot x}{n-1} + \frac{(n-2)}{n-1} I_{n-2} \qquad \ldots (2)$$

which is valid for n > 2.

**Example 2:** Integrate $\int \text{cosec}^4 x\,dx$ .

**Solution:** On putting n = 4 in equation (2), we get

$$\int \text{cosec}^4 x\,dx = \frac{-\cot x\, \text{cosec}^2 x}{3} + \frac{2}{3} \int \text{cosec}^2 x\,dx$$

$$= \frac{-\cot x\, \text{cosec}^2 x}{3} - \frac{2}{3} \cot x$$

### 4.3: Reduction formula for $\int \cos^n x\,dx$, n being a positive integer greater than unity

Let

$$I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$$

using method of integration by parts, we get

$$I_n = cos^{n-1} x \int \cos x\,dx - \left( \int \left( \frac{d\left(\cos^{n-1} x\right)}{dx} \int \cos x\,dx \right) dx \right)$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin x \sin x\,dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \left(1 - \cos^2 x\right) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, dx - (n-1) \int \cos^{n} x\, dx$$

$$= \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$\therefore \quad I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2} \quad \ldots (3)$$

which is valid for n ≥ 2.

**Example 3:** Evaluate (i) $\int \cos^4 x \, dx$ (ii) $\int \cos^5 x \, dx$

**Solution:** *(i)* Putting n = 4 and 2 in equation (3), we get

$$\int \cos^4 x \, dx = \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \int \cos^2 x \, dx$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4}\left[\frac{\cos x \sin x}{2} + \frac{1}{2} x\right]$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3 \cos x \sin x}{8} + \frac{3}{8} x$$

*(ii)* putting *n* = *5* and *3* in equation (3), we get

$$\int \cos^5 x \, dx = \frac{\cos^4 x \sin x}{5} + \frac{4}{5} \int \cos^3 x \, dx$$

$$= \frac{\cos^4 x \sin x}{5} + \frac{4}{5}\left[\frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx\right]$$

$$= \frac{\cos^4 x \sin x}{5} + \frac{4 \cos^2 x \sin x}{15} + \frac{8}{15} \sin x$$

**4.4: Reduction formula for $\int \sec^n x \, dx$, n being a positive integer greater than unity**

Let

$$I_n = \int \sec^n x \, dx = \int sec^{n-2} x \; sec^2 x \, dx$$

using method of integration by parts, we get

$$I_n = \sec^{n-2} x \int sec^2 x \, dx - \left(\int \left(\frac{d\left(\sec^{n-2} x\right)}{dx} \int sec^2 x \, dx\right) dx\right)$$

$$= sec^{n-2} x \tan x - (n-2) \int sec^{n-3} x \sec x \tan x \tan x \, dx$$

$$= sec^{n-2} x \tan x - (n-2) \int sec^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= sec^{n-2} x \tan x - (n-2) \int sec^n \; x \; dx + (n-2) \int sec^{n-2} \; x \; dx$$

$$= sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\therefore \qquad I_n = \frac{sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{n-1} I_{n-2} \qquad \text{... (4)}$$

which is valid for n > 2.

**Example 4:** Integrate $\int \sec^6 x \, dx$

**Solution:** Putting n = 6, 4 and 2 in equation (4), we get

$$\int \sec^6 x \, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \int \sec^4 x \, dx$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \left[ \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \int \sec^2 x \, dx \right]$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4 \sec^2 x \tan x}{15} + \frac{8}{15} \tan x$$

### 4.5: Reduction formula for $\int \tan^n x \, dx$, n being a positive integer greater than unity

Let

$$I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \, \tan^2 x \, dx$$

$$= \int \tan^{n-2} x \left( \sec^2 x - 1 \right) dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx \qquad \text{...(5)}$$

| **Value Addition: Do you know ?** |
| --- |
| $\int \left( f(x) \right)^m f'(x) \, dx = \frac{\left[ f(x)^{m+1} \right]}{m+1} + k$ <br> Similarly, $\int \tan^{n-2} x \sec^2 x \, dx = \frac{\tan^{n-1} x}{n-1} + k$ |

Therefore, from equation (5), we have,

$$I_n = \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \qquad \text{...(6)},$$

which is the required reduction formula for $\int \tan^n x \, dx$ .

