Discipline Course-I

Semester-II

Paper No: Electricity and Magnetism

Lesson: Electric Field and Calculation of Electric Field for various systems

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Learning Objectives

After going through this chapter, the student would be able to

- Know the concept of 'Electric field'
- Get familiar with the 'physical' and 'mathematical ' definition of 'Electric field'
- Develop an understanding about the various types of Charge Distributions
- Calculate the Electric field due to different types of charge distribution

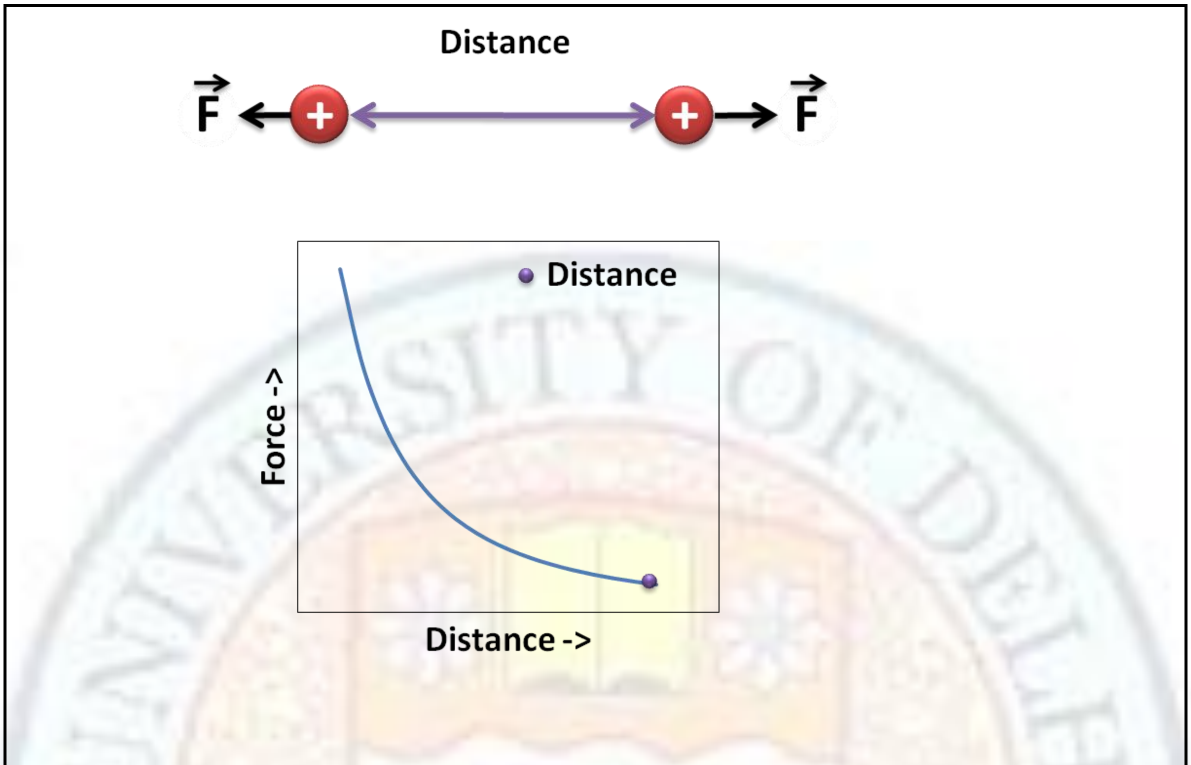
1.1 Introduction of electric field

Coulomb's law gives the interaction between two point charges separated by a distance. A very famous relation between Coulomb's force, charges and distance between then is given by the relation

$F= (1/4\pi\varepsilon_o)Q_1Q_2/r^2$

Where, Q1, Q2, ε_o and r are the charges on two particles, permittivity of free space, and the distance between two charges, respectively. We know that the opposite charges attract each other while similar charges oppose each other. The behavior of Coulomb's repulsive force F between two similar charges as a function of distance has been shown in the following animation.

