Discipline Course - I

Mathematics

Paper: Analysis - I

Lesson: Subsequences and Bolzano- Weierstrass Theorem

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Subsequences and Bolzano-Weierstrass Theorem

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Subsequences and Bolzano-Weierstrass Theorem

1. Learning Outcomes

After reading this chapter, you will be able to

- Define a subsequence
- Find whether or not a given sequence is a subsequence of another given sequence
- Construct a subsequence having a certain property
- Define a Cauchy sequence
- Decide whether or not a given sequence is convergent or divergent by applying either Divergence criterion or Cauchy convergence criterion.

2. Introduction

In this chapter we will consider sequences of real numbers, Subsequences, Cauchy sequences and prove few important results about them. The convergence and divergence of a sequence in terms of its subsequences is presented. There are some deeper results concerning the convergence of a sequence. These include The Monotone subsequence Theorem, The Bolzano-Weierstrass Theorem and the Cauchy Criterion for convergence of sequences. There are exercises after each section. The purpose of exercises is to provide practice working with the definitions.

3. Subsequences

3.1. Definition : Let $\langle x_n \rangle_{n=1}^{\infty}$ be any given sequence of real numbers and let $n_1 < n_2 < n_3 < \ldots < n_k < \ldots.$ be a strictly increasing sequence of natural numbers then the sequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$ given by $\langle x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots \rangle$ is called a subsequence of $\langle x_n \rangle$.

In other words for any sequence $\langle x_n \rangle$ if its terms are picked in any way, **but preserving the original order** we get a subsequence. In order to define a subsequence all its terms (or elements) must come from the original sequence and index of the terms of the subsequence must be strictly increasing.

Example 1 : If $\langle s_1, s_2, \ldots, s_n, \ldots \rangle$ be a given sequence then

(a) $\langle s_1, s_5, s_9, s_{10}, \ldots \rangle$ is a subsequence of $\langle s_n \rangle$.

(b) $\langle s_2, s_4, s_6, \ldots . s_{2n}, \ldots \rangle$ is a subsequence of $\langle s_n \rangle$.

(c) $\langle s_5, s_{15}, s_{20}, s_{23} \ldots . \rangle$ is a subsequence of $\langle s_n \rangle$.

(d) But $\langle s_3, s_1, s_5, s_{13}, s_8 \ldots . \rangle$ is not a subsequence of $\langle s_n \rangle$ because here the index sequence is not strictly increasing. This is so because here order of the terms is not preserved.

I.Q. 1.

I.Q. 2.

Example 2 : Consider the sequence $\left\langle \frac{1}{n} \right\rangle$, then

(a) $\left\langle 1, \frac{1}{3}, \frac{1}{5}, \ldots . \frac{1}{2k-1}, \ldots \right\rangle$ is a subsequence of $\langle x_n \rangle$.

(b) $\left\langle \frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \ldots ., \frac{1}{(2k)!}, \ldots \right\rangle$ is a subsequence of $\langle x_n \rangle$.

But

(c) $\left\langle \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \ldots \right\rangle$ is not a subsequence of $\left\langle \frac{1}{n} \right\rangle$ because here order of the terms is not preserved.

(d) $\left\langle 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\right\rangle$ is not a subsequence of $\left\langle \frac{1}{n} \right\rangle$ because here '0' does not belong to the given sequence $\langle x_n \rangle = \left\langle \frac{1}{n} \right\rangle$.

Example 3 : Consider $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$. This sequence is given by

$\langle 1, 0, -1, 0, 1, 0, -1, 0, \ldots\rangle$.

Here

(a) $\langle x_{2k} \rangle_{k=1}^{\infty} = \langle 0, 0, 0, 0, 0, \ldots\rangle$ is a subsequence.

(b) $\langle x_{2k-1} \rangle_{k=1}^{\infty} = \langle 1, -1, 1, -1, \ldots\rangle$ is a subsequence

(c) $\langle x_{4k+1} \rangle_{k=1}^{\infty} = \langle 1, 1, 1, 1, 1, \ldots\rangle$ is a subsequence

Example 4. : Construct a subsequence of $\langle x_n \rangle_{n=1}^{\infty}$ where $x_n = \sin\left(\sqrt{n}\,\frac{\pi}{2}\right)$ with all terms equal to 1.