### 4.6: Reduction formula for $\int \cot^n x \, dx$, n being a positive integer greater than unity

Let

$$I_n = \int \cot^n x\, dx = \int \cot^{n-2} x\, \cot^2 x\, dx$$
$$= \int \cot^{n-2} x \left(\csc^2 x - 1\right) dx$$
$$= \int \cot^{n-2} x \csc^2 x\, dx - \int \cot^{n-2} x\, dx \qquad \text{...(7)}$$

As from the previous example, from equation (7), we get

$$I_n = \int \cot^n x\, dx = \frac{-\cot^{n-1} x}{n-1} - I_{n-2} \qquad \text{...(8)},$$

which is the required reduction formula for $\int \cot^n x\, dx$ .

**Example 5:** Evaluate (i) $\int \tan^8 x\, dx$, (ii) $\int \cot^5 x\, dx$

**Solution:** (i) Putting n = 8, 6 and 4 in the reduction formulae obtained in equation (6), we get

$$\int \tan^8 x\, dx = \frac{\tan^7 x}{7} - \int \tan^6 x\, dx = \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} - \int \tan^4 x\, dx$$
$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - \int \tan^2 x\, dx$$
$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - \int \left(\sec^2 x - 1\right) dx$$
$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - \tan x + x$$

(ii) Putting n = 5 and 3 in the reduction formulae obtained in equation (8), we get

$$\int \cot^5 x\, dx = \frac{-\cot^4 x}{4} - \int \cot^3 x\, dx = \frac{-\cot^4 x}{4} + \frac{\cot^2 x}{2} + \int \cot x\, dx$$
$$= \frac{-\cot^4 x}{4} + \frac{\cot^2 x}{2} + \log \sin x$$

### 4.7: Reduction formula for $\int (\log x)^n\, dx$

Let

$$I_n = \int (\log x)^n\, dx$$

By using method of integration by parts, we get,

$$I_n = \int (\log x)^n \, dx = (\log x)^n \int dx - \int \left( \frac{d(\log x)^n}{dx} \int dx \right) dx$$

$$= x(\log x)^n - n\int (\log x)^{n-1} \frac{1}{x} x \, dx = x(\log x)^n - n\, I_{n-1}$$

$$\therefore I_n = \int (\log x)^n \, dx = x(\log x)^n - n\, I_{n-1}, \qquad \ldots(9)$$

Which is the required reduction formula for $\int (\log x)^n \, dx$ .

**Example 6:** Write reduction formula for $\int x^m (\log x)^n \, dx$

**Solution:** On integrating by parts, we have

$$\int x^m (\log x)^n \, dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^{m+1} (\log x)^{n-1} \frac{1}{x} dx$$

$$\therefore \int x^m (\log x)^n \, dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1} \, dx$$

is the required reduction formula for $\int x^m (\log x)^n \, dx$ .

## 4.8: Reduction formula for $\int \sin^n x \cos^m x \, dx$

Let

$$I_{n,m} = \int \sin^n x \cos^m x \, dx$$

If m = 1, then

$$I_{n,1} = \int \sin^n x \cos x \, dx = \begin{cases} \dfrac{\sin^{n+1} x}{n+1} + k, \text{where } n \neq -1 \\ 1n|\sin x| + k, \text{where } n = -1 \end{cases}$$

So, we take m > 1, and we get,

$$I_{n,m} = \int \sin^n x \cos^m x \, dx = \int \cos^{m-1} x \left(\sin^n x \cos x\right) dx$$