Value addition: Did you Know
Coulomb's law
Body text: Animate the images to change in closed loop type arrangement (continuously)



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If we have more than two point charges, we repeat the process again and again to find the interaction between a given charge and other charges present in space. In order to avoid this repetition, we associate a characteristic known as its electric field, with any given charge. We can then say that each charge will experience the presence of other charges due to their electric fields.

In figure 1.1, we consider the mutual repulsion of two positively charged bodies A and B. Let $\vec{F}$ be the force and Q be the charge on B. We remove body B and label its former position as point 'P' in figure 1.2. We can then say that the force, $\vec{F}$, experienced by the charged body B, is due to the electric field, produced at point P (as shown in Fig. 1.3), by the charged body A. Since B would experience a force at any point in space around A, this electric field exists at all points in the region around A.

We can, therefore, say: "An electric field is said to exist at a point, if a stationary charged body placed at that point, experiences a force."

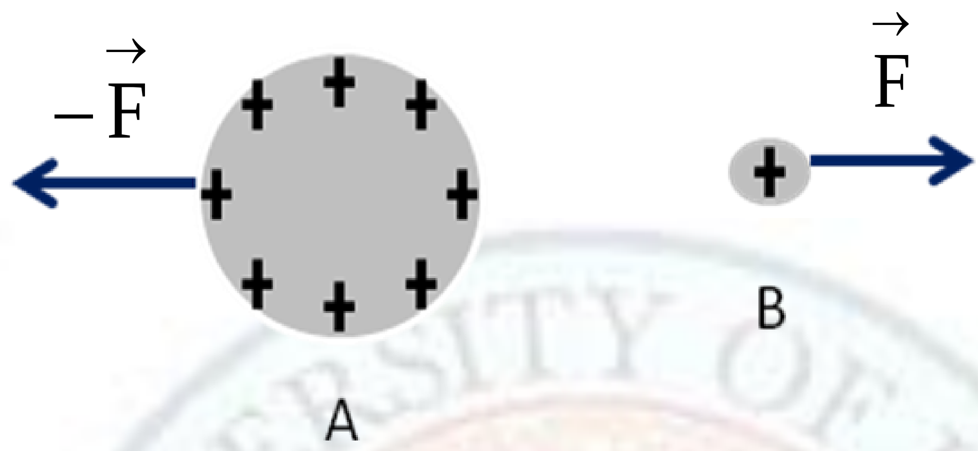


Figure: 1.1 Interaction between two charged bodies (A and B) .

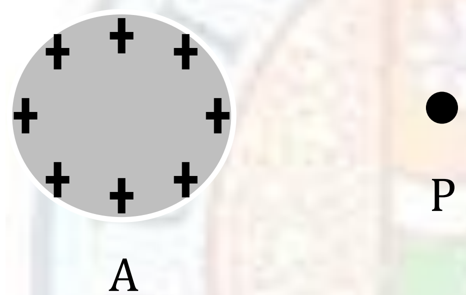


Figure: 1.2

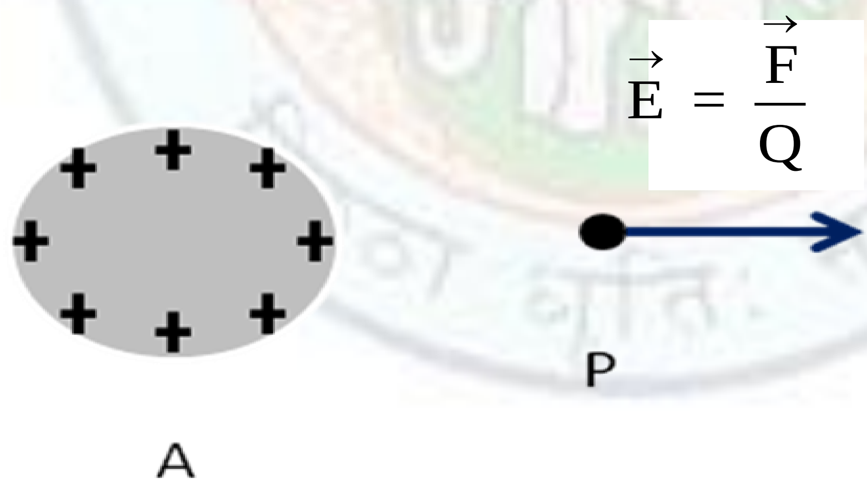


Figure: 1.3 Electric field at a point P due to a charged body A.

