Discipline Course - I

Mathematics

Paper: Analysis - I

Lesson: Infinite Series

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Infinite Series

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1. Learning Outcomes

After you have read this chapter, you should be able to

- Define the infinite series,
- Sequence of partial sums of an infinite series,
- Convergence and Divergence of an infinite series,
- Comparison Series (p - series).
- Cauchy's General Principle of convergence for series.
- Comparison Tests

2. Introduction :

The purpose of this chapter is to discuss sums that contain infinitely many terms. Our first objective is to define what is meant by the "Sum" of infinitely many real numbers. We begin with some terminology.

3. Infinite Series:

An expression of the form

$$u_1 + u_2 + u_3 + \text{----------} + u_n + \text{------}$$

where each u_n is a real number, is called an infinite series of real numbers. It is denoted by $\sum_{n=1}^{\infty} u_n$ *or* $\sum u_n$, where u_n is called nth term of the series $\sum u_n$.

3.1. Sequence of Partial sums of an infinite series:

Let $u_1 + u_2 + u_3 + \text{----------} + u_n + \text{------}$ be an infinite series. If S_n denotes the sum of the first n terms of this series so that

$$S_n = u_1 + u_2 + u_3 + \text{----------} + u_n$$

then $<S_n>$ is called the sequence of partial sums of the given series.

3.2. Convergence and Divergence of an infinite series:

Definitions:

(i) An infinite series $\sum u_n$ is said to be convergent, if the sequence $<S_n>$ of partial sums of $\sum u_n$ is convergent. If $\lim_{n\to\infty} S_n = S$, then S is called the sum of the series $\sum u_n$.

(ii) An infinite series $\sum u_n$ is said to be divergent, if the sequence $<S_n>$ of partial sums of $\sum u_n$ is divergent.

(iii) An infinite series $\sum u_n$ is said to oscillate, if the sequence $<S_n>$ of partial sums of $\sum u_n$ oscillates.

Example 1: Show that series $1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + --- + \left(\frac{1}{3}\right)^{n-1} + ---$ converges.

Solution : Here, $S_n = 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + ---- + \left(\frac{1}{3}\right)^{n-1}$

$$= \frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}} = \frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}}$$

$$= \frac{3}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^n$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{3}{2} - \frac{3}{2}\left(\frac{1}{3}\right)^n\right) = \frac{3}{2}$$

Since the sequence $<S_n>$ converges to $\frac{3}{2}$, therefore the given series is convergent.

Example 2: Prove that series $1 + 2 + 3 + ---- + n --$ diverges.

Solution : Here,

$$S_n = 1 + 2 + 3 + --- + n$$
$$= \frac{n(n+1)}{2}$$

$$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty$$

Hence $<S_n>$ diverges and consequently the given series diverges.

Example 3: Show that series 2 - 2 + 2 + ---- oscillates

Solution : Here,

$$S_n = \begin{cases} 0, & if\ \ n\ is\ even \\ 2, & if\ \ n\ is\ odd \end{cases}$$

Therefore, the sequence $<S_n>$ oscillates and consequently the given series oscillates.

Example 4 : Show that series 1 + r + r^2 + ------ (r > 0) converges if 0 < r < 1 and diverges if r ≥ 1.

Solution : Since r > 0, therefore, the series of positive terms

(i) if $0 < r < 1$, then

$$S_n = 1 + r + r^2 ----+ r^{n-1}$$

$$= \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

$$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-r} - \frac{1}{1-r}\left(\lim_{n\to\infty} r^n\right)$$

$$= \frac{1}{1-r} \left[\lim_{n\to\infty} r^n = 0, \ if \ |r|<1\right]$$

Thus <S_n> is a convergent sequence consequently the given series is convergent for 0<r<1

(ii) When r = 1, then

S_n = 1 + 1 + 1 + ----- + 1 = n

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} n \to \infty$$

Thus sequence <S_n> is divergent consequently the given series is divergent for r = 1.

(ii) When r > 1, then

S_n = 1 + r + r^2 + - - - + r^{n-1} > n

$$\Rightarrow \lim_{n\to\infty} S_n = \infty$$

Thus <S_n> is divergent consequently the given series is divergent when r > 1.

Value Addition: Remark

An infinite series of the form a + a.r + a.r^2 + ------, is called a geometric series with common ration r.

As an application of Example 4, we see that

(a) $\sum \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ---$ is convergent $\left(\text{since } r = \frac{1}{3} < 1\right)$

(b) The series $1+1+1+---$ is divergent $(\text{since } r = 1)$

(c) $\sum 4^n = 4 + 4^2 + 4^3 + ---$ is divergent $(\text{since } r = 4 > 1)$

Example 5 : Show that series $\sum_{n=1}^{\infty}(-1)^n n$ is not convergent.

Solution : We have

$$\sum_{n=1}^{\infty}(-1)^n.n = -1+2-3+4-5+6----$$

$$\Rightarrow \quad S_1=-1,\ S_2=-1+2=1,\quad S_3=-1+2-3=-2$$
$$S_4=-1+2-3+4=2,\quad S_5=-3---- \quad etc$$

$\Rightarrow \langle S_n\rangle = \langle -1,\ 1, -2,\ 2-3,------\rangle$ which is not bounded

Hence the sequence $<S_n>$ is not convergent consequently the given series is not convergent.

Example 6 : Show that series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+----$ is convergent

Solution : We have

$$S_n = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+----+\frac{1}{n(n+1)}$$
$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+----+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$= 1-\frac{1}{n+1}$$
$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1-\frac{1}{\infty} = 1$$

Therefore, the given series converges to 1.

3.3: Comparison Series (p - series)

Theorem 1 : The series $\sum \frac{1}{n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+---+\frac{1}{n^p}+---$ is convergent if p>1 and divergent if $p \le 1$.

Proof : Let S_n denote the sum of the first n terms of the given series.

Case I, when p>1

Now,

$$\frac{1}{1^p} = 1$$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{\left(2^n\right)^p} + \frac{1}{\left(2^n+1\right)^p} + ----+ \frac{1}{\left(2^{n+1}-1\right)^p} + \frac{2^{n+1}-2^n}{\left(2^n\right)^p} = \frac{1}{\left(2^n\right)^{p-1}} = \left(\frac{1}{2^{p-1}}\right)^n$$

Adding,

$$S_{2^{n+1}-1} \langle 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + ----+ \left(\frac{1}{2^{p-1}}\right)^n$$

$$= \frac{1-\left(\frac{1}{2^{p-1}}\right)^{n+1}}{1-\frac{1}{2^{p-1}}} = \frac{2^{p-1}\left[1-\left(\frac{1}{2^{p-1}}\right)^{n+1}\right]}{\frac{1}{2^{p-1}}-1} < \frac{2^{p-1}}{2^{p-1}-1}, \ \forall \ n$$

we know that when n is any positive integer

$$2^{n+1}-1 \rangle \ 2^n \rangle \ n$$

$$\therefore \ S_n \langle \ S_{2^n} \langle \ S_{2^{n+1}-1} \langle \ \frac{2^{p-1}}{2^{p-1}-1}$$

Since for a given p, $\frac{2^{p-1}}{2^{p-1}-1}$ is a fixed number.

Hence the sequence $<S_n>$ is bounded above and hence the given series converges for p>1

Case II when $p \le 1$

We know, if n is any positive integer and $p \le 1$, then

$$n^p \le n \Rightarrow \frac{1}{n^p} \ge 1/n$$

$$\Rightarrow 1+\frac{1}{2^p} \geq 1+\frac{1}{2} > 1/2$$

$$\Rightarrow \frac{1}{3^p}+\frac{1}{4^p} \geq \frac{1}{3}+\frac{1}{4} \geq \frac{2}{4}=\frac{1}{2}$$

$$\Rightarrow \frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p} \geq \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{4}{8}=\frac{1}{2}$$

$$\frac{1}{\left(2^{m-1}+1\right)}+\frac{1}{\left(2^{m-1}+2\right)^p}+---+\frac{1}{\left(2^m\right)^p} \geq \frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}---+\frac{1}{2^m}$$

$$> \frac{2^m-2^{m-1}}{2^m}=\frac{1}{2}$$

Adding ,

$$S_{2^m} > \frac{m}{2}$$

We shall show that <S_n> is not bounded above if G be any number, however large then there exist m∈N such that

$$\frac{m}{2} > G$$

Let $n > 2^m$

$$\therefore \quad S_n > S_{2^m} > G$$

Thus the sequence <S_n> of partial sums of the given series is not bounded above and hence the series diverges for p ≤ 1.

Thus the given series $\sum\frac{1}{n^p}$ converges if p > 1 and diverges if p ≤ 1.

Value addition : Note
The p-series is extensively used to examine the convergence and divergence of the large number of series. The students are strongly advised to remember the conditions for the convergence and divergence of the p-series.

Example 7:

1. $\Sigma \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ----$ *converges* $(\because p = 2 > 1)$

2. $\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + -------$ *diverges* $(\because p = 1)$

3. $\Sigma \frac{1}{n^{3/2}}$ *converges* $\left(\because p = 3/2 > 1\right)$

4. $\Sigma \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + -----$ *diverges* $\left(\because p = \frac{1}{2} < 1\right)$

3.4. A necessary condition for convergence:

Theorem 2 : A necessary condition for a series Σu_n to converge is that $u_n \to 0$ *as* $n \to \infty$.

Proof : Let $S_n = u_1 + u_2 + --- + u_n$

then $u_n = S_n - S_{n-1}$ __________________ (i)

If $S_n \to S$ *as* $n \to \infty$, then taking limits of both sides of (i), we obtain

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(S_n - S_{n-1}\right)$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

Value Addition: Remark

The students should note that the above condition is only necessary, not sufficient. There exist non-convergent series for which $u_n \to 0$ *as* $n \to \infty$, for example, consider the series

$$\Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ---- + \frac{1}{n} + -----$$

Hence $u_n = \frac{1}{n}$ which tends to zero as $n \to \infty$ but the series $\Sigma \frac{1}{n}$ is divergent.

Corollary : If $\lim_{n\to\infty} u_n \neq 0$, then the series Σu_n cannot converge.

Example 8: Show that the series $\sum \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + ---$ is not convergent.

Solution : We have

$$u_n = \frac{n}{n+1} = \frac{1}{1+1/n}$$

$$\Rightarrow \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1 \neq 0$$

Hence the given series is not convergent by above corollary.

Example 9 : Show that the series

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + ---- + \sqrt{\frac{n}{2(n+1)}} + -----$$

does not converge.

Solution : Here,

$$u_n = \left(\sqrt{\frac{n}{2(n+1)}}\right) = \frac{1}{\sqrt{2}} \times \sqrt{\frac{n}{n+1}}$$

$$= \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1+1/n}}$$

$$\Rightarrow \lim_{x\to\infty} u_n = \lim_{x\to\infty} \frac{1}{\sqrt{2}} \sqrt{\frac{1}{1+1/n}} = \frac{1}{\sqrt{2}} \neq 0$$

Hence the given series does not converge.

Example 10 : Test for convergence the series $\sum \cos\left(\frac{1}{n}\right)$

Solution : We have

$$u_n = \cos\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \cos\frac{1}{n}$$

$$= \cos 0 = 1 \neq 0$$

Hence the given series does not converge.

4. Cauchy's General Principle of convergence for series:

Theorem 3: A necessary and sufficient condition for a series $\sum u_n$ to converge is that to each $\in$>0, there exist a positive integer m, such that

$$\left|u_{m+1} + u_{m+2} + --- + u_n\right| < \varepsilon,\ for\ \ all\ \ n \geq m.$$

Proof : The series $\sum u_n$ converges iff the sequence <S_n> of its partial sums converges.

By Cauchy's general participle of convergence for sequences, a necessary and sufficient condition for the convergence of <S_n> is that given any $\in$>0 there exists a positive integer m such that.

$\left|s_n - S_m\right| < \varepsilon,$ whenever $n \geq m$

$$\Rightarrow \left|u_{m+1} + u_{m+2} + --- + u_n\right| < \varepsilon, whenever\ \ n \geq m.$$

Hence the theorem.

Example 11 : Show the series

$\sum\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ---- + \frac{1}{n} + -----$ does not converge.

Solution : Suppose that the given series converges. Then the sequence <S_n> of partial sums of the given series is convergent. By Cauchy's principle of convergence for sequences for $\varepsilon = 1/2$, there exist a positive integer m such that

$$\left|s_n - S_m\right| < \frac{1}{2}, \forall\ n \geq m.$$

$$\Rightarrow \quad \left|\left(1+\frac{1}{2}+\frac{1}{3}+----+\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}----+\frac{1}{n}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+----+\frac{1}{m}\right)\right| < \frac{1}{2}, \forall\ n \geq m$$

$$= \frac{1}{m+1}+\frac{1}{m+2}+---+\frac{1}{n} < \frac{1}{2}, \forall\ n \geq m$$

On taking n = 2m, we have

$$\frac{1}{m+1}+\frac{1}{m+2}+---+\frac{1}{2m} > \frac{1}{2m}+\frac{1}{2m}.\ .\ .\ +\frac{1}{2m} = m.\frac{1}{2m} = \frac{1}{2}$$

$$\therefore \quad \frac{1}{m+1}+\frac{1}{m+2}+---+\frac{1}{n} > \frac{1}{2} \qquad (\because n = 2m > m)$$

Thus we have a contradiction. Hence the given series does not converge.

Example 12 : Show the series

$$\sum\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+----+\frac{1}{n}+-----$$ does not converge.

Solution : Suppose that given series is convergent. Therefore the sequence $<S_n>$ of partial sums of the given series is convergent.

$\therefore$ By Cauchy's general principle of convergence for any $\varepsilon = \frac{1}{2}$, there exist a positive integer m such that

$$|s_n - S_m| < \frac{1}{2}, \forall\ n \geq m$$

$$\Rightarrow \quad \left|\left(1+\frac{1}{2}+----+\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}----+\frac{1}{n}\right) - \left(1+\frac{1}{2}+----+\frac{1}{m}\right)\right| < \frac{1}{2}, \forall\ n \geq m$$

$$\Rightarrow \quad \frac{1}{m+1}+\frac{1}{m+2}+---+\frac{1}{n} < \frac{1}{2}, \forall\ n \geq m \quad \_\_\_\_\_\_\_\_ (1)$$

on taking n = 2m, we have

$$\frac{1}{m+1}+\frac{1}{m+2}+---+\frac{1}{n}=\frac{1}{m+1}+\frac{1}{m+2}+...+\frac{1}{2m}$$

$$>\frac{1}{2m}+\frac{1}{2m}+-----+\frac{1}{2m}=m.\frac{1}{2m}=\frac{1}{2}$$

Hence

$$\frac{1}{m+1}+\frac{1}{m+2}+---+\frac{1}{n}>\frac{1}{2}\quad (n=2m>m) \_\_\_\_\_ (2)$$

Since (1) and (2) are contradictory statements, hence the given series does not converge.

5. Comparison Tests:

5.1. First Comparison Test : Let $\sum u_n$ and $\sum v_n$ be two positive terms series such that

$$u_n \leq kv_n, \forall n \geq m \quad ------ (1)$$

Where k is fixed positive number and m is a fixed positive integer then

(i) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges,

(ii) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges

Proof : Let $\langle S_n \rangle$ and $\langle T_n \rangle$ be the sequences of partial sums of the series $\sum u_n$ and $\sum v_n$ respectively for $n \geq m,$ we have

$$S_n - S_m = \left(u_1 + u_2 + --- + u_m + u_{m+1} + u_{m+2} + --- + u_n\right) - \left(u_1 + u_2 + --- + u_m\right)$$

$$\Rightarrow \quad S_n - S_m = u_{m+1} + u_{m+2} + --- + u_n \quad ----- (2)$$

$$Similarly, \; T_n - T_m = v_{m+1} + v_{m+2} + --- + v_n \quad ----- (3)$$

From equation (1) and (2), we have

$$S_n - S_m \leq k\left(v_{m+1} + v_{m+2} + --- + v_n\right)$$

$$\Rightarrow \quad S_n - S_m \leq k\left(T_n - T_m\right)$$

$\Rightarrow \quad S_n - S_m \le kT_n + a \text{ -------- } (4)$

where $a = S_m - kT_m$ *is a fixed number*

(i) Suppose $\sum v_n$ converges

By theorem, the sequence $\langle T_n \rangle$ of partial sums of $\sum v_n$ is bounded above i.e. there exists a positive real number t such that

$$T_n \le t, \forall n \text{ -------- } (5)$$

From equations (4) and (5), we have

$$S_n \le kt, + a, \forall n$$

Thus the sequence $\langle S_n \rangle$ of partial sums of $\sum u_n$ is bounded above. Hence $\sum u_n$ is convergent.

(ii) Suppose $\sum u_n$ diverges, then

$$\lim_{x \to \infty} S_n = +\infty \quad \text{------ } (6)$$

From equations (4) and (6), we have

$$T_n \ge \frac{1}{k}(s_n - a)$$

$\Rightarrow \quad \lim_{x \to \infty} T_n = +\infty \quad (\because k > 0)$

$\Rightarrow$ The sequence $\langle T_n \rangle$ of partial sums of the series $\sum v_n$ diverges.

Hence $\sum v_n$ is divergent

Example 13 : Show the series $\sum e^{-n^2}$ is convergent.

Solution : We know that

$$\frac{1}{e^{n^2}} < \frac{1}{n^2}$$

$$\Rightarrow \quad e^{-n^2} < \frac{1}{n^2}, \ \forall n$$

But $\sum \frac{1}{n^2}$ is convergent since p = 2 > 1

Hence by first comparison Test $\sum e^{-n^2}$ is also convergent.

Example 14 : Show the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is divergent.

Solution : We know that $\log n < n, \ \forall \ n \geq 2$

$$\Rightarrow \quad \frac{1}{\log n} > \frac{1}{n}, \ \forall n \geq 2$$

$$\Rightarrow \quad \frac{1}{n} < \frac{1}{\log n}, \ \forall n \geq 2$$

But $\sum \frac{1}{n}$ is divergent since p=1 consequently by First comparison Test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is also divergent.

Example 15 : Test for convergence the series

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + --- + \frac{1}{n^n} + ----$$

Solution : We know that

$$n^n > 2^n, \ \forall n > 2$$

$$\Rightarrow \quad \frac{1}{n^n} < \frac{1}{2^n}, \ \forall n > 2$$

Since $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ----$ is a geometric series with common ratio

$$r = \frac{1}{2} < 1$$

Since $\sum \frac{1}{2^n}$ is convergent. Therefore by first comparison test, $\sum \frac{1}{n^n}$ is also convergent.

5.2. Second Comparison Test: Let $\sum u_n$ and $\sum v_n$ are two positive term series such that $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}, \ \forall n \geq m \ ----- (1)$

then (i) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges

(ii) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges

Proof : For $n \geq m,$ we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \ ---- \ \frac{u_{n-1}}{u_n}$$

$$\geq \frac{v_m}{v_{m+1}} \cdot \frac{v_{m+1}}{v_{m+2}} \cdot \frac{v_{m+2}}{v_{m+3}} \ ---- \ \frac{v_{n-1}}{v_n}, \ by \ (1)$$

$$\geq \frac{v_m}{v_n}$$

Hence $\frac{u_m}{u_n} \geq \frac{v_m}{v_n} \ or \ \frac{u_n}{u_m} \leq \frac{v_n}{v_m}, \ \forall n \geq m$

$\Rightarrow \ u_n \leq k v_n, \ \forall n \geq m$

where $k = \frac{u_m}{v_m}$ is a fixed positive number ------- (2)

Applying First comparison Test in (2), we obtain

(i) $\sum v_n$ converges $\Rightarrow \sum u_n$ converges,

(ii) $\sum u_n$ diverges $\Rightarrow \sum v_n$ diverges

5.3. Comparison Test (Limit form):

Let $\sum u_n$ and $\sum v_n$ be two positive term series such that $\lim\limits_{x\to\infty}\frac{u_n}{v_n}=\ell$, ($\ell$ is non zero finite number) then $\sum u_n$ and $\sum v_n$ both converge or diverge together.

Proof : Since $\frac{u_n}{v_n}>0,\ \forall n$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{u_n}{v_n}\geq 0$$

Thus $\ell>0,\ as\ \ell\neq 0$

Let $\varepsilon > 0$ be any number such that $\ell - \varepsilon > 0$

since $\lim\limits_{x\to\infty}\frac{u_n}{v_n}=\ell$, so there exist a positive integer m such that

$$\left|\frac{u_n}{v_n}-\ell\right|<\varepsilon,\ \forall n\geq m$$

$$\Rightarrow \quad \ell-\varepsilon<\frac{u_n}{v_n}<\ell+\varepsilon,\ \forall n\geq m,\ where\ v_n>0, \forall n$$

$$\Rightarrow \quad (\ell-\varepsilon)\,v_n<u_n<(\ell+\varepsilon)\,v_n\ \ \forall n\geq m\ \ -----(1)$$

From equation (1), we consider

$$u_n<(\ell+\varepsilon)\,v_n\ \ \forall n\geq m\ \ -----(2)$$

Where $\ell + \varepsilon$ is a fixed positive number

Applying first comparison test in (2), we obtain

$$\left.\begin{array}{l}\sum v_n\ converges\ \Rightarrow\ \sum u_n\ converges\\ \sum u_n\ diverges\ \ \Rightarrow\ \sum v_n\ diverges\end{array}\right\}---\ (A)$$

From equation (1), we consider

$$(\ell-\varepsilon)\,v_n<u_n\quad \forall n\geq m$$

$$v_n<k\,u_n\quad \forall n\geq m\quad ....\ (3)$$

where $k = \frac{1}{(\ell - \varepsilon)} > 0$ is a fixed number.

Applying first comparison test in (3), we get

$$\left.\begin{array}{l} \sum u_n \text{ converges } \Rightarrow \sum v_n \text{ converges} \\ \sum v_n \text{ diverges } \Rightarrow \sum u_n \text{ diverges} \end{array}\right\} --- (B)$$

From (A) and (B), we have

$$\left.\begin{array}{l} \sum u_n \text{ converges } \Leftrightarrow \sum v_n \text{ converges} \\ \sum u_n \text{ diverges } \Leftrightarrow \sum v_n \text{ diverges} \end{array}\right\}$$

Hence, the two series $\sum u_n$ and $\sum v_n$ converge or diverge together.

Value Addition
If $\ell = 0 \, or \, \ell = \infty$, then the conclusion of the above test may not hold good.
(1) Let $\sum u_n = \sum \frac{1}{n^2}$ and $\sum v_n = \sum \frac{1}{n}$, then $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$, Thus $\ell = 0$, but in this case $\sum u_n$ converges and $\sum v_n$ diverges.
(II) Let $\sum u_n = \sum \frac{1}{n}$ and $\sum v_n = \sum \frac{1}{n^2}$, then $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty$, Thus $\ell = \infty$, but in this case $\sum u_n$ diverges and $\sum v_n$ converges.

Example 16: Test for convergence the series $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11} + \ldots$

Solution: Here, $u_n = \frac{\sqrt{n}}{2n+3} > 0, \quad \forall \, n$

for large values of n, u_n behaves as $\frac{\sqrt{n}}{2n}$ *i.e. as* $\frac{1}{2\sqrt{n}}$.

Let $\quad v_n = \frac{1}{\sqrt{n}}$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n} \times \sqrt{n}}{(2n+3) \times 1} = \lim_{n\to\infty} \frac{n}{(2n+3)} = \frac{1}{2}$$

since $\lim\limits_{n\to\infty}\frac{u_n}{v_n}$ is non-zero finite number. Therefore, the series $\sum u_n$ and $\sum v_n$ converge and diverse together. But the series $\sum\frac{1}{\sqrt{n}}$ diverges. Hence the given series also diverges.

Example 17: Test for convergence the series $\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+\frac{1}{6.13}+...$

Solution: We have

$$u_n=\frac{1}{(n+2)(2n+5)}=\frac{1}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$$

Let $v_n=\frac{1}{n^2}$, then

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}=\frac{1}{2}$$

since $\lim\limits_{n\to\infty}\frac{u_n}{v_n}\neq 0$ is a non-zero finite number, therefore, the series $\sum u_n$ and $\sum v_n$ converge and diverse together. But the series $\sum v_n=\sum\frac{1}{n^2}$ converges. Hence the given series also converges.

Example 18: Test the convergence of the series whose n^{th} term is $\sqrt{n^4+1}-n^2$.

Solution: Here, $u_n=\sqrt{n^4+1}-n^2>0, \quad \forall\, n$

therefore, the given series is of positive terms

Also,

$$u_n = \sqrt{n^4+1} - n^2$$
$$= n^2\left(1+\frac{1}{n^4}\right)^{1/2} - n^2$$
$$= n^2\left(1+\frac{1}{2n^4}+\ldots\right) - n^2$$
$$= \frac{1}{2n^2}+\ldots$$

for sufficiently large values of n, u_n behaves as $\frac{1}{2n^2}$. Therefore,

Let $v_n = \frac{1}{n^2}$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{2}$$

since $\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{2}$, which is non-zero finite number, therefore, by comparison test series $\sum u_n$ and $\sum v_n$ converge or diverse together. But the series $\sum v_n$ converges. Hence the given series also converges.

Example 19: Show that the series $\sum \frac{1}{\sqrt{n}}\tan\frac{1}{n}$ is convergent.

Solution: Here, $u_n = \frac{1}{\sqrt{n}}\tan\frac{1}{n}$

Let $v_n = \frac{1}{n\sqrt{n}}$

$$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$$

since $\lim_{n\to\infty}\frac{u_n}{v_n} = 1$, which is non-zero finite number, therefore, by comparison test series $\sum u_n$ and $\sum v_n$ converge or diverse together. But the series $\sum v_n$ converges. Hence the given series also converges.

Example 20: Test for convergence the series $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+...$

Solution: We have

$$u_n = \frac{2n-1}{\text{n}(n+1)(n+2)}$$

for large values on n, $u_n \approx \frac{2}{n^2}$

Let $v_n = \frac{1}{n^2}$, then

$$\frac{u_n}{v_n} = \frac{(2n-1)n^2}{n(n+1)(n+2)} = \left(\frac{n}{n+1}\right)\left(\frac{2n-1}{n+2}\right)$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)\left(\frac{2-\frac{1}{n}}{1+\frac{2}{n}}\right) = 2$$

since $\lim_{n\to\infty}\frac{u_n}{v_n} = 2 \neq 0$ is a non-zero finite number, therefore, by the comparison test (Limit form), the series $\sum u_n$ and $\sum v_n$ converge or diverse together. Since the series $\sum v_n = \sum \frac{1}{n^2}$ converges. Hence the given series also converges.

Exercise:

Test each of the series 1- 10 for convergence

1. $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+...$

2. $\frac{1}{1.3.5}+\frac{1}{2.4.6}+\frac{1}{3.5.7}+...$

3. $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ $\left[\text{Hint } u_n = \frac{1}{2n-1},\ v_n = \frac{1}{n}\right]$

4. $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$ $\left[\text{Hint: } v_n = \frac{1}{n}\right]$

5. $\frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \ldots$

6. $\frac{1}{1+2} + \frac{1}{1+2^2} + \frac{1}{1+2^3} + \ldots$

7. $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$

8. $\frac{1}{4.6} + \frac{\sqrt{3}}{6.8} + \frac{\sqrt{5}}{8.10} + \frac{\sqrt{7}}{10.12} + \ldots$

9. $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \frac{1}{4.5^2} + \ldots$

10. $\sum_{n=1}^{\infty} \frac{1}{2^n + x}$ for all positive values of x.

11. Test for convergence the series whose n^{th} term is

    (a) $\sqrt{n+1} - \sqrt{n}$

    (b) $\sqrt{n^3+1} - \sqrt{n^3}$

    (c) $\sqrt{n^4+1} - \sqrt{n^4-1}$

    (d) $\sqrt{n^2+1} - n$

12. Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$ converges.

Summary:

In this lesson we have emphasize on the followings:

- Define the infinite series,
- Sequence of partial sums of an infinite series,

Infinite Series

- Convergence and Divergence of an infinite series,
- Comparison Series (p - series).
- Cauchy's General Principle of convergence for series.
- Comparison Tests

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