By using method of integration by parts, we get,

$$I_{n,m} = \frac{\cos^{m-1} x \sin^{n+1} x}{n+1} - (m-1)\int \cos^{m-2} x(-\sin x)\frac{\sin^{n+1} x}{n+1}dx, when\, n \neq -1$$

$$= \frac{\cos^{m-1} x \sin^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^n x \cos^{m-2} x\left(1-\cos^2 x\right)dx,$$

$$= \frac{\cos^{m-1} x \sin^{n+1} x}{n+1} + \frac{m-1}{n+1}\left[I_{n,m-2} - I_{n,m}\right],$$

$$\therefore I_{n,m} + \frac{m-1}{n+1}I_{n,m} = \frac{n+m}{n+1}I_{n,m} = \frac{\cos^{m-1} x \sin^{n+1} x}{n+1} + \frac{m-1}{n+m}I_{n,m-2}$$

$$or,\ I_{n,m} = \frac{\cos^{m-1} x \sin^{n+1} x}{n+m} + \frac{m-1}{n+m}I_{n,m-2} \qquad ...(10)$$

**Value Addition: Cautions**

This formula is not valid when m+n = 0. So, when m+n=0, we take n = -m and get

$$I_{n,-n} = \int \sin^{-m} x \cos^m x\, dx = \int \cot^m x\, dx,$$

which we have already derived in section (4.6).

Similarly if we consider n > 1 instead of m >1, and proceeding in the same way, we can derive

$$I_{n,m} = \frac{-\cos^{m+1} x \sin^{n-1} x}{n+m} + \frac{n-1}{n+m}I_{n-2,m} \ if\ n+m \neq 0. \qquad ...(11)$$

**Example 7:** Evaluate

$$\int \sin^4 x \cos^2 x\, dx$$

**Solution:** Putting n = 4 and m = 2 in equation (11), we get

$$\int \sin^4 x \cos^2 x\, dx = \frac{-\cos^3 x \sin^3 x}{6} + \frac{3}{6}\int \sin^2 x \cos^2 x\, dx$$

$$= \frac{-\cos^3 x \sin^3 x}{6} + \frac{1}{2}\left\{\frac{-\cos^3 x \sin x}{4} + \frac{1}{4}\int \cos^2 x\, dx\right\}$$

$$= \frac{-\cos^3 x \sin^3 x}{6} - \frac{\cos^3 x \sin x}{8} + \frac{1}{8}\int \frac{1}{2}\left(1+\cos 2x\right) dx$$

$$= \frac{-\cos^3 x \sin^3 x}{6} - \frac{\cos^3 x \sin x}{8} + \frac{1}{16}x + \frac{1}{32}\sin 2x$$

$$= \frac{1}{16}x + \frac{1}{6}\cos x\left\{-\cos^2 x \sin^3 x - \frac{6}{8}\cos^2 x \sin x + \frac{6}{16}\sin x\right\}$$

$$= \frac{1}{16}x + \frac{1}{6}\cos x\left\{-\left(1-\sin^2 x\right)\sin^3 x - \frac{6}{8}\left(1-\sin^2 x\right)\sin x + \frac{6}{16}\sin x\right\}$$

$$= \frac{1}{16}x + \frac{1}{6}\cos x\left\{\sin^5 x + \sin^3 x\left(-1+\frac{3}{4}\right) + \sin x\left(-\frac{3}{4}+\frac{3}{8}\right)\right\}$$

$$= \frac{1}{16}x + \frac{1}{6}\cos x\left\{\sin^5 x - \frac{1}{4}\sin^3 x - \frac{3}{8}\sin x\right\}$$