Solution: We put $\sin\left(\sqrt{n}\,\frac{\pi}{2}\right) = 1$.

$\Rightarrow \quad \frac{\pi}{2}(\sqrt{n}) = \frac{\pi}{2} + 2k\pi$

$\Rightarrow \quad \sqrt{n} = 1 + 4k$

$\Rightarrow \quad n = (1 + 4k)^2$

Therefore the required subsequence is $\left\langle x_{(1+4k)^2} \right\rangle_{k=1}^{\infty}$ and is given by $\langle 1, 1, 1, 1, 1, \ldots\rangle$.

I.Q. 3.

Value Addition: Note

Every sequence is a subsequence of itself.

3.2. Definition: Tail of a sequence : If $\langle x_n \rangle$ is a real sequence and if m is a given natural number then m-tail of $\langle x_n \rangle$ is the sequence

$$X_m = \langle x_{m+1}, x_{m+1} ..... \rangle$$

for eg. The 3-tail of the sequence $\langle 2,4,6,8,10,12.... \rangle$ is the sequence

$$X_3 = \langle 8,10,12,...2n... \rangle .$$

Value addition : Remark
From definition it is very much obvious that tail of a sequence is always a subsequence. But not every subsequence of a given sequence need to be a tail of the sequence.

I.Q. 4.

Lemma 1: If $\langle x_k \rangle$ is a strictly increasing sequence of natural numbers then for all $k \in N$, $n_k \geq k$.

Proof: Clearly $n_1 \geq 1$ because no term can have index smaller than 1. Suppose $n_k \geq k$ then as $n_{k+1} > n_k$ and these are natural numbers, it follows that $n_{k+1} \geq n_k + 1$ or $n_{k+1} \geq k + 1$ therefore by induction $n_k \geq k \;\; \forall \;\; k \in N$.

A subsequence is a sequence in its own right and therefore we can speak of convergent and divergent subsequences and also it is natural to ask whether every sequence has any convergent subsequence?

Example 5 : The sequence $\langle x_n \rangle$ with $x_n = n$ has no convergent subsequence for if $\langle x_{n_k} \rangle$ is a subsequence then

$x_{n_k} = n_k$ and from above Lemma we know that

$n_k \geq k \ \forall \ k \in N$

$\Rightarrow \quad x_{n_k} \geq k$

$\Rightarrow \quad \langle x_{n_k} \rangle$ is not bounded. Therefore $\langle x_{n_k} \rangle$ cannot converge.

The above example shows that not all sequences have convergent subsequences. Is there any condition to have a convergent subsequence?

I.Q. 5.

Theorem 1: If $\langle x_n \rangle$ converge to a real number '*l*' then every subsequence of $\langle x_n \rangle$ also converge to '*l*'.

Proof. Let $\langle x_{n_k} \rangle$ be a subsequence of $\langle x_n \rangle$. Let $\in$ > 0 be given. Then there exist a natural number n_0 such that

$|x_n - l| < \in$ for all $n \geq n_0$

Consider any $k \geq n_0$. Since $n_1 < n_2 < n_3 .... < n_k ...$ is an increasing sequence of natural numbers, from Lemma we get

$n_k \geq k \geq n_0$

so that $\quad |x_{n_k} - l| < \in$

Therefore $\{x_{n_k}\}$ also converge to '*l*' as required.

Value Addition: Note
The converse of the theorem is trivially true. Suppose every subsequence of $\langle x_n \rangle$ converges to the same number say '*l*'. Then, since every sequence is a subsequence of itself, the sequence $\langle x_n \rangle$ must converge to '*l*'. Certainly a sequence need not be convergent to have convergent subsequences, for e.g. the sequence $\langle x_n \rangle$ where $x_n = (-1)^n$. Here the subsequence $\langle x_{2n} \rangle = \langle 1, 1, 1, ... \rangle$ converges to 1 and the subsequence

$\langle x_{2n-1} \rangle = \langle -1, -1, -1, ... \rangle$ converges to –1.

Example 6: Show that $\lim_{n \to \infty} b^n = 0$ if 0 < b < 1.