We usually think of the electric field $\vec{E}$, at a point, as the ratio obtained when the force $\vec{F}$, acting on a positive test charge, is divided by the magnitude, Q, of that test charge.

$$\vec{E} = \frac{\vec{F}}{Q} \qquad (1.1)$$

The direction of $\vec{E}$ is same as the direction of $\vec{F}$, for a positive test charge.

Electric field is also known as electric field intensity. In SI units, when F = 1 N and Q = 1 C, E = 1 NC^{-1}.

Value addition: Did you Know
Electric Field is a vector field.
Body text: The force experienced, by the same test charge Q, would vary from point to point. Hence, the electric field would also be different at different points. In general then $\vec{E}$ is not a single vector quantity, but an infinite set of vectors, one associated with each point in space. We can, therefore, think of $\vec{E}$ as a vector field.
Reference: David J. Griffiths, Introduction to Electrodynamics, 3rd edition.

We need here to pause and think of an important point. It is very likely that the presence of the test charge Q may affect the charge distribution A. The electric field around A, when Q is present, would, then, be not the same as when it is absent. This is certainly neither desirable nor permissible. We can minimize this disturbing effect by realizing that it Q were very small, its effect on the redistribution of charge, on body A, would also be very small. We, therefore, prefer to think of the electric field as being defined by

$$E = \lim_{\Delta Q \to 0} \left(\frac{\Delta F}{\Delta Q} \right) \qquad (1.2)$$

1.2 Relation between Electric Field and Coulomb's Force

To find the magnitude of the electric field at a point P, at a distance r from a point charge Q, we consider a positive test charge ΔQ_0 to be placed at P. We then have

$$\Delta F = \frac{1}{4\pi \in_0} \left(\frac{Q \Delta Q_0}{r^2} \right) \qquad \text{(eq. 1.3)}$$

The electric field at P, then has a magnitude, E, given by

$$E = \frac{\Delta F}{\Delta Q_0} = \frac{1}{4\pi \in_0}\left(\frac{Q}{r^2}\right) \qquad \text{(eq. 1.4)}$$

1.3 Direction of Electric Field

The direction of the electric field, caused by a point charge, is represented by the use of an appropriate unit vector. Let $\hat{r}$ be a unit vector, directed along the line, from the charge Q to the point P, at which the field is to be determined. Let us write

$$\vec{E} = \frac{1}{4\pi \in_0}\left(\frac{Q}{r^3}\right)\vec{r} \qquad \text{(eq. 1.5)}$$

When Q is positive, the direction of $\vec{E}$ is along $\hat{r}$, i.e., directed away from Q. When Q is negative the direction of $\vec{E}$ would be opposite to $\hat{r}$, i.e., towards Q. This choice of the direction of $\vec{E}$, is, therefore, consistent with the basic law of electrostatics: 'Like charges repel; unlike charges attract each other.' In terms of the co-ordinates (x, y, z), of the field point, we can write:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

And $r^3 = (x^2+y^2+z^2)^{3/2}$

These values are based on the implicit assumption that the location of the charge Q is the origin of our co-ordinate system. Hence

$$\vec{E}(x,y,z) = \frac{1}{4\pi \in_0}\frac{Q}{\left(x^2+y^2+z^2\right)^{3/2}}(x\hat{i}+y\hat{j}+z\hat{k})$$

We can always write

$$\vec{E}(x,y,z) = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$

Thus the x-component, of the electric field, is given by

$$E_x = \frac{1}{4\pi \in_0}\frac{Q.x}{\left(x^2+y^2+z^2\right)^{3/2}}$$

We would have similar expressions for the y and z components, (E_y and E_z), of the electric field.