**Example 8:** Write reduction formula for $\int x^m \cos nx\, dx$

**Solution:** On integrating by parts, we have

$$\int x^m \cos nx\, dx = x^m \frac{\sin nx}{n} - \frac{m}{n}\int x^{m-1}\sin nx\, dx$$

$$= x^m \frac{\sin nx}{n} - \frac{m}{n}\left[-x^{m-1}\frac{\cos nx}{n} + \frac{m-1}{n}\int x^{m-2}\cos nx\, dx\right]$$

$$= x^m \frac{\sin nx}{n} + \frac{m x^{m-1}\cos nx}{n^2} - \frac{m\left(m-1\right)}{n^2}\int x^{m-2}\cos nx\, dx$$

$$\int x^m \cos nx\, dx = \frac{n x^m \sin nx + m x^{m-1}\cos nx}{n^2} - \frac{m\left(m-1\right)}{n^2}\int x^{m-2}\cos nx\, dx$$

is the required reduction formula for $\int x^m \cos nx\, dx$ .

## 5: Determination of the value of definite integrals

### 5.1: Reduction formula for $\int_0^{\pi/2} \sin^n x\, dx$,

we have

$$\int_0^{\pi/2} \sin^n x\, dx = \left.\frac{-\sin^{n-1} x \cos x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx, n \ge 2.$$

by repeating this formula we have

$$\int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} dx, \text{when } n \text{ is even and } n \ge 2. \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2} \sin x\, dx, \text{when } n \text{ is odd and } n \ge 3. \end{cases}$$

$$or, \int_0^{\pi/2} \sin^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{when } n \text{ is even and } n \ge 2. \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}, \text{ when } n \text{ is odd and } n \ge 3. \end{cases}$$

**Example 9:** *If* $I_n = \int_0^{\pi/2} x^n \sin x\, dx$ *and* $n > 1$, *then prove that* $I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

**Solution:** On integrating $I_n$ by parts, $x^n$ as first function, we get

$$I_n = \left[x^n(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1}(-\cos x)\, dx$$

$$= 0 + n\left[\left\{\left(x^{n-1}\sin x\right)\right\}_0^{\pi/2} - \int_0^{\pi/2}(n-1)x^{n-2}\sin x\, dx\right] = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\, dx$$

$$\Rightarrow I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$$

$$\Rightarrow I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

## 5.2: Reduction formula for $\int_0^{\pi/2} \cos^n x\, dx,$

In the same way we can derive and get

$$\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, \textit{when n is even and } n\geq 2. \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}, \textit{ when n is odd and } n\geq 3. \end{cases}$$

**Example 10:** Evaluate (i) $\int_0^{\pi/2} \sin^9 x\,dx$, (ii) $\int_0^{\pi/2} \sin^{10} x\,dx$, (iii) $\int_0^{\pi/2} \cos^5 x\,dx$, (iv) $\int_0^{\pi/2} \cos^6 x\,dx$

Solution: (i) We have

$$\int_0^{\pi/2} \sin^9 x\,dx = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{128}{315}$$

(ii) we have

$$\int_0^{\pi/2} \sin^{10} x\,dx = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{63}{512}\pi$$

(iii) we have

$$\int_0^{\pi/2} \cos^5 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$$

(iv) we have

$$\int_0^{\pi/2} \cos^6 x\,dx = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{32}\pi$$

## 5.3: Reduction formula for $\int_0^{\pi/2} \sin^n x \cos^m x\,dx$, *where n and m are positive integers*

By equation (11), we have,

$$\int \sin^n x \cos^m x\,dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x \cos^m x\,dx$$

Taking the limits 0 to $\dfrac{\pi}{2}$ on both sides, we get,

$$\mathrm{I}_{n,m} = \int_0^{\pi/2} \sin^n x \cos^m x\,dx = \left|\frac{-\cos^{m+1} x \sin^{n-1} x}{n+m}\right|_0^{\pi/2} + \frac{n-1}{n+m}\int_0^{\pi/2} \sin^{n-2} x \cos^m x\,dx \qquad ...(12)$$

$$\Rightarrow \mathrm{I}_{n,m} = \frac{n-1}{n+m}\mathrm{I}_{n-2,m} \qquad ...(13)$$