Solution : Let $x_n = b^n$

Since 0 < b < 1

$\Rightarrow \quad x_{n+1} = b^{n+1} < b^n = x_n$.

$\Rightarrow \quad \langle x_n \rangle$ is a monotonically decreasing sequence.

Also $0 < x_n < 1$, therefore $\langle x_n \rangle$ is a bounded sequence.

From Monotone Convergence Theorem, it follows that $\langle x_n \rangle$ is convergent.

Let $\lim_{n \to \infty} x_n = x$. Let $\langle x_{2n} \rangle$ be a subsequence of $\langle x_n \rangle$. Then we must have

$$\lim_{n \to \infty} x_{2n} = x$$

Now $x_{2n} = b^{2n} = (b^n)^2 = x_n^2$

Therefore $\lim_{n \to \infty} x_{2n} = \lim_{n \to \infty} (x_n)^2 = \lim_{n \to \infty} x_n . \lim_{n \to \infty} x_n = x^2$

or $\quad x = \lim_{n \to \infty} x_{2n} = x^2$

this implies x = 0 or 1. Since $\langle x_n \rangle$ is monotonically decreasing and bounded above by 1 we deduce that

$$x = 0$$

that is $\lim_{n \to \infty} x_{2n} = 0$

$\Rightarrow \quad \lim_{n \to \infty} x_n = 0$.

Example 7. Show that $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = e^{1/2}$.

Solution: The first few terms of the sequence are

$$\left(1+\frac{1}{2}\right)^1, \left(1+\frac{1}{4}\right)^2, \left(1+\frac{1}{6}\right)^3, \ldots$$

We can rewrite these terms as

$$\sqrt{\left(1+\frac{1}{2}\right)^2}, \sqrt{\left(1+\frac{1}{4}\right)^4}, \sqrt{\left(1+\frac{1}{6}\right)^6}, \ldots$$

Thus $\left\{\left(1+\frac{1}{2n}\right)^n\right\}$ is a subsequence of $\sqrt{\left(1+\frac{1}{n}\right)^n}$

$$\text{Therefore} \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty}\sqrt{\left(1+\frac{1}{n}\right)^n}$$

$$= \sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

$$= e^{1/2}$$

The following result is based on negation of the definition of $\lim_{n\to\infty} x_n = x$. It provides a convenient way to establish the divergence of a sequence.

I.Q. 6.

Theorem 2: Let $\langle x_n \rangle$ be a real sequence. Then the following are equivalent

(i) The sequence $\langle x_n \rangle$ does not converge to x $\in$ R.

(ii) There exists an $\in$ >0 such that for any k $\in$ N, there exist $n_k \in N$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \in$.

(iii) There exist an $\in > 0$ and a subsequence $\langle x_{n_k} \rangle$ such that $| x_{n_k} - x | \geq \in_0 \ \forall k \in N$.

3.3. Divergence criterion: If a sequence $\langle x_n \rangle$ of real numbers has either of the following properties then $\langle x_n \rangle$ is divergent.

(i) $\langle x_n \rangle$ has two convergent subsequences, converging to different limits.

(i) $\langle x_n \rangle$ is unbounded.

I.Q. 7.

Example 8: (a) The sequence $\langle x_n \rangle$ where $x_n = (-1)^n$ is divergent. Here the subsequence $\langle x_{2n} \rangle$ given by $\langle 1, 1, 1, ... \rangle$ converges to 1 and the subsequence $\langle x_{2n-1} \rangle$ given by $\langle -1, -1, -1, ... \rangle$ converges to –1. Therefore $\langle x_n \rangle$ is divergent.

(b) The sequence $\langle a_n \rangle$ where $a_n = \dfrac{1 + (-1)^n}{2}$ is divergent. Here the subsequence $\langle a_{2n-1} \rangle$ given by $\langle 0, 0, 0, 0, ... \rangle$ converges to 0 and subsequence $\langle a_{2n} \rangle$ given by $\langle 1, 1, 1, ... \rangle$ converges to 1. Therefore $\langle a_n \rangle$ is divergent.

(3) The sequence $\left\langle 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, ... \right\rangle$ is unbounded. Therefore it is divergent.

Value Addition :
We know that not every sequence is a monotone sequence. An interesting property of real sequences is the fact that every real sequence has a monotone subsequence.