Value addition: Did you Know
Position vector
Body text: It is very important to realize that the vector $\vec{r}$, in the expression for $\vec{E}$, is the position vector, of the field point P, with respect to the point of location of the source charge Q. We can, therefore, write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ only by implicitly assuming that we are considering the location of the source charge as the origin. In general $\vec{r} = (x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}$ where (x,y,z) is the field point and (x_0,y_0,z_0) is the point where the source charge is located.
Reference: David J. Griffiths, Introduction to Electrodynamics, 3rd edition.

1.4 Introduction of Principle of Superposition

It states that the interaction between any two charges is completely unaffected by the presence of others.

If a number of point charges q_1, q_2, etc. are at distance r_1, r_2, etc. from a given point P. The resultant electric field is taken as the vector sum of the individual electric fields. We refer to this statement as the principle of superposition. We thus have

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \ldots\right) \qquad \text{(Eq. 1.6)}$$

Hence $\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i$

Value addition: Did you Know
Electric field due to a point charge

Body text:

We now prefer to think of the electric field, due to a point charge, q_i, as being defined through the mathematical expression

$$\vec{E}(r) = \frac{1}{4\pi \in_0} \frac{q_i}{r_i^3} \vec{r_i}$$

Reference: David J. Griffiths, Introduction to Electrodynamics, 3rd edition.

1.5 Types of charge distribution

The principle of superposition assumes that the source of the field is a collection of discrete point charges. When the charge is distributed continuously over a region, the sum (in eq. 6) has to be replaced by an integral. In general, a continuous charge distribution can be of three types: a linear charge distribution, a surface charge distribution and a volume charge distribution.

Line charge distribution

When the charge is distributed (continuously) over a line, with a linear charge density λ (charge per unit length), an elemental charge, dq, over an elemental length, dl, would be given by

$$dq = \lambda.dl$$

Surface charge distribution

When the charge is distributed (continuously) over a surface with a surface charge density σ (charge per unit area), an elemental charge, dq, over an elemental surface area, da, would be given by

$$dq = \sigma.da$$

Volume charge distribution

When the charge is distributed over a volume, with a volume charge density ρ (charge per unit volume), an elemental charge, dq, over an elemental volume, dV, would be given by

$$dq = \rho.dV$$

For any continuous charge distribution, the electric field must be calculated by supposing the charge distribution to be subdivided into many such small elements of charge ΔQ.

$$\vec{E} = \frac{1}{4\pi \in_0} \sum \left(\frac{\Delta Q}{r^2} \right) \hat{r} \qquad \text{(Eq. 1.7)}$$

In the limit as $\Delta Q \to 0$

$$\vec{E} = \frac{1}{4\pi \in_0} \lim_{\Delta Q \to 0} \sum \left(\frac{\Delta Q}{r^2} \right) \hat{r} \qquad \text{(Eq. 1.8)}$$

The limit of the vector sum, however, is the vector integral.

$$\vec{E} = \frac{1}{4\pi \in_0} \int \hat{r} \frac{dQ}{r^2} \qquad \text{(Eq. 1.9)}$$

Hence, the electric field of a continuous linear distribution of charge, would be given by

$$\vec{E}(r) = \frac{1}{4\pi \in_0} \int_L \frac{\lambda.dl}{r^2} \hat{r}$$

For a continuous surface charge distribution, we would have

$$\vec{E}(r) = \frac{1}{4\pi \in_0} \int_S \frac{\sigma.da}{r^2} \hat{r}$$

For a continuous volume charge distribution, the corresponding result would be

$$\vec{E}(r) = \frac{1}{4\pi \in_0} \int_V \frac{\rho.dV}{r^2} \hat{r}$$

We now use these results for a few simple continuous charge distributions.