interchanging n to n-2, n-4, n-6 and so on, we obtain

$$I_{n-2,m}=\frac{n-3}{m+n-2}I_{n-2,m}, I_{n-4,m}=\frac{n-5}{m+n-4}I_{n-6,m}, I_{n-6,m}=\frac{n-7}{n+m-6}I_{n-8,m}\ \ldots$$

$$I_{3,m}=\frac{2}{3+m}I_{1,m}, \text{ when } n \text{ is odd}, I_{2,m}=\frac{1}{2+m}I_{0,m}, \text{ when } n \text{ is even},$$

$$\text{Also}, I_{1,m}=\int_0^{\pi/2}\sin x\cos^m x\,dx=\frac{1}{m+1}, \text{ and } I_{0,m}=\int_0^{\pi/2}\sin^0 x\cos^m x\,dx=\int_0^{\pi/2}\cos^m x\,dx$$

*Substituting for* $I_{0,m}, I_{2,m}$, and $I_{1,m}, I_{3,m}$, *we get*,

$$I_{n,m}=\frac{n-1}{n+m}\cdot\frac{n-3}{n+m-2}\cdot\frac{n-5}{n+m-4}\cdots\frac{1}{2+m}\int_0^{\pi/2}\cos^m x\,dx, \text{ when } n \text{ is even}$$

$$\text{and } I_{n,m}=\frac{n-1}{n+m}\cdot\frac{n-3}{n+m-2}\cdot\frac{n-5}{n+m-4}\cdots\frac{2}{3+m}\cdot\frac{1}{m+1}, \text{ when } n \text{ is odd}$$

$$\Rightarrow I_{n,m}=\frac{n-1}{n+m}\cdot\frac{n-3}{n+m-2}\cdot\frac{n-5}{n+m-4}\cdots\frac{1}{2+m}\times\frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, \text{ when } n \text{ is even}, m \text{ is even}$$

$$\text{and } I_{n,m}=\frac{n-1}{n+m}\cdot\frac{n-3}{n+m-2}\cdot\frac{n-5}{n+m-4}\cdots\frac{1}{2+m}\times\frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdots\frac{2}{3}, \text{ when } n \text{ is even}, m \text{ is odd}$$

**Example 11:** Write down the value of (i) $\int_0^{\pi/2}\sin^4 x\cos^5 x\,dx$, (ii) $\int_0^{\pi/2}\sin^6 x\cos^8 x\,dx$

Solution: (i) $\int_0^{\pi/2}\sin^4 x\cos^5 x\,dx=\frac{3}{9}\cdot\frac{1}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}=\frac{8}{315}$,

(ii) $\int_0^{\pi/2}\sin^6 x\cos^8 x\,dx=\frac{5}{14}\cdot\frac{3}{12}\cdot\frac{1}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{5\pi}{4096}$,

## Summary:

Reduction formula is a formula which expresses a given integral in terms of another simple integral. In this chapter we have derived reduction formulae for the followings.

- $\int\sin^n x\,dx=\frac{-sin^{n-1}x\cos x}{n}+\frac{(n-1)}{n}\int\sin^{n-2}x\,dx$,
- $\int\cos^n x\,dx=\frac{\cos^{n-1}x\sin x}{n}+\frac{(n-1)}{n}\int\cos^{n-2}x\,dx$,
- $\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx$,