3.4. Monotone subsequence Theorem:

Subsequences and Bolzano-Weierstrass Theorem

Theorem 3: If $\langle x_n \rangle$ is a real sequence then there is a subsequence of $\langle x_n \rangle$ that is monotone.

Proof: We say that the nth term is dominant if it is greater than every term which follows it

i.e. $s_m < s_n \qquad \forall m > n$ (1)

Case 1. Suppose that there are infinitely many dominant terms. Let $\langle s_{n_k} \rangle$ be any subsequence consisting of only dominating terms.

Then $s_{n_{k+1}} < s_{n_k} \qquad \forall k$ by (1)

Therefore $\langle s_{n_k} \rangle$ is a decreasing sequence.

Case 2. Suppose that there are only finitely many dominant terms. Select n_1 so that s_{n_1} is beyond all the dominant terms of the sequence. Then

given $N \geq n_1$ there exists $m > N$ such that $s_m \geq s_n$ (2)

Applying (2) with $N = n_1$ we select $n_2 > n_1$ such that $s_{n_2} > s_{n_1}$.

Suppose that $n_1, n_2, \ldots, n_k$ have been selected so that

$n_1 < n_2 < \ldots < n_k$ (3)

and $s_{n_1} \leq s_{n_2} \leq \ldots \leq s_{n_k}$ (4)

Applying (2) with $N = n_k$, we select $n_{k+1} > n_k$ such that $s_{n_{k+1}} \geq s_{n_k}$. Then (3) and (4) hold with $k+1$ in place of *k*. Continuing in this manner we obtain a increasing subsequence $\langle s_{n_k} \rangle$.

Since bounded monotone sequences converge, a monotone subsequence of a bounded sequence must converge. This is a very important property of sequences and is used in the proof of the theorem known as Bolzano-Weierstrass Theorem.

Example 9: Consider the sequence $\{s_n\}$ where $s_n = \cos\left(\frac{n\pi}{3}\right)$ given by $\left\langle \frac{1}{2}, \frac{-1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, \frac{-1}{2}, \frac{1}{2}, 1, \ldots \right\rangle$. Here $\langle s_n \rangle$ is not monotone where as the subsequence $\left\langle \cos\frac{\pi}{3} \right\rangle$ given by $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots. \right\rangle$ is monotone.

Example 10: Consider $\{x_n\}$ where $x_n = (-1)^n + \frac{1}{n}$ given by $\left\langle -1+1, 1+\frac{1}{2}, -1+\frac{1}{3}, 1+\frac{1}{4}, \ldots \right\rangle$.

Here $\langle x_n \rangle$ is not monotone whereas the subsequence $\langle x_{2n} \rangle$ given by $\left\langle 1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{6}, \ldots. \right\rangle$ is monotone.

Value Addition :
Certainly a sequence need not be convergent to have a convergent subsequence as we have seen with $x_n = (-1)^n$.

In fact a famous Theorem of Bolzano and Weierstrass provides a very elementary sufficient condition for the existence of a convergent subsequence. Before we consider the Bolzano-Weierstrass Theorem we state Nested Interval Theorem. This result is extremely useful in many applications.

3.5. Definition: Nested Intervals: A sequence of intervals. In, $n \in \mathbb{N}$ is said to be nested if the following chain of inclusions holds

$$I_1 \supseteq I_{21} \supseteq I_{31} \supseteq \ldots \supseteq I_n \supseteq I_{n+11} \supseteq \ldots$$

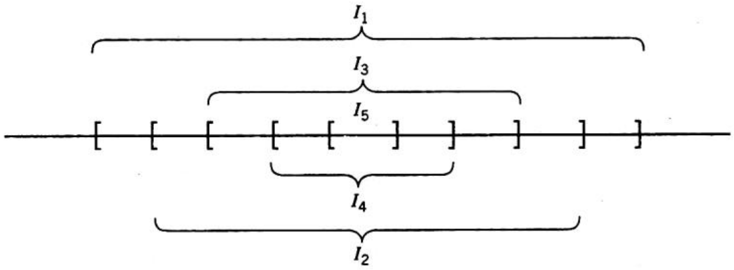

Fig. : Nested Intervals

3.6. Nested Interval Property : If $I_n = [a_n, b_n] \quad n \in \mathbb{N}$ is a nested sequence of closed and bounded intervals then there exist $\xi \in \mathbb{R}$ such that $\xi \in I_n \;\; \forall\, n \in \mathbb{N}$.