1.6 Calculation of electric field for various systems: Electric Field, at an axial point, due to a circular ring of charge

We consider that the given circular ring, in figure 1.4, is made up of small elements of length dl each having a charge dQ. At an axial point P, assuming the axis of the circular ring to be along x-axis, the element of charge dQ causes an electric field contribution $d\vec{E}$, having magnitude dE given by

$$dE = \frac{1}{4\pi \in_0} \frac{dQ}{x^2 + a^2} \qquad \text{(Eq. 1.10)}$$

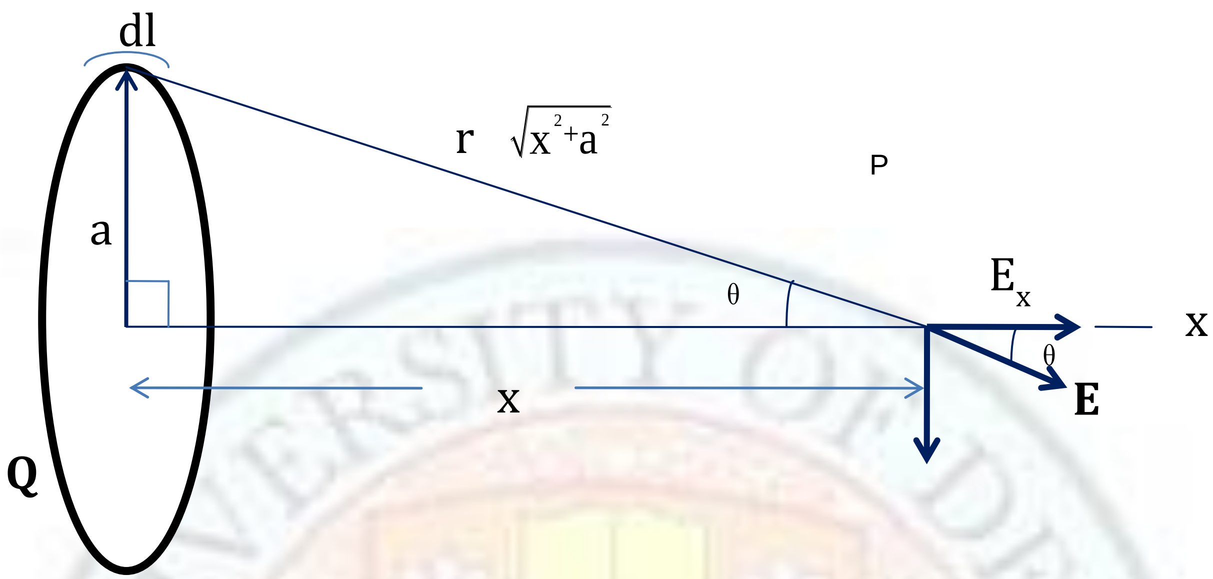


Figure: 1.4 Electric field on the axis of charged ring.

The component dE_x of this field, along the x-axis, is

$$dE_x = dE\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{dQ}{x^2+a^2}\frac{x}{\sqrt{x^2+a^2}} \qquad \text{(Eq. 1.11)}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{x\,dQ}{\left(x^2+a^2\right)^{3/2}}$$

To find the total x-component of the field, E_x, we integrate this expression:

$$E_x = \frac{1}{4\pi\epsilon_0}\int\frac{x\,dQ}{\left(x^2+a^2\right)^{3/2}}$$

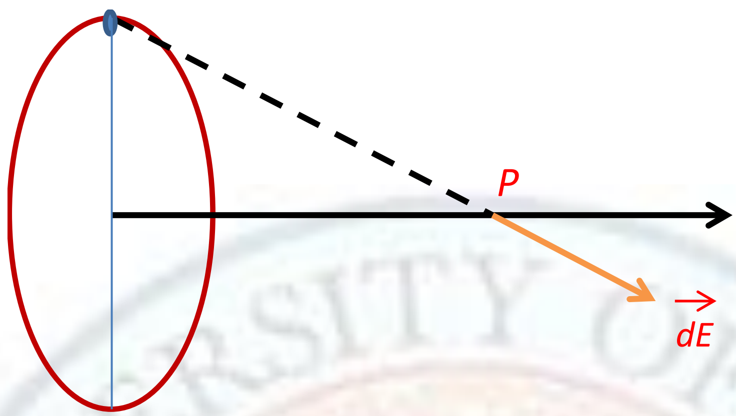
Here x and a are constants for all the elements of the charged ring. Also, the integral of dQ is simply the total charge Q. Therefore

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Qx}{\left(x^2+a^2\right)^{3/2}} \qquad \text{(Eq. 1.12)}$$

The components perpendicular to the x-axis add to zero due to symmetry.

Value addition: Did you Know
Calculating the Electric field of a ring of charge.

Body text:



Animate the above figures

Source: self

At the center of the ring (x=0), the total field is zero.

For large values of x, i. e., for x >>a, the field due to ring approximates the field due to a point charge.

$$E = \frac{Q}{(4\pi \in_0) x^2}. \qquad \text{(Eq. 1.13)}$$

This is understandable, as from a large distance, a (small) ring of charge would look almost like a point charge.

1.7 Electric Field due to a charged disc

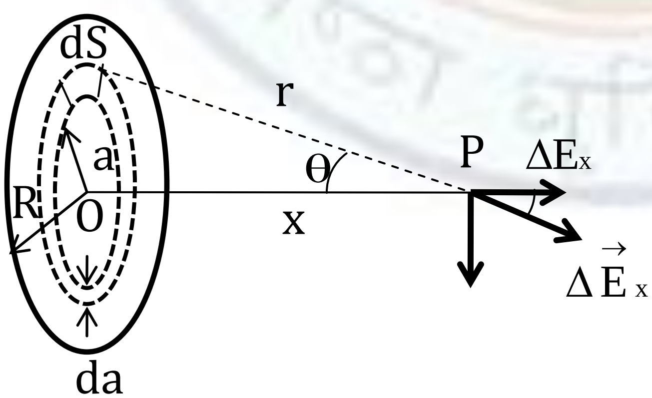


Figure: 1.5 Electric field on the axis of charged disc.

Let us calculate the field strength on the axis of a disc of radius R, at a distance x from its centre O. For this let us divide the disc into annular rings. In figure 1.5 the field, at a point P, due to charge element, σdS, on the annular ring, of radius a and of thickness da, is given by

$$\Delta \vec{E} = \frac{1}{4\pi \epsilon_0}\frac{\sigma dS}{r^2}\hat{r}$$

Its component, along the axis of the disc, or along the x-direction, is $\Delta E_x = \frac{1}{4\pi \epsilon_0}\frac{\sigma dS}{r^2}\cos\theta = \frac{1}{4\pi \epsilon_0}\frac{\sigma dS}{r^2}\frac{x}{r}$

By symmetry the total field is along x-direction and can be obtained by integrating. If the surface density of charge σ is constant, then

$$E = E_x = \frac{\sigma}{4\pi \epsilon_0}\int_0^R \frac{x dS}{r^3} = \frac{\sigma}{4\pi \epsilon_0}\int_0^R \frac{x 2\pi a da}{(x^2+a^2)^{3/2}}$$

Using integration by parts to solve the above integral one obtains

$$E = \frac{\sigma}{2\epsilon_0}\left[-\frac{x}{(x^2+a^2)^{1/2}}\Bigg|_0^R - \left(\frac{1}{(x^2+a^2)^{1/2}}\frac{dx}{da}\right)\Bigg|_0^R\right]$$

$$\Rightarrow E = \frac{\sigma}{2\epsilon_0}\left[-\frac{x}{(x^2+a^2)^{1/2}}\Bigg|_0^R - 0\right]$$

$$\Rightarrow E = \frac{\sigma}{2\epsilon_0}\left[1 - \frac{x}{(x^2+R^2)^{1/2}}\right]$$

1.8 Electric Field due to a charged wire

We now try to find the electric field at a distance 'd' (point P), above the midpoint (point O) of a charged wire of length 2L (Fig. 1.6), having a uniform linear charge density λ. We consider the charged wire along x-axis, origin as its midpoint and think of it as made up of small elements of length dx each having a charge dQ. At a point P, an element, of charge dQ, causes an electric field contribution $d\vec{E}$. Due to symmetry all the horizontal components of the field contributions cancel out. This is so because a pair of charge elements, located symmetrically around the origin, would cancel each other as they would produce equal contributions on either side oppositely directed. We can, therefore, say, for each such 'pair':

$$d\mathbf{E} = 2\frac{1}{4\pi \in_0}\left(\frac{\lambda dx}{r^2}\right)\cos\theta\,\hat{y} \qquad \text{(Eq. 1.14)}$$

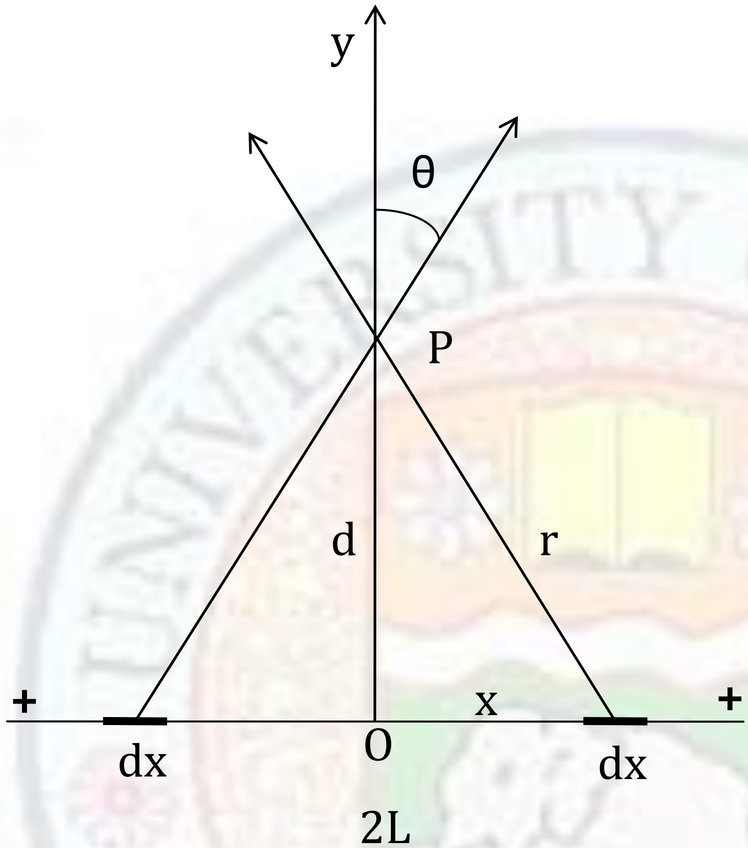


Figure: 1.6 Electric field at a distance 'd' from the mid-point of a uniformly charged wire.

Here cosθ = d/r,

and r = $\sqrt{d^2 + x^2}$ and x varies from 0 to L.

$$E = \frac{1}{4\pi \in_0}\int_{-L}^{L}\frac{\lambda d}{\left(d^2 + x^2\right)^{3/2}}dx$$

$$\therefore \qquad \text{(Eq. 1.15)}$$

$$= \frac{1}{4\pi \in_0}\int_{0}^{L}\frac{2\lambda d}{\left(d^2 + x^2\right)^{3/2}}dx$$

Here the limits have been changed from (-L,L) to (0,L) and the integrand multiplied by a factor of two as the integral will lead to an even function of 'x'.

Put x = d.tanθ => dx = d.sec²θ.dθ

$$\therefore E = \frac{2\lambda d}{4\pi \in_0} \int_0^{\theta_{max}} \frac{d \sec^2\theta . d\theta}{d^3\left(\tan^2\theta + 1\right)^{3/2}} \qquad \text{(Eq. 1.16)}$$

where $\tan\theta_{max} = \frac{L}{d}$

$$= \frac{2\lambda d}{4\pi \in_0} \int_0^{\theta_{max}} \frac{\cos\theta d\theta}{d^2}$$

$$= \frac{2\lambda d}{4\pi \in_0} \left. \frac{\sin\theta}{d^2} \right|_0^{\theta_{max}}$$

$$= \frac{2\lambda d}{4\pi \in_0} \left[\frac{x}{d^2\sqrt{d^2 + x^2}} \right]_0^L$$

$$E = \frac{2\lambda L}{4\pi \in_0 d\sqrt{d^2 + L^2}} \qquad \text{(Eq. 1.17)}$$

(it points in the y-direction)

Case (i) When d ≫ L

$$E = \frac{1}{4\pi \in_0} \frac{2\lambda L}{d^2}$$

Case (ii) For an infinite straight wire (i.e. L>>d), we have

$$\sqrt{d^2 + L^2} \sim L$$

Hence

$$E = \frac{1}{4\pi \in_0} \frac{2\lambda}{d}$$

Question Number	**Type of question**

1	MCQ

Question

(1) The unit of electric field intensity is ______________.

Option(s)	a. Newton, b. Newton/Coulomb, c. N/cm^2, d. None.

Question

(2) Electric field intensity can be calculated by using electrostatic potential.

Option(s)	(a) E=gradϕ(r) (b) $E = -\int grad\ \phi(r)dr$ (c) E=-grad ϕ(r), (d) None.

Correct Answers	1) b 2) c

Question Number	**Type of question**
2	Subjective

Question

1) Obtain the expression for electric field from Coulomb's law.
2) Point charges of 3 x 10^{-9} C are situated at each of these corners of a square whose side is 15 cm. Find the direction and magnitude of the electric field at the vacant corners.

Correct Answers	(2) Along diagonal, 2,296 V/m

Question

3) Obtain the expression of potential difference between two point distant r_1 and r_2 from an infinitely long line charge of linear charge density λ.

Correct Answers/Solution	
	Let E be the field strength at a distance r from an infinite linear charge density λ, $$E = \frac{q}{4\pi\varepsilon_o r^2}$$ We know λ=q/l, q=λl Then, $$E = \frac{2\lambda}{4\pi\varepsilon_o r}$$ By definition r_{ref} potential 'ϕ' at a distance r from wire $$E = -grad\phi$$ $$\phi = -\int_{r_{ref}}^{r} E.dr$$ Here, r_{ref} denotes the reference for zero potential. Here, reference distance cannot be taken as infinity since the wire itself extends to infinity. For r_1 and r_2 $$\Delta\phi = -\int_{r_2}^{r_1} \frac{2\lambda}{4\pi\varepsilon_o r} dr$$ $$\phi = \frac{2\lambda}{4\pi\varepsilon_o} log\frac{r_1}{r_2}$$

Question Number	Type of question
3	Unsolved problem

Question

A right circular cylinder of radius R and height L is oriented along z-axis. It has a non-uniform volume density of charge given by $\rho(z) = \rho_o + \beta(z)$ with reference to an origin at the centre of cylinder. Find the electric field at the centre of cylinder.

Correct Answers/Solution	$E = \frac{\beta}{2\varepsilon_o}\left[\frac{L}{2}\left\{\frac{L}{2} - \sqrt{\left(\frac{l^2}{4} + R^2\right)}\right\} + R^2 log\left\{\frac{L}{2R} + \sqrt{\left(1 + \frac{L^2}{4R^2}\right)}\right\}\right]$

References:

David J. Griffiths, Introduction to Electrodynamics, 3rd edition.