Discipline Course-I

Semester-II

Paper No: Electricity and Magnetism

Lesson: Method of image

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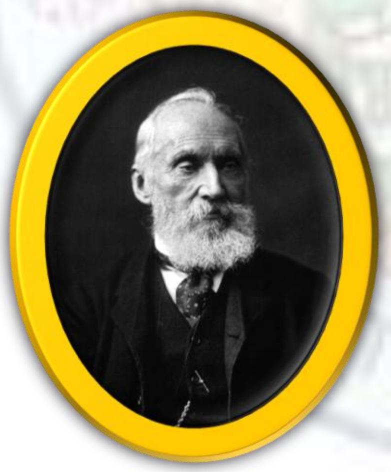
Learning Objectives

This lesson aims at the following student learning objectives.

- Know the uniqueness theorem
- About the technique method of electrical image for the:
  - Calculation of electric field at some point in the ambient of a system when charge is placed in front of
    - Conducting plane
    - Conducting sphere
  - Calculation of electric field at some point when dipole is placed
    - Conducting plane
- Calculation of other properties of the system like
  - Nature of force between the charge and object placed close to it
  - Work done in removing the charge from the electric field of the system

14.1. Introduction

The method of images is a mathematical technique to solve problems of one or more point charges which are situated close to boundary surfaces. Lord Kelvin (1824-1907) had developed the method of electrical images for the solution of complex problem involving charges and conducting surfaces. The idea of electrical images involves that the complicated charge distribution can be replaced, for simplicity, by either a set of point charges or simply a point charge leaving no effect on the boundary conditions of the problems. In this way, this is the technique to place a point charge or a set of point charges in place of complex charge distribution (such as how much charge is induced on the system and how it is distributed) of opposite polarity in place of infinite grounded conducting plane. Thus a point charge or a set of point charges of appropriate magnitude are placed in peripheral to the region of interest at appropriate distance of boundary and can mimic the required boundary conditions, are called as image charges and the technique of replacement of boundaries with image charges in the extended region is called method of images.



Lord Kelvin (1824-1907)
http://en.wikipedia.org/wiki/William_Thomson,_1st_Baron_Kelvin

For example, if we consider a system of charge $+q$ placed at some distance d from an infinite grounded conducting plane. Now if do not know the distribution and quantity of induced negative charge generated due to $+q$ on the conducting plane, then it becomes difficult to know the potential at some point in the scenario of this system.

Now let us make it more generalize, if we consider a system of point charges say q_1, q_2, q_3...q_i...... q_n, as shown in Fig. 1(a) then the potential due to these charges at some point is

$V = \frac{1}{4\pi\varepsilon_o}\sum_{i=1}^{n}\frac{q_i}{r_i}$ (1)

And the potential at some point due to grounded conducting plane is **zero**.

Now we take the combination of a system of charges and a grounded conducting plane then the grounded conductor may be replaced with a system of charges of opposite polarity which is called as image of the actual charges. The combination of imaginary and actual charge should produce the zero potential on the surface of conducting plane which fulfills the boundary condition of grounded state of conducting plane. Now we can ignore the presence of conducting plane and can take into account a system of actual and image charges as shown in Fig. 1(b). This combination of charge and conducting plane is one of several problems which can be solved by this technique of method of electrical image. The other systems can be a point charge or linear charges placed in front of conducting sphere or more systems such as sphere etc. In this chapter, we shall discuss some cases such as point charge placed in front of grounded conducting plane, sphere, dipole placed in front of grounded conducting plane, and conducting sphere placed in a uniform electric field.

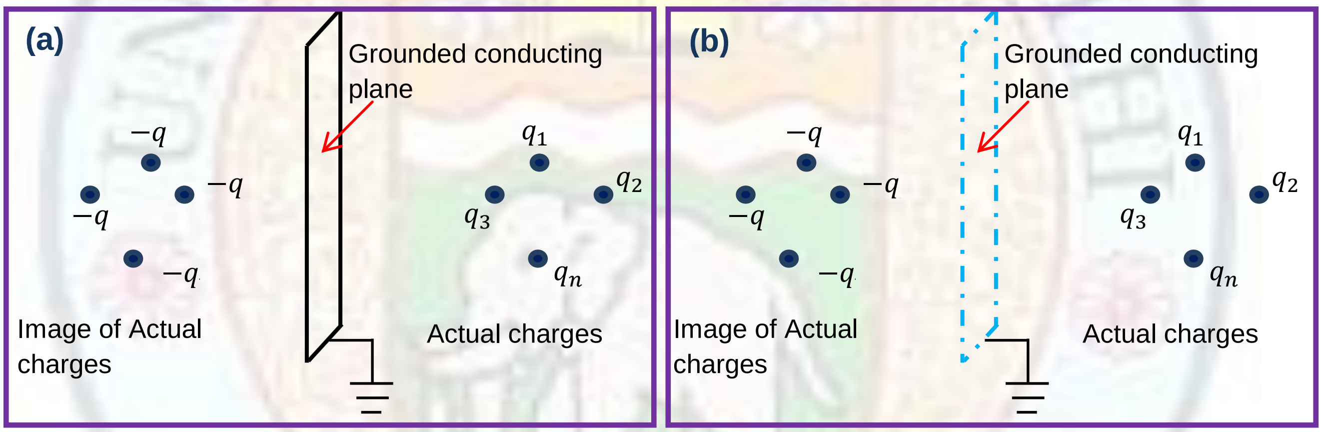


Fig. 1. (a) The actual system with image charges, and (b) the proposed system one can work with by ignoring the presence of plane.

14.1.1. Induced surface charge

It is straight forward method to calculate the induced surface charge on the surface of conductor. We know that electric field is the negative gradient of potential, then we can write

$\vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial r}$ (2)

Where $\vec{r}$ is the position vector in the direction of point where potential is to be calculated. Then, the relation between surface charge density on the conductor and $\vec{E}$ and hence V also by using Eq. (2) is

$\frac{\sigma}{\varepsilon_o} = \vec{E} = -\frac{\partial V}{\partial r}$

$\sigma = -\varepsilon_o\frac{\partial V}{\partial r}$ (3)

This shows that the induced charge on the surface of conducting plane will be of **opposite polarity** than the actual charge. Thus by knowing the potential at point due to combination of image and actual charges, one can get the surface charge

density.
The case can be extended to the other surfaces also, such as spherical, cylindrical surface.