3.7. Nested Interval Theorem : If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed and bounded intervals such that the lengths $(b_n - a_n)$ of I_n satisfy.

$$\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$$

then the real number $\xi \in I_n \;\; \forall\, n$ is unique.

Example 11: $I_n = \left[0, \frac{1}{n}\right], n \in \mathbb{N}$ is a nested sequence of intervals such that

$$\inf\left\{\left(\frac{1}{n} - 0\right) \mid n \in \mathbb{N}\right\} = 0.$$

Here $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

3.8. Bolzano-Weierstrass Theorem:

Theorem 4: Every bounded sequence of real numbers has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence. Then the range set $S=\{x_n \mid n\in\mathbb{N}\}$ is also bounded. Since *S* is bounded, it is contained in an interval. $I_1=[a,b]$ we take $n_1 = 1$.

We now bisect I_1 into two equal intervals I_1' and I_1'' and divide the set of indices $\{n\in\mathbb{N} \mid n>1\}$ into two parts

$$A_1=\{n\in\mathbb{N} \mid n>n_1,\, x_n\in I_1'\}$$

$$B_1=\{n\in\mathbb{N} \mid n>n_1,\, x_n\in I_1''\}$$

If A_1 is infinite, we take $I_2=I_1'$ and let n_2 be the smallest natural number in A_1 (Existence ensured by Well Ordering Principle).

If A_1 is a finite set then B_1 must be infinite and we take $I_2=I_1''$ and let n_2 be the smallest natural number in B_1.

Again, we bisect I_2 into two equal subintervals I_2' and I_2'' and divide the set $\{n\in N \mid n>n_2\}$ into two parts

$$A_2=\{n\in N \mid n>n_2, x_n\in I_2'\}$$

$$B_2=\{n\in N \mid n>n_2, x_n\in I_2''\}$$

If A_2 is infinite, we take $I_3=I_2'$ and let n_3 be the smallest natural number in A_2.

If A_2 is a finite set, then B_2 must be infinite and we take $I_3=I_2''$ and let n_3 be the smallest natural numbers in B_2

We continue in this way to obtain a sequence of nested intervals,

$$I_1\supseteq I_2\supseteq I_3\supseteq\ldots\supseteq I_k\supseteq\ \ldots$$

and a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Now length of $I_k = \dfrac{b-a}{2^{k-1}}$ (because at each subdivision the interval is divided into two equal parts)

Since length $(I_k) \to 0$ as k → ∞ therefore by Nested Interval Theorem, there is a unique real number $\xi \in I_k$ for all k ∈ N.

Since x_{n_k} and ξ both belong to I_k therefore we must have

$$|x_{n_k} - \xi| \leq \frac{b-a}{2^{k-1}}$$

$$\Rightarrow \quad |x_{n_k} - \xi| \to 0 \text{ as } k \to \infty$$

Hence the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to ξ.

Theorem 5: Let $\langle x_n \rangle$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ be such that every convergent subsequence of $\langle x_n \rangle$ converges to x. Then the sequence $\langle x_n \rangle$ converges to *x*.

Proof. Suppose *M* > 0 is a bound for the sequence $\langle x_n \rangle$ so that

$$|x_n| \leq M \ \forall \ n \in \mathbb{N}.$$

Let if possible $\langle x_n \rangle$ does not converge to *x*. Then there exist an $\epsilon_0 > 0$ and a subsequence $\langle x_{n_k} \rangle$ such that

$$|x_{n_k} - x| \geq \epsilon_0 \quad \forall k \in \mathbb{N}. \qquad (1)$$

Now $\langle x_{n_k} \rangle$ s a subsequence of $\langle x_n \rangle$ and M is a bound for $\langle x_n \rangle$ therefore M is also a bound for $\langle x_{n_k} \rangle$. Hence $\langle x_{n_k} \rangle$ is a bounded subsequence. Therefore by Bolzano-Weierstrass Theorem $\langle x_{n_k} \rangle$ has a convergent subsequence $\langle x_{p_k} \rangle$. Now $\langle x_{p_k} \rangle$ is also a sequence of $\langle x_n \rangle$ therefore $\langle x_{p_k} \rangle$ also converges to x by hypothesis. Thus terms of $\langle x_{p_k} \rangle$ belongs to $\in_0$ neighbourhood of x, which contradicts (1). Hence the result.