- $\int \csc^n x\,dx = \frac{-csc^{n-2} x \cot x}{n-1} + \frac{(n-2)}{n-1}\int \csc^{n-2} x\,dx$
- $\int \sec^n x\,dx = \frac{sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{n-1}\int \sec^{n-2} x\,dx,$
- $\int \cot^n x\,dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$
- $$\int \sin^n x \cos^m x\,dx = \frac{\cos^{m-1} x \sin^{n+1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x \cos^{m-2} x\,dx,\ \mathrm{m} > 1$$
  $$= \frac{-\cos^{m+1} x \sin^{n-1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x \cos^m x\,dx,\ n > 1$$
- $\int (\log x)^n\,dx = \int (\log x)^n\,dx = x(\log x)^n - n\int (\log x)^{n-1}\,dx,$
- $$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{when } n \text{ is even and } n \geq 2. \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}, \text{ when } n \text{ is odd and } n \geq 3. \end{cases}$$
- $$\int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, \text{when } n \text{ is even and } n \geq 2. \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}, \text{ when } n \text{ is odd and } n \geq 3. \end{cases}$$
- $$\int_0^{\pi/2} \sin^n x \cos^m x\,dx = \begin{cases} \frac{n-1}{n+m}\cdot\frac{n-3}{n+m-2}\cdot\frac{n-5}{n+m-4}\cdots\frac{1}{2+m} \times \frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, \\ \text{when } n \text{ is even, } m \text{ is even} \\ \frac{n-1}{n+m}\cdot\frac{n-3}{n+m-2}\cdot\frac{n-5}{n+m-4}\cdots\frac{1}{2+m} \times \frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdots\frac{2}{3}, \\ \text{when } n \text{ is even, } m \text{ is odd} \end{cases}$$

We have used method of integration by parts to derive reduction formulae for integrals. These are the formulas that express an integral involving a power of a function in terms of an integral that involves a lower power of that function. It has been also observed that by using this method we can derive many integrations related to algebraic and trigonometric functions.

**Exercises:**

(1) Evaluate :

$(a)\int \sin^5 x\,dx$ $\quad$ (b) $\int \cos^{10} x\,dx$ $\quad$ (c) $\int \tan^{11} x\,dx$

(d) $\int \csc^8 x\,dx$ $\quad$ (e) $\int \sec^5 x\,dx$ $\quad$ (f) $\int \cot^9 x\,dx$

(2) Show that

$$\int \sin^2 x\cos^6 x\,dx = \frac{5}{128}x + \frac{1}{8}\sin x\left\{-\cos^7 x + \frac{1}{6}\cos^5 x + \frac{5}{24}\cos^3 x + \frac{5}{16}\cos x\right\}$$

(3) Show that

$$\int \tan^2 x\sec^4 x\,dx = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x$$

(4) Evaluate :

$(a)\int_0^{\pi/2} \sin^{15} x\,dx$ $\quad$ (b) $\int_0^{\pi/2} \cos^{10} x\,dx$

(c) $\int_0^{\pi/2} \sin^4 x\cos^6 x\,dx$ $\quad$ (d) $\int_0^{\pi/2} \sin^6 x\cos^7 x\,dx$

(5) Evaluate

$(a)\int_0^{\pi/4} \tan^5 x\,dx$ $\quad$ (b) $\int_{-\pi}^{\pi} \cos^2 5x\,dx$

(c) $\int_{\pi/4}^{\pi/2} \csc^4 x\,dx$ $\quad$ (d) $\int_0^{\pi/4} \sec^6 x\,dx$

(6) Prove that

(a) $\int_0^1 x^2\left(1-x^2\right)^{3/2} dx = \frac{\pi}{32}$ $\quad$ (b) $\int_0^a x^4\sqrt{\left(a^2-x^2\right)}\,dx = \frac{\pi}{32}a^2$

(7) Prove that $\int x^m \sin nx\,dx = -\frac{x^m\cos nx}{n} + \frac{m x^{m-1}\sin nx}{n^2} - \frac{m(m-1)}{n^2}\int x^{m-2}\sin nx\,dx$

(8) Obtain a reduction formula for $\int \cos^m x\sin nx\,dx$

(9) Obtain a reduction formula for $\int \cos^m x\sin nx\,dx$

(10) $If\ I_n = \int_0^{\pi/2} x\sin^n x\ dx\ and\ n>1, then\ prove\ that\ I_n = \frac{n-1}{n}I_{n-2} + \frac{1}{n^2}$