14.1.2. Force and energy

Now since the induced charge developed on the surface of conducting plane is of opposite polarity, therefore, there would an attractive force between the actual charge and the imaginary charge of equal magnitude. Let us consider a system in which the distance between conducting plane and point charge q is d then the total distance between actual and image of charge $-q$ is $2d$ as shown in Fig.2. Thus, the force generated between them will be

$$\vec{F} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2}\hat{x} \qquad (4)$$

The exerted force will be in the direction of line passing through both the point charges.
Then, the energy of this system can be calculated by integrating the force to find the work required to bring q from ∞ to d i.e.

$$U = \int_{\infty}^{d} \vec{F}.dl' \qquad (5)$$

Substituting the expression of $\vec{F}$ in Eq. 5, we get

$$U = -\frac{1}{4\pi\varepsilon_o}\int_{\infty}^{d}\frac{q^2}{(2l\prime)^2}dl'$$

$$U = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)} \qquad (6)$$

Eq. 6 represents the energy of the system.

14.2. Uniqueness theorem and method of image

In case of conductors, there is zero electric field strength inside the conductor in equilibrium state. Therefore, the whole charge density is confined within few atomic layers of conductors i.e. at the surface of conductor. Then, the surface charge density is responsible for the electric field strength at exterior point close to the conductor. The electric field around the conductor can be obtained by knowing the potential around it. This can be achieved by solving the Laplace's equation for potential under the appropriate boundary conditions over the surface of conductor. The Uniqueness theorem is related to boundary value problems in electrostatics and says that the there is only one solution to the Laplace's equation which satisfies the given boundary conditions.
Let us discuss the ***Uniqueness theorem***. The theorem states that *the Laplace's equation satisfying given boundary conditions have one and only one solution i.e. unique solution.*
Proof: Let us consider a closed volume V. Consider that the surfaces $S_1, S_2, S_3, \ldots., S_k$ be the surfaces of the conductors enclosed within the V. All the conductors are bounded by a common surface S. Let us suppose $\varphi_1$and φ_2 are the two distinct solutions Laplace's equation in volume V. These solutions satisfy the same boundary conditions on surfaces $S_1, S_2, S_3, \ldots., S_k$. Let $\varphi_1 = \varphi_2$ using the boundary conditions. Also $(\nabla\varphi_1)_n = (\nabla\varphi_2)_n$ using the same boundary conditions, where $\nabla\varphi$ is the normal component to the surface and also called as *normal derivative*.

Fig. 2 Schematic of the conductors within a volume V.

Now let us substitute $\varphi\nabla\varphi$ into Gauss divergence theorem, we get
$\int_{S1,S2,S3,....,Sk} \varphi\nabla\varphi . dS = \int_V \nabla.(\varphi\nabla\varphi)dv$ (7)
Where dS and dv are the surface and volume elements respectively.
Using vector identity,
$\nabla.\varphi\nabla\varphi = \varphi\nabla^2\varphi + (\nabla\varphi)^2$
Substituting this relation in Eq. (7)
$\int_{S1,S2,S3,....,Sk} \varphi\nabla\varphi . dS = \int_V [\varphi\nabla^2\varphi + (\nabla\varphi)^2]dv$ (8)
But form Laplace equation, we have $\nabla^2\varphi = 0$. Thus, Eq. (8) becomes
$\int_{S1,S2,S3,....,Sk} \varphi\nabla\varphi . dS = \int_V (\nabla\varphi)^2 dv$ (9)
As we have considered two possible solution for Laplace's equation $\varphi_1$and φ_2, so Their linear combination $\emptyset = \varphi_1 - \varphi_2$ must be the solution of Laplace's equation satisfying the same boundary conditions. Then substituting the $(\varphi_1 - \varphi_2)$ for φ in Eq. (9). Then we get,
$\int (\varphi_1 - \varphi_2)\nabla(\varphi_1 - \varphi_2). dS = \int_V \nabla(\varphi_1 - \varphi_2)^2 dv$ (10)
Now applying the boundary condition $\varphi_1 = \varphi_2$ or $(\nabla\varphi_1)_n = (\nabla\varphi_2)_n$. This shows that the L.H.S. becomes equal to zero. Then
$\int_V \nabla(\varphi_1 - \varphi_2)^2 dv = 0$ (11)
Now again $\varphi_1 = \varphi_2$ shows that integrand must be zero so that integrand vanishes. Thus

$\nabla(\varphi_1 - \varphi_2) = 0$ (12)
Or $\nabla\varphi_1 = \nabla\varphi_2$
This implies that $\varphi_1 = \varphi_2 + constant$ the integral constant has no contribution in the calculation of electric field. Hence the two potential φ_1 and φ_2 give the same electric field distribution under the same boundary conditions. In other words, they are the same or unique solution of Laplace's equation. Thus, the uniqueness theorem is proved.

Let us see its use in method of images. Suppose, the potential can be written as
$\varphi(r) = \varphi_1(r) + \frac{1}{4\pi\varepsilon_o}\int_S \frac{\sigma(r\prime)}{|r-r\prime|}da'$ (13)
Where $\varphi_1(r)$ is a potential that can be calculated easily and the integral part is the contribution from the surface charge density of the conducting surface. In this problem σ is not known but this is the important part of the method of images. Thus the last term in Eq. (13) can be replaced by φ_2. The φ_2 potential is due to definite charge contribution. This situation can be possible if the surfaces of all the conductors coincide with the equipotential surface of the combined potential $\varphi_1 + \varphi_2$. The definite charges producing φ_2 potential are called ***image charges***. These charges do not really exists. The potential $\varphi_1 + \varphi_2$ is valid solution to the problem only in exterior region because their existence appears inside the conductors.
Thus this is a method for obtaining the solution of electrostatic problems using boundary conditions without solving a differential equation. However, this is not applicable to all problems but a large variety of problem come into this category.
Limitations of Uniqueness theorem: Following are the limitations of Uniqueness theorem under which it is valid for the problem solutions:

1. The potential φ should be well defined at all the boundaries i.e. $\varphi = 0$, as $\varphi_1 = \varphi_2$ and the corresponding surface integral vanishes as discussed in Eq. (11).
2. The gradient of net potential, $\nabla\varphi$, should be well defined at all the boundaries such that $\nabla\varphi_1 = \nabla\varphi_2$ as discussed in Eq. (12).
3. For boundaries specifies for conductors, $\nabla\varphi$ should be well defined by using Gauss's law.
4. If the boundary conditions and volume charge density are specified in a system where volume charges are present, then potential at a point can be uniquely specified.