Exercises:

(1) Give an example of an unbounded sequence that has a convergent subsequence.

(2) Express each of the following subsequences of $\langle x_n \rangle_{n=1}^{\infty} = \langle n^{1/6} \rangle_{n=1}^{\infty}$ in the form $\langle x_{n_k} \rangle$ for a suitably chosen index sequence $\langle n_k \rangle$

(a) < 1, 2, 3, 4, 5, 6, 6, ...>

(b) < 1, $2^{1/3}$, $3^{1/3}$, $4^{1/3}$, $5^{1/3}$, ...>

(3) For each of the following sequences determine two subsequences, each converging to a different limit and express each subsequence in the form $\langle x_{n_k} \rangle$ for a suitably chosen index sequence $\langle n_k \rangle$

(a) $\langle x_n \rangle_{n=1}^{\infty}$, $x_n = \cos\left(n\dfrac{\pi}{2} \right)$

(b) $\langle x_n \rangle_{n=1}^{\infty}$, $x_n = \sin\left(n\dfrac{\pi}{4} \right)$

(4) Give an example of a sequence that has both convergent and divergent subsequences.

(5) Find the limits of the following sequences :

(a) $\left\{ \left(1 + \dfrac{2}{n} \right)^{n} \right\}$ (b) $\left\{ \left(1 + \dfrac{1}{n^2} \right)^{2n} \right\}$

(6) Show that the following sequences are divergent

(a) $\left\{\sin\left(n\frac{\pi}{4}\right)\right\}$ (b) $\left\{(-1)^n n^2\right\}$ (c) $\left\{1-(-1)^n+\frac{1}{n}\right\}$

(7) Provide an example of a sequence with the given property

(a) a sequence that has subsequences that converge to 1, 2 and 3

(b) a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence.

(c) a sequence that has no convergent subsequence.

4. The Cauchy Criterion

It is often possible to prove that a sequence converges without knowing its actual limit. The Monotone Convergence Theorem provides one tool for doing that. Since convergent sequences are not necessarily monotone, it would be nice to have a condition for convergence that does not require the sequence to be monotone. Fortunately there is such a condition and sequences that satisfy this condition are known as Cauchy sequences in honour of Augustin-Louis Cauchy (1789-1867). Cauchy sequences play an important role in analysis.

4.1. Definition: A sequence $\langle x_n \rangle$ of real numbers is said to be a Cauchy sequence if for every $\in$ > 0 there exist a natural number $N(\in)$ such that

$|x_n - x_m| < \in \quad \forall$ n, m ≥ N.

Value Addition :
The definition of Cauchy sequence is similar to the definition of a convergent sequence. The only difference is that in convergence we say terms are close to some real number L whereas in the definition of Cauchy sequence we say that terms are eventually close to each

other.

Example 12: Consider the sequence $\{x_n\}$ where $x_n = \frac{n+1}{n},\ n \in$ □

$$| x_n - x_m | = \left| \frac{n+1}{n} - \frac{m+1}{m} \right| = \frac{|m-n|}{mn} \le \frac{m+n}{mn} = \frac{1}{n} + \frac{1}{m}$$

Let ∈ > 0 be given, we can choose a positive integer N such that p ≥ N

⇒ $\frac{1}{p} < \frac{\in}{2}$. Then if $m,n \ge N$ we have

$$| x_n - x_m | < \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence $\langle x_n \rangle$ is a Cauchy sequence.

Example 13. Consider the sequence $\langle s_n \rangle$ where $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$, $n \in N$.

Let ∈ > 0 be given we can choose a positive integer N such that $\frac{1}{N} < \in$. If $m,n \ge N$ with $n > m$ then

$$| s_n - s_m | = \left| \sum_{k=m+1}^{n} \frac{1}{k^2} \right| \le \sum_{k=m+1}^{n} \left| \frac{1}{k^2} \right|$$

$$< \sum_{k=m+1}^{n} \frac{1}{k(k-1)} = \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right)$$

$$= \frac{1}{m} - \frac{1}{n} < \frac{1}{N} < \in$$

Hence the sequence $\{s_n\}$ is a Cauchy sequence.

