Discipline Course-I

Semester-II

Paper No: Electricity and Magnetism

Lesson: Magnetic Field Lesson 3.1: Magnetic Field

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Learning Objectives

After going through this chapter, the reader will

i. Get some idea about the discovery and history of magnetism
ii. Know about the initial theories of magnetism
iii. Understand the meaning of 'magnetic field'
iv. Know how two current elements exerts forces on each other
v. Be able to state and understand 'Biot-Savart' law
vi. Know the method of applying Biot-Savart law for calculating the magnetic field due to a current distribution
vii. Learn to calculate the magnetic field
    a. due to a straight current carrying wire
    b. due to a circular loop
viii. Know about the equivalence of a current loop to a magnetic dipole

Introduction

We all know that the phenomenon of magnetism was probably discovered by a Creton shepherd named Magnus. He, while taking is sheep along, in Magnesia, in northern Greece, was startled to find his shoe nails and his iron stick getting stuck on a black rock. Subsequently, investigations revealed that the block rock had magnetite – a magnetic material (Fe_3O_4) in it. The property of this rock, of attracting iron, soon had many interesting applications.

It was the Chinese, who in the 12th century, discovered that the magnetite rock also had an interesting directional property. When free to move, suitably designed pieces of this rock, always moved so as to have the designed point stay along the north-south direction. One could, therefore, use this rock for navigational purpose. Because of this directional use, the rock was also named as loadstone.

Initial theories

In the initial stages, it was thought that the theories of the phenomenon of magnetism could be developed and studied on the same lines as the corresponding theories in electricity. This belief stemmed, perhaps, from the fact that there appeared to be two distinct 'poles' on a magnet, their behavior was similar to that of the two kinds of charges – the positive and the negative charges – in electricity. The law: '*like poles repel while unlike poles attract each other*', appeared to be an exact analogue of: '*like charges repel while unlike charges attract each other*'. This analogy led to the suggestion that the north pole of a magnet could be regarded as the equivalents of the positive charge while the South Pole could be regarded as the equivalent of a negative charge. Following this suggestion, a 'Coulomb's Law' was postulated for the force between the two magnetic poles. The theories of the magnetic dipole and other magnetic entities were developed, in a way, similar to the corresponding theories of the 'electric dipole' and other electrical entities. For quite some time it was firmly believed that magnetism and electricity were two parallel phenomena having essentially the same basic kind of theories.

This smug belief, about the phenomenon of magnetism received a severe jolt when it was discovered that the north and the south poles could not be separated out. Irrespective of the number of times a magnet was cut into two halves, each half continued to remain a 'complete magnet' having its own north and south poles. There never were any' isolated north pole' or' isolated south pole' in nature!

Once it was realized that isolated magnetic poles do not exist in nature, the very basis of the 'similar to electricity' theory of magnetism seemed to collapse. It was necessary to think afresh about the basic cause or source of magnetism. For quite some time, there was no clear cut answer to this query.

Oersted's discovery

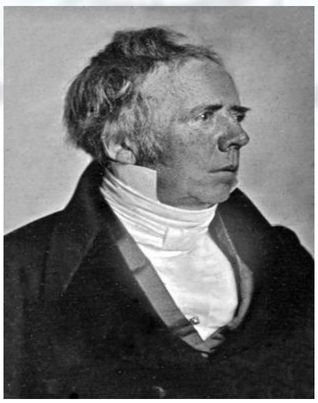
It was in the year 1819-20 that Hans Christian Oersted, a professor at the University of Copenhagen (somewhat accidentally) made a chance discovery that later on led to a revision of the theories of magnetism. He discovered that a current carrying wire affects a nearby compass needle in much the same way as a magnet affects it. Apparently, a current carrying wire behaves like a magnet. One could think of electric currents, or moving charges, as a source of magnetic field. It is this idea that now forms the basis of all our theories of magnetism. We now say:
'It is electric currents or moving charges that have to be regarded as the basic source of magnetism'.

The basic law, for the magnetic field due to a current element, was suggested, in 1820, by Jean Baptiste Biot and Felix Savart. Both of them came up with an equation that gives the magnetic field, at any point, due to a current carrying wire.

The year 1820 also saw Andre'-masie Ampere making an important discovery that is now contained in what is now referred to as 'Ampere's Circuital Law'. This law related the current flowing through the perimeter of a closed path with the magnetic field circulating in that path. Ampere's circuital law, as this law is now referred to, provides us with a simple and quick method of calculating magnetic field due to current distributions having a high degree of symmetry in them.

Did you know?

1. We all know that certain birds have a built-in sense of direction. They fly over vast distances, in different weathers and yet never seem to lose their way. It has now been discovered that this sense of direction associated with some bird families is not restricted to birds alone. Certain butterflies and insects also have a similar built-in sense of direction. All this has now been associated with 'magnetism' – the magnetism in their bodies and the magnetism of earth.
2. Brief life sketch of Oersted