14.3. Point charge close to conducting plane

Consider a conducting plane which is placed in the plane $x - z$ plane of coordinate axis. This plane is grounded so that zero potential could be maintained at its surface. Now consider a charge q on $+y$ axis at distance d form the sheet which is placed at origin. This charge $+q$ is expected to induce $-q$ charge on the surface of conducting plane. But the problem is that it is not concentrated at a point so it is required to take charge density in to account which we do not know quantitatively.

Thus as discussed in the introduction section of this chapter we can consider the $-q$ on the opposite side of the conducting sheet which is called the image of actual $+q$ charge, as shown in Fig. 3. Since the image and actual charge are of equal magnitude and placed at the same distance from origin, so the electric potential will be zero on the conducting plane. Now if we ignore the presence of sheet then still the boundary condition is satisfied and can be worked with the actual charge and its image.

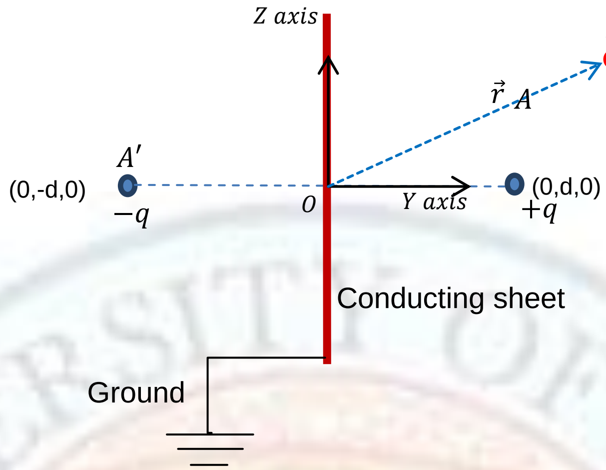


Fig. 3

Now let us consider a point P at position $\vec{r}$ from origin O where we want to calculate the electric field strength and the direction of electric field. The electric potential at P is

$$V = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{|\vec{r}-\vec{d}|} - \frac{q}{|\vec{r}+\vec{d}|}\right] \qquad (14)$$

The unit vectors of the xyz co-ordinate axis is i, j, k. Then the position vector of $+q$ and $-q$ are $+d\hat{\jmath}$ and $-d\hat{\jmath}$ from origin O respectively. Now, $\vec{r}-\vec{d}$ and $\vec{r}+\vec{d}$ are the position vector of P from $+q$ and $-q$. Then

$$V = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{\{x^2+(y-d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(y+d)^2+z^2\}}}\right] \qquad (15)$$

We know that $\vec{E} = -\vec{\nabla}V$, then the electric field can be written as the derivative of potential. So

$$\vec{E}_x = -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{\{x^2+(y-d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(y+d)^2+z^2\}}}\right]$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{x}{\{x^2+(y-d)^2+z^2\}^{3/2}} - \frac{x}{\{x^2+(y+d)^2+z^2\}^{3/2}}\right] \qquad (16)$$

$$\vec{E}_y = -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial y}\left[\frac{1}{\sqrt{\{x^2+(y-d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(y+d)^2+z^2\}}}\right]$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{y-d}{\{x^2+(y-d)^2+z^2\}^{3/2}} - \frac{y+d}{\{x^2+(y+d)^2+z^2\}^{3/2}}\right] \qquad (17)$$

$$\vec{E}_z = -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{\{x^2+(y-d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(y+d)^2+z^2\}}}\right]$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{z}{\{x^2+(y-d)^2+z^2\}^{3/2}} - \frac{z}{\{x^2+(y+d)^2+z^2\}^{3/2}}\right] \qquad (18)$$

The electric field strength

$$\vec{E} = \vec{E_x}\hat{\imath} + \vec{E_y}\hat{\jmath} + \vec{E_z}\hat{k} \qquad (19)$$

The electric field strength can be calculated at some special cases such as if P **is on the conducting sheet** itself then the electric field can be calculated by as

$$\vec{E}_x = -\left(\frac{\partial V}{\partial x}\right)_{y=0}$$

$$= -\left(\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{\{x^2+(d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(d)^2+z^2\}}}\right]\right)_{y=0}$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{x}{\{x^2+d^2+z^2\}^{3/2}} - \frac{x}{\{x^2+d^2+z^2\}^{3/2}}\right] = 0 \qquad (20)$$

$$\vec{E}_y = -\left(\frac{\partial V}{\partial y}\right)_{y=0}$$

$$= -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial y}\left[\frac{1}{\sqrt{\{x^2+(y-d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(y+d)^2+z^2\}}}\right]_{y=0}$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{d}{\{x^2+d^2+z^2\}^{3/2}} + \frac{d}{\{x^2+d^2+z^2\}^{3/2}}\right]_{y=0}$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{2d}{\{x^2+d^2+z^2\}^{3/2}}\right]_{y=0} \qquad (21)$$

$$\vec{E}_z = -\left(\frac{\partial V}{\partial z}\right)_{y=0}$$

$$= -\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{\{x^2+(y-d)^2+z^2\}}} - \frac{1}{\sqrt{\{x^2+(y+d)^2+z^2\}}}\right]_{y=0}$$

$$= \frac{q}{4\pi\varepsilon_o}\left[\frac{z}{\{x^2+d^2+z^2\}^{3/2}} - \frac{z}{\{x^2+d^2+z^2\}^{3/2}}\right] = 0 \qquad (22)$$

Thus, the magnitude of electric field at $y = 0$ is equal to $\frac{qd}{2\pi\varepsilon_o\{x^2+d^2+z^2\}^{3/2}}$ and is directed towards the normal of the conducting surface of the plane i.e. in the direction of $\hat{j}$.

Surface charge density on the conductor

If σ is the induced surface charge density of on the conducting sheet at $P(x, 0, z)$, then the electric field at P can be written as

$$(E)_{y=o} = \frac{\sigma}{\varepsilon_o}$$

$$\sigma = \varepsilon_o(E)_{y=o}$$

$$= \varepsilon_o\frac{-qd}{2\pi\varepsilon_o\{x^2+d^2+z^2\}^{3/2}}$$

$$= -\left(\frac{qd}{2\pi\{r^2+d^2\}^{3/2}}\right)_{y=0} \qquad (23)$$

Induced charge on the conducting plane

We know that the surface charge density is equal to the total charge per unit area i.e.

$$\sigma = \frac{Charge}{Area} \qquad (24)$$

In order to specify the area, let us consider a small ring of radius r and thickness dr, as shown in Fig. 4.

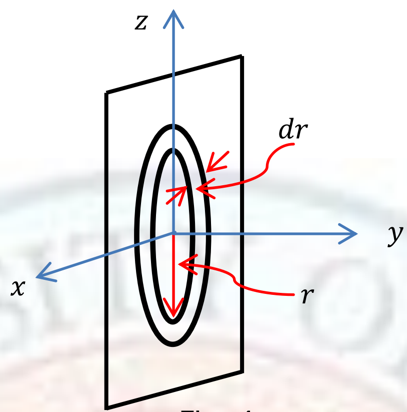


Fig. 4

Then

$$charge\ on\ the\ ring = dq = \sigma \times (area\ of\ ring). \qquad (25)$$

Area of ring is $2\pi rdr$ as shown in Fiq. 3. Now, substituting the expression of area and charge density [Eq. (23)] in Eq. (25). We get,

$$dq = -\left(\frac{qd}{2\pi\{r^2+d^2\}^{3/2}}\right)_{y=0} \times 2\pi rdr \qquad (26)$$

For the total charge on the sheet, we need to integrate Eq. (26) from 0 to ∞

$$dq = -\int_0^\infty \left(\frac{qd}{2\pi\{r^2+d^2\}^{3/2}}\right) \times 2\pi rdr$$

$$= -qd\int_0^\infty \frac{rdr}{\{r^2+d^2\}^{3/2}} \qquad (27)$$

Now solving the integral in Eq. (27) by putting $r = d \times tan\theta$; $dr = d \times sec^2\theta \times d\theta$ in the above Eq. (27). Then,

$$= -qd\int_0^{\pi/2} \frac{d.tan\theta.d.sec^2\theta d\theta}{\{d^2tan^2\theta+d^2\}^{3/2}}$$

$$= -qd\int_0^{\pi/2} \frac{d.tan\theta.d.sec^2\theta d\theta}{d^3\{tan^2\theta+1\}^{3/2}}$$

$$= -q\int_0^{\pi/2} \frac{tan\theta.sec^2\theta d\theta}{\{sec^2\theta\}^{3/2}}$$

$$= -q\int_0^{\pi/2} \frac{tan\theta.sec^2\theta d\theta}{sec^3\theta}$$

$$= -q\int_0^{\pi/2} \frac{tan\theta d\theta}{sec\theta}$$

$$= -q\int_0^{\pi/2} sin\theta d\,\theta$$

$= -q$ (28)

This shows that the total charge on the conducting plane is equal in magnitude and opposite sign to the inducing charge q.

Since the induced charge on the conducting plane is equal and opposite to the actual charge placed closed to the plane, therefore there would be an attractive force in between the conducting plane and the actual charge.

The force between conducting sheet and charge

The nature of force between grounded conductor plane and $+q$ is attractive because the induced charge is of negative and the positive and opposite polarity charges attract each other. Thus the force between charge q and the grounded conducting sheet is same as the force between $+q$ and image charge $-q$. Then, it can be written as

$$\vec{F} = \frac{1}{4\pi\varepsilon_o}.\frac{q(-q)}{(A'A)^2}\hat{j}$$

$$= -\frac{1}{4\pi\varepsilon_o}.\frac{q^2}{(2d)^2}\hat{j}$$

$$= -\frac{1}{16\pi\varepsilon_o}.\frac{q^2}{d^2}\hat{j} \qquad (29)$$

The negative sign of force clarifies the attractive nature of force between conducting plane and the actual charge which is directed towards perpendicular to the plane of conductor i.e. normal to the plane of conductor which is directed along $\hat{j}$.

Work done in removing the charge q to infinity

We have obtained the force between conductor and charge along the line passing through $-q$, plane (grounded) and $+q$. Then, the work done against this force in removing the charge from the distance between two charges, d to infinity, ∞ will be

$$W = -\int_d^\infty F.dr$$

Substituting the expressions of force and position vector r.

$$W = -\int_d^\infty \left(-\frac{1}{16\pi\varepsilon_o}.\frac{q^2}{y^2}\hat{j}\right).\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \frac{q^2}{16\pi\varepsilon_o}.\int_d^\infty \left(\frac{1}{y^2}\right)dy$$

$$= \frac{q^2}{16\pi\varepsilon_o d} \qquad (30)$$

The work done in removing the charge is given by $\frac{q^2}{16\pi\varepsilon_o d}$. Thus, the work done in bringing the charge from infinity to the position d will $-\frac{q^2}{16\pi\varepsilon_o d}$.

14.3. Dipole and conducting planes

Consider a dipole of dipole moment $\vec{p}$ is situated a distance d above an infinite grounded conducting plane (Fig. 5). The dipole makes and angle φ with the perpendicular to the plane. Now we shall try to know the torque on $\vec{p}$. If the dipole is free to rotate, in what orientation will it come to rest?

As shown in Fig. 5, the dipole is situated above the grounded plane at a distance d and make angle φ with plane normal. There is an induced charge development due to presence of $\vec{p}$ on the conducting plane, which would apply the electric field on $\vec{p}$. This electric field will apply a force on the $\vec{p}$ due to which a torque will be a torque on $\vec{p}$.

$\vec{\tau} = \vec{p} \times \vec{E}$ (31)

As the electric field $\vec{E}$ is due to induced charge on conducting plane so there must be equivalent field due to the field, which should be applied by the image. This image dipole will be developed across the conducting plane. Then, let us consider an image $\vec{p}'$ of dipole moment $\vec{p}$ at a distance d in $-z$ direction from conducting plane. Then the electric field at $\vec{p}$ due to $\vec{p}'$ will be

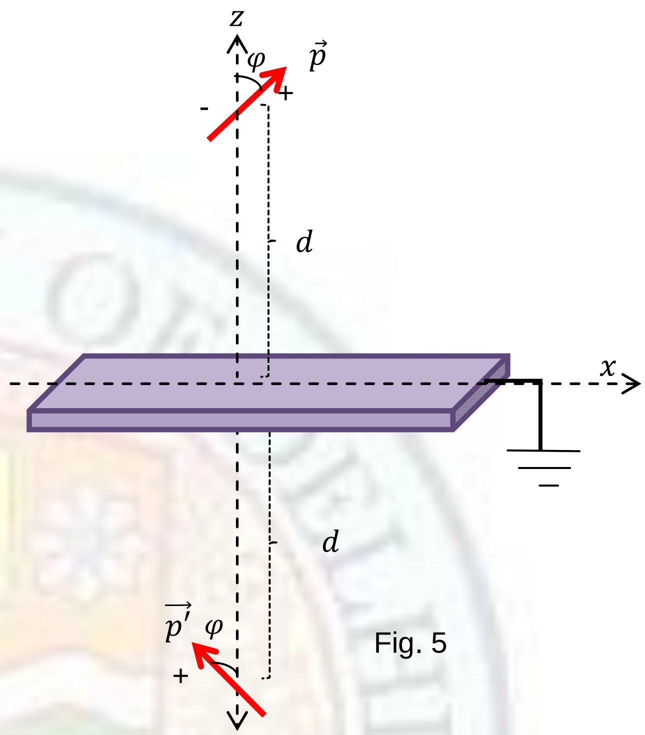


Fig. 5

$$\vec{E} = \frac{1}{4\pi\varepsilon_o r^3}\left[3(\vec{p}'.\hat{r})\hat{r} - \overrightarrow{p'}\right]$$

The vector $\hat{r}$ is along $\vec{p}$ making an angle with φ. The image dipole moment can be expressed as

$$\overrightarrow{p'} = -p' \sin\varphi\ \hat{x} + p' \cos\varphi\ \hat{z}$$

So the electric field can be as

$$\vec{E} = \frac{1}{4\pi\varepsilon_o r^3}[3p' \cos\varphi\ \hat{z} - (-p' \sin\varphi\ \hat{x} + p' \cos\varphi\ \hat{z})]$$

$$= \frac{1}{4\pi\varepsilon_o r^3}[2p' \cos\varphi\ \hat{z} + p' \sin\varphi\ \hat{x}]$$

Putting $r = 2z$, therefore

$$\vec{E} = \frac{1}{4\pi\varepsilon_o (2z)^3}[2p' \cos\varphi\ \hat{z} + p' \sin\varphi\ \hat{x}] \qquad (32)$$

The expression of actual dipole can be written as

$$\vec{p} = p \sin\varphi\ \hat{x} + p \cos\varphi\ \hat{z} \qquad (33)$$

Thus the torque in Eq. (31) can be written as

$$\vec{\tau} = \vec{p} \times \vec{E}$$

In component form $\vec{p} = \hat{x}p_x + \hat{y}p_y + \hat{z}p_z$; and $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$ then torque can be written as

$\vec{\tau} = \hat{x}(p_y E_z - p_z E_y) - \hat{y}(p_x E_z - p_z E_x) + \hat{z}(p_x E_y - p_y E_x)$

Since electric field dipole moment in y direction are absent. Then

$\vec{\tau} = -\hat{y}(p_x E_z - p_z E_x)$ (34)

Substituting the expressions of dipole moment from Eq. (33) and electric field from Eq. (32) in Eq. (34) expression

$\vec{\tau} = -\hat{y}\left(p \sin\varphi \ \frac{2p\cos\varphi}{4\pi\varepsilon_o (2z)^3} - p\cos\varphi \ \frac{p\sin\varphi}{4\pi\varepsilon_o (2z)^3}\right)$

$= \frac{-p^2}{32\pi\varepsilon_o z^3}[2\sin\varphi \ .\cos\varphi - \sin\varphi \, .\cos\varphi\,]\hat{y}$

$= \frac{-p^2}{32\pi\varepsilon_o z^3}[\sin\varphi \ .\cos\varphi\,]\hat{y}$ (35)

Expression in Eq. (35) suggests that the torque has the direction along $y\, axis$ (out of page) and then produces the twist as shown in Fig. 5.

In order to find the orientation of dipole let us check the angular part of the torque in Eq. (35)

$\sin\varphi \ .\cos\varphi = \frac{1}{2}\sin(2\varphi)$

Then the torque becomes

$\vec{\tau} = \frac{-p^2}{64\pi\varepsilon_o z^3}[\sin(2\varphi)]\hat{y}$

If $\varphi = 0 \; or\; \pi$ then there will be no torque in the dipole $\vec{p}$. But there will be a torque if a small change in the angle is made due to which dipole tends to rotate back to its equilibrium state.

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14.4. Point charge close to conducting sphere

Consider a grounded sphere of radius R at origin and a $+q$ charge placed at a distance l from the origin outside and close to this sphere as shown in Fig. 6. Now our purpose is to find out the electric field at a point P outside the sphere at position r. Once the electric field is known then other properties like surface charge density, force on the sphere etc. can also be calculated by using the electric field. For this purpose, first we need to calculate the potential.

Now consider an image charge q' inside the sphere at a distance of s from the origin. Let us consider the following:

OM=s, OL=l, OP'=R.

Since the sphere is grounded so the potential at the surface of the sphere is zero i.e. $V = 0$ at position $r = R$. Also the potential at infinity is zero i.e. $V = 0$ at $r = \infty$.

So the expression for potential at P' is given as

$V(P') = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{LP'} + \frac{q'}{MP'}\right] = 0$

$q' = -q\frac{MP'}{LP'}$ (36)

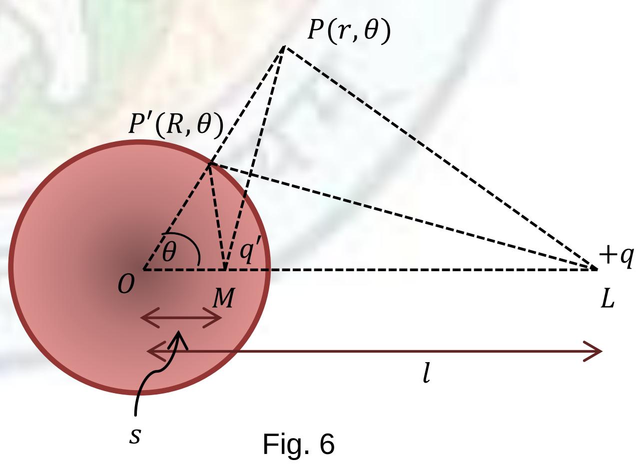


Fig. 6

In order to know MP' and LP', consider the triangle OLP' and

$(LP')^2 = (OP')^2 + (OL)^2 - 2(OP')(OL)cos\theta$

$(LP')^2 = (R)^2 + (l)^2 - 2(R)(l)cos\theta$

$LP' = \sqrt{(R)^2 + (l)^2 - 2(R)(l)cos\theta}$ (37)

And in triangle OMP'

$(MP')^2 = (OP')^2 + (OM)^2 - 2(OP')(OM)cos\theta$

$(MP')^2 = (R)^2 + (s)^2 - 2(R)(s)cos\theta$

$MP' = \sqrt{(R)^2 + (s)^2 - 2(R)(s)cos\theta}$ (38)

Substituting Eq. (37) and (38) in Eq. (36), we get

$q' = -q\frac{\sqrt{(R)^2+(s)^2-2(R)(s)cos\theta}}{\sqrt{(R)^2+(l)^2-2(R)(l)cos\theta}}$ (39)

Now let us choose that $s = \frac{R^2}{l}$, then Eq. (39) becomes

$$q' = -q\frac{\sqrt{(R)^2+\left(\frac{R^2}{l}\right)^2-2(R)(\frac{R^2}{l})cos\theta}}{\sqrt{(R)^2+(l)^2-2(R)(l)cos\theta}}$$

$$q' = -q\frac{R}{l}\frac{\sqrt{(d)^2+(R)^2-2(R)(l)cos\theta}}{\sqrt{(R)^2+(l)^2-2(R)(l)cos\theta}}$$

$q' = -q\frac{R}{l}$ (40)

Eq. (40) reveals that the magnitude of image charge is not equal to the actual charge but it has the magnitude of $-q\frac{R}{l}$ situated at a distance of $\frac{R^2}{l}$ from the center of the sphere in the line joining the center to actual charge $+q$.

Now let us consider a point $P(r,\theta)$ outside the sphere where the potential is to be find out. The potential at this point can be written as

$$V(P) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{LP} + \frac{q'}{MP}\right]$$

But $LP = \sqrt{(r)^2 + (l)^2 - 2(r)(l)\ cos\theta}$ and $MP = \sqrt{(r)^2 + (s)^2 - 2(r)(s)\ cos\theta}$

Therefore

$$V(P) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{(r)^2+(l)^2-2(r)(l)cos\theta}} + \frac{q'}{\sqrt{(r)^2+(s)^2-2(r)(s)cos\theta}}\right]$$

$$V(P) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{(r)^2+(l)^2-2(r)(l)cos\theta}} + \frac{-q\frac{R}{l}}{\sqrt{(r)^2+\left(\frac{R^2}{l}\right)^2-2(r)(\frac{R^2}{l})cos\theta}}\right]$$

$$V(P) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{(r)^2+(l)^2-2(r)(l)cos\theta}} - \frac{q}{\sqrt{\left(\frac{rl}{R}\right)^2+(R)^2-2(r)(l)cos\theta}}\right] \quad (41)$$

Now we can calculate the electric field strength at point $P(r,\theta)$ in the component form along the normal OP. Then

$$E_r = -\frac{\partial V}{\partial r}$$

$$E_r = -\frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{(r)^2+(l)^2-2(r)(l)cos\theta}} - \frac{q}{\sqrt{\left(\frac{rl}{R}\right)^2+(R)^2-2(r)(l)cos\theta}}\right]$$

$$E_r = \frac{q}{4\pi\varepsilon_o}\left[\frac{(r-l.cos\theta)}{[(r)^2+(l)^2-2(r)(l)cos\theta]^{3/2}} - \frac{r\frac{l^2}{R^2}-l.cos\theta}{\left[\left(\frac{rl}{R}\right)^2+(R)^2-2(r)(l)cos\theta\right]^{3/2}}\right] \quad (42)$$

And $E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}$

$$E_\theta = \frac{q}{4\pi\varepsilon_o}\frac{1}{r}\left[\frac{l.sin\theta}{[(r)^2+(l)^2-2(r)(l)cos\theta]^{3/2}} - \frac{l.sin\theta}{\left[\left(\frac{rl}{R}\right)^2+(R)^2-2(r)(l)cos\theta\right]^{3/2}}\right] \quad (43)$$

Then, the electric field on the surface of the sphere can be calculated by substituting the $r = R$ in Eq. (42) and (43)

$$E_r = \frac{q}{4\pi\varepsilon_o}\left[\frac{R^2-l^2/R}{[(l)^2+(R)^2-2(R)(l)cos\theta]^{3/2}}\right]_{r=R} \quad (44)$$

And

$$E_\theta = \frac{q}{4\pi\varepsilon_o}\frac{1}{R}\frac{\partial}{\partial\theta}\left[\frac{l.sin\theta}{[(r)^2+(l)^2-2(r)(l)cos\theta]^{3/2}} - \frac{l.sin\theta}{\left[\left(\frac{rl}{R}\right)^2+(R)^2-2(r)(l)cos\theta\right]^{3/2}}\right]_{r=R}$$

$$E_\theta = 0$$

Thus the magnitude of electric field at r=R is $\frac{q}{4\pi\varepsilon_o}\left[\frac{R^2-l^2/R}{[(l)^2+(R)^2-2(R)(l)cos\theta]^{3/2}}\right]_{r=R}$.

Surface charge density of induced charge on the surface of sphere:

Let σ be the surface charge density on the surface of the sphere. Then,

$$E = (E)_{r=R} = \frac{\sigma}{\varepsilon_o}$$

$$\therefore\ \sigma = \varepsilon_o(E)_{r=R}$$

$$= -\varepsilon_o.\frac{q}{4\pi\varepsilon_o}\left[\frac{R^2-l^2/R}{[(l)^2+(R)^2-2(R)(l)cos\theta]^{3/2}}\right]_{r=R} \quad (45)$$

Force between the sphere and the point charge:

The force between the sphere of radius R and point charge $+q$ is the same as between $+q$

and $q' = -q\frac{R}{l}$. Therefore, the force

$$F = \frac{qq'}{4\pi\varepsilon_o (ML)^2}$$

$$= \frac{q\left(-q\frac{R}{l}\right)}{4\pi\varepsilon_o (l-s)^2}$$

$$= \frac{q\left(-q\frac{R}{l}\right)}{4\pi\varepsilon_o \left(l-\frac{R^2}{l}\right)^2}$$

$$= -\frac{q^2 Rl}{4\pi\varepsilon_o (l^2-R^2)^2} \text{ Newton.} \qquad (46)$$

The force is ***attractive*** as confirmed by negative sign.

14.5. Conducting sphere in a uniform electric field

We can consider a uniform electric field between two point charges of magnitude $\pm q$ placed at infinity. If we consider the position of two charges at $z = \mp l$ i.e. $+q$ is placed at $-l$ and $-q$ is placed at $+l$ in the direction of z axis as shown in the Fig. 7. Fig. 7(a) shows the direction of electric field at a point due to the combine effect of two charges. The net electric field E_o will be approximately $2q/4\pi\varepsilon_o l^2$ parallel to z axis.

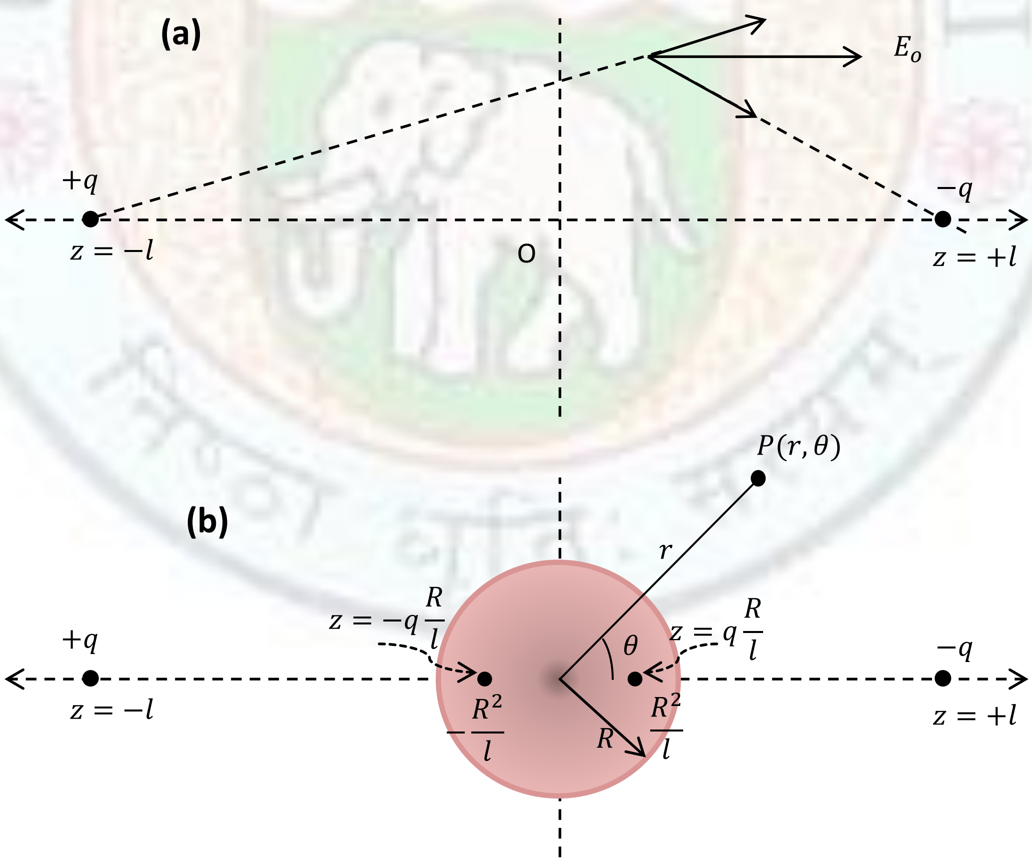


Fig. 7

Now consider a sphere [Fig. 7(b)] at origin in the system shown in Fig. 7(a). We have to find out the **potential** at any arbitrary point $P(r,\theta)$ outside the sphere placed at position r and an angle θ with respect to $z\,axis$. The potential at this point will be due to the combine effect of $\pm q$ and their images inside the sphere at $\mp\frac{R^2}{l}$. The charges of the images will be $\pm q\frac{R}{l}$ respectively as discussed in the above section. Then the potential developed at $P(r,\theta)$ can be written as

$V(P) = Potential\ due\ to\ real\ charges + Potential\ due\ to\ their\ images\ inside\ the\ sphere$

$$= \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{(r^2+l^2+2rl.cos\theta)}} - \frac{q}{\sqrt{(r^2+l^2-2rl.cos\theta)}} - \frac{q\frac{R}{l}}{\sqrt{\left(r^2+\frac{R}{l^2}^4+2r\frac{R^2}{l}.cos\theta\right)}} + \frac{q\frac{R}{l}}{\sqrt{\left(r^2+\frac{R}{l^2}^4-2r\frac{R^2}{l}.cos\theta\right)}}\right] \qquad (47)$$

Since, l is much less than position vector r so we can expand the denominators of first two terms taking l^2 common and factor the r after l as

$$\frac{1}{\sqrt{(r^2+l^2+2rl.cos\theta)}} = \frac{1}{l}\left[1+\frac{r^2}{l^2}+2\frac{r}{l}.cos\theta\right]^{-1/2}$$

$$\approx \frac{1}{l}\left[1-\frac{r}{l}.cos\theta\right]$$

And $\frac{1}{\sqrt{(r^2+l^2-2rl.cos\theta)}} = \frac{1}{l}\left[1+\frac{r^2}{l^2}-2\frac{r}{l}.cos\theta\right]^{-1/2}$

$$\approx \frac{1}{l}\left[1+\frac{r}{l}.cos\theta\right]$$

Also the denominators in last two term of Eq. (47) can be expanded by taking r^2 common as

$$\frac{1}{\sqrt{\left(r^2+\frac{R}{l^2}^4+2r\frac{R^2}{l}.cos\theta\right)}} = \frac{1}{r}\left[1+\frac{R}{r^2l^2}^4+2\frac{R^2}{rl}.cos\theta\right]^{-1/2}$$

$$\approx \frac{1}{r}\left[1-\frac{R^2}{rl}.cos\theta\right]$$

And $\frac{1}{\sqrt{\left(r^2+\frac{R}{l^2}^4-2r\frac{R^2}{l}.cos\theta\right)}} = \frac{1}{r}\left[1+\frac{R}{r^2l^2}^4-2\frac{R^2}{rl}.cos\theta\right]^{-1/2}$

$$\approx \frac{1}{r}\left[1+\frac{R^2}{rl}.cos\theta\right]$$

Using these expansions in Eq. (47), we get

$$V(P) = \frac{1}{4\pi\varepsilon_o}\left[\frac{2qR^3}{l^2r^2}cos\theta - \frac{2qr}{l^2}cos\theta\right] \qquad (48)$$

At $l\to\infty$, the electric field is **uniform**. Let us denote this field as $E_o = 2q/4\pi\varepsilon_o l^2$. Then Eq. (48) can be written as

$$V(P) = -E_o\left[r - \frac{R^3}{r^2}\right]cos\theta \qquad (49)$$

As $z = rcos\theta$, so Eq. (49) can be written as

$$V(P) = -E_o z + E_o\frac{R^3 z}{r^3} \qquad (50)$$

In Eq. (50), the first term is potential due to the actual charges whereas the second term is potential due to the images of actual charges i.e. due to the induced surface charge density on the sphere.

The surface charge density can be calculated as

$$\sigma = \varepsilon_o E = -\varepsilon_o \frac{\partial V}{\partial r}\bigg|_{r=R}$$

Using Eq. (50), we get

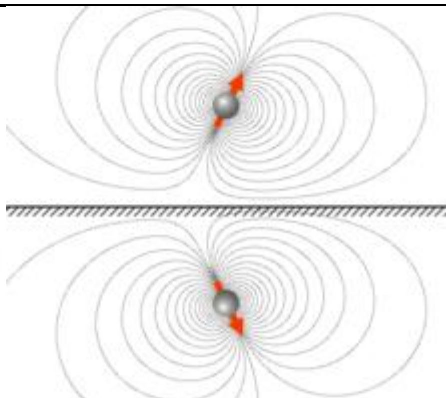
$$\sigma = -3\varepsilon_o E_o cos\theta \qquad (51)$$

The surface integral of surface charge density will vanish representing that there will be no difference a grounded and an insulated sphere.

Value addition: Did you Know

Magnetostatic application of method of image

Body text: The method of electrical image may also be applied to magnetic field calculation in the case of magnetostatics other than electrostatics. Recently, the method of image technique has been utilized to find out the magnetic field of a magnet in the vicinity of superconducting surface. It has been assumed that if the superconductor is an ideal diamagnet (into which the magnetic field does not penetrate), the mirror image of the magnet will be of same sign and will have a magnetization vector that is mirrored, but of the same sign. Here the case is different as the mirror image of magnet is of same sign so the force between superconductor and magnet is repulsive.



A magnetic dipole over the superconducting surface. The field between the magnet and surface is the same as between this magnet and a symmetric one.

Source: http://en.wikipedia.org/wiki/Method_of_images

14.6. Problems

Question Number	Type of question
1	Multiple Choice Questions

Question

14.1.1. The induced charge is assumed at ____________ from the grounded conducting plane but in opposite direction to the actual charge.
1) Unequal distance
2) Oblique distance
3) In plane to the conduction plane
4) Equal distance

14.1.2. In general, the sign and magnitude of induced image charge in a system of grounded conducting plane and a point charge are
1) Same and equal to actual charge
2) Same and unequal to actual charge
3) Opposite and unequal to the actual charge
4) Opposite and equal to the actual charge

14.1.3. The sign and magnitude of induced charge in a system of sphere and point charge are
1) Same and equal to actual charge Non-polar
2) Same and unequal to actual charge
3) Opposite and unequal to the actual charge
4) None of the above

14.1.4. The induced charge on the surface of a sphere of radius R, which is

placed at a distance of l from a point charge in the direction of $+z$ axis, can be assumed at 1) center 2) R^2/l 3) $-R^2/l$ 4) None

Answers	1) 4 2) 4 3) 3 4) 3

14.7.

Question Number	Type of question
2	True/False

Questions

14.1.1. In all cases of method of images, the image charge is always found to be of opposite nature real charge available in the system. True/False

14.1.2. In the case of conducting sphere and a real charge placed close to it, the magnitude of image charge is independent the position of real and image charge. True/False

14.1.3. The method of electrical image is not limited to a system of point charges only but it can be extended to line charges also. True/False

14.1.4. The method of electrical images is applicable to conducting object placed close to a point charge and not applicable to dielectric objects placed closed to a point charge. True/False

14.1.5. The force developed between a conducting object and a point charge is always attractive. True/False

Answers	1) True 2) False 3) True 4) False 5) True

Question Number	Type of question
3	Subjective

Question

1. What is an electrical image method? A point charge is situated near an infinite plane earthed conductor. Apply the method of image to calculate: a) surface charge density induced on the plane, b) the force between plane and the charge.

2. An electron is at distance of 10 Angstrom from a n infinite plane conductor. Calculate the force experienced by the electron and the work done in moving it to the infinite from conductor.

References and further readings

1. **J. R. Reitz, F. J. Milford, and R. W. Cristy, "Foundations of electromagnetics theory",Narosa Publishing House, New Delhi.**
2. **David J. Griffiths, Introduction to Electrodynamics, 3rd edition.**
3. **J. D. Jackson (1998). *Classical Electrodynamics* (3rd ed.). John Wiley & Sons. ISBN 978-0-471-30932-1.**
4. **http://en.wikipedia.org/wiki/Dielectric**