Example 14: show that the sequence $\langle b_n \rangle$, where $b_n = 1 + \frac{1}{4} + \frac{1}{7} + ... + \frac{1}{3n-2}$ is not a Cauchy sequence.

Sol. : Consider $|b_{2n} - b_n| = \left| \frac{1}{3n+1} + \frac{1}{3n+4} + ... + \frac{1}{3\times 2n-2} \right|$

> $n \times$ smallest term.

$= n \times \frac{1}{6n-2}$

$> \frac{n}{6n} = \frac{1}{6}$

Since $|b_{2n} - b_n| > \frac{1}{6}$. Hence $\langle b_n \rangle$ is not a Cauchy sequence.

Theorem 6: If $\langle x_n \rangle$ is a convergent sequence of real numbers then $\langle x_n \rangle$ is a Cauchy sequence.

Proof: Suppose that $\lim_{n\to\infty} x_n = x$. The idea is that since the terms x_n are close to x for large n, they must also be close to each other.

Consider $|x_n - x_m| = |x_n - x + x - x_m| \le |x_n - x| + |x - x_m|$

Let $\in$ > 0 be given. Then there exist a natural number *N* such that *n* > *N* implies $|x_n - x| < \in/2$.

Similarly, *m* > *N* implies $|x_m - x| < \in/2$.

therefore m, n > N implies $|x_n - x_m| \le |x_n - x| + |x_m - x| = \in/2 + \in/2 = \in$

Thus $\langle x_n \rangle$ is a Cauchy sequence.

Value Addition:
It is a simple consequence of the definition that every convergent sequence is a Cauchy sequence. But Cauchy sequence need not be convergent

Theorem 7: Every Cauchy sequence of real numbers is bounded.

Proof: Let $\langle x_n \rangle$ be a Cauchy sequence. Let $\in$ = 1. Then by definition there exist a natural number N such that

$$m, n > N \text{ implies } |x_n - x_m| < 1$$

In particular, $|x_n - x_{N+1}| < 1$ for $n > N$

therefore $|x_n| < |x_{N+1}| + 1$ for $n > N$

Let $M = \max\{|x_{N+1}| + 1, |x_1|, |x_2|, \ldots |x_N|\}$

then $|x_n| \leq M \quad \forall\, n \in \mathbb{N}$

Hence $\langle x_n \rangle$ is bounded.

I.Q. 8.

Value Addition:
A bounded sequence may not be Cauchy. E.g. $x_n = \langle (-1)^n \rangle$ is a bounded sequence which is not Cauchy.

I.Q. 9.

4.2. Cauchy Convergence Criterion: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof: We have already proved that every convergent sequence is a Cauchy sequence. Conversely suppose that the sequence $\langle x_n \rangle$ is Cauchy. Then $\langle x_n \rangle$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem $\langle x_n \rangle$ must have a convergent subsequence, say $\langle x_n \rangle$ with $\lim_{k \to \infty} x_{n_k} = l$. We shall show that $\lim_{n \to \infty} x_n = l$.

Let $\in$ > 0 be given. Since $\{x_n\}$ is a Cauchy sequence there exist a natural number N such that $|x_n - x_m| < \in / 2$, for $m, n \geq N$.

Since the subsequence $\langle x_{n_k} \rangle$ converges to '*l*' there exists a natural number $p \geq N$ such that

$$| x_{n_p} - l | < \in / 2$$

Therefore, for any $n \geq N$, we have

$$| x_{n_p} - l | = | x_n - x_{n_p} + x_{n_p} - l | \leq | x_n - x_{n_p} | + | x_{n_p} - l | < \in / 2 + \in / 2 = \in$$

$\Rightarrow \quad \langle x_n \rangle$ converges to '*l*'.

Value Addition:
The fact that every Cauchy sequence of real numbers converges to some real number is a very important property of the set of real numbers. It distinguishes the real numbers from the rational numbers since, a Cauchy sequence of rational numbers may not converge to a rational number.

The advantage of Cauchy sequences lies in the fact that one can establish the convergence of a sequence without finding a value for the limit or requiring the sequence to be monotone. Most important is that the definition of a Cauchy sequence only involves the distance between points in the sequence. It is an internal property of sequences.

I.Q. 10.

Example 15: Let $\langle a_n \rangle$ be a sequence of positive numbers such that

$a_0 = 1,\ a_1 = 1,\ a_n = \frac{1}{2}(a_{n-1} + a_{n-2}) \quad \forall\ n \geq 2$ show that $\langle a_n \rangle$ is a Cauchy sequence.

Solution: We can write $2a_n = a_{n-1} + a_{n-2}$

$$2(a_n - a_{n-1}) = -(a_{n-1} - a_{n-2})$$

or $$a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2}) = \left(-\frac{1}{2}\right)^2 (a_{n-2} - a_{n-3})$$

$$= \left(-\frac{1}{2}\right)^3 (a_{n-3} - a_{n-4}) = \ldots = \left(-\frac{1}{2}\right)^{n-1} (a_1 - a_0) = \left(-\frac{1}{2}\right)^{n-1} (1-0)$$

Thus $|a_n - a_{n-1}| = \left(\frac{1}{2}\right)^{n-1}$ or $|a_{n+1} - a_n| = \left(\frac{1}{2}\right)^n$

Consider $$|a_n - a_m| = |(a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \ldots + (a_{m+1} - a_m)|$$

$$\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots + |a_{m+1} - a_m|$$

$$= \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^{n-2} + \ldots + \left(\frac{1}{2}\right)^m$$

$$= \left(\frac{1}{2}\right)^m \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{n-1-m}\right]$$

$$= \left(\frac{1}{2}\right)^{m-1} \left[1 - \left(\frac{1}{2}\right)^{n-m}\right] < \left(\frac{1}{2}\right)^{m-1}$$

Let $\in > 0$ be given. If m is chosen so large that $\left(\frac{1}{2}\right)^{m-1} < \in$ then for $n \geq m$ we get

$|a_n - a_m| < \in \Rightarrow \langle a_n \rangle$ is Cauchy sequence.

I.Q. 11.

Exercises:

1. Prove directly from definition that the following are Cauchy sequences.

   (a) $\left\{\frac{1}{n}\right\}$ (b) $\left\{\frac{n+1}{n}\right\}$

2. Show directly from the definition that the following are not Cauchy sequences.

   (a) $\{(-1)^n\}$ (b) $\left\{\frac{n^2+1}{n}\right\}$

3. Give an example of a bounded sequence that is not a Cauchy sequence.

4. Show directly from the definition that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then $\{x_n+y_n\}$ and $\{x_n y_n\}$ are Cauchy sequences.

5. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.

6. Show that the sequence defined by $a_n = 1+\frac{1}{3}+\frac{1}{5}+....+\frac{1}{2n-1}$ is not convergent

   [Hint : $|a_{2n}-a_n| = \frac{1}{2n+1}+\frac{1}{2n+3}+...+\frac{1}{4n-1}$

   $> n\times\frac{1}{4n-1} > \frac{n}{4n}$

   Since $|a_{2n}-a_n| > \frac{1}{4}$ Cauchy Convergence Criterion is violated. Hence $\langle a_n\rangle$ cannot converge.

8. Given a sequence $\langle a_n\rangle$ such that $|a_n-a_{n+1}|\le\frac{1}{n}$. Do you think that $\langle a_n\rangle$ is a Cauchy sequence?

   [Hint : No, let $a_n = 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}$ then $|a_n-a_{n+1}| = \frac{1}{n+1}\le\frac{1}{n}$.

   However, $\langle a_n\rangle$ is not Cauchy as

$$|a_{2n} - a_n| = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} > n \times \frac{1}{2n} = \frac{1}{2}$$

Subsequences and Bolzano-Weierstrass Theorem

Summary

In this chapter we have discussed:

- Definition of a subsequence
- To construct a subsequence satisfying certain property.
- To find whether a given sequence is convergent or divergent by means of its subsequences.
- To find whether a given sequence is convergent or divergent by using Cauchy Convergence Criteria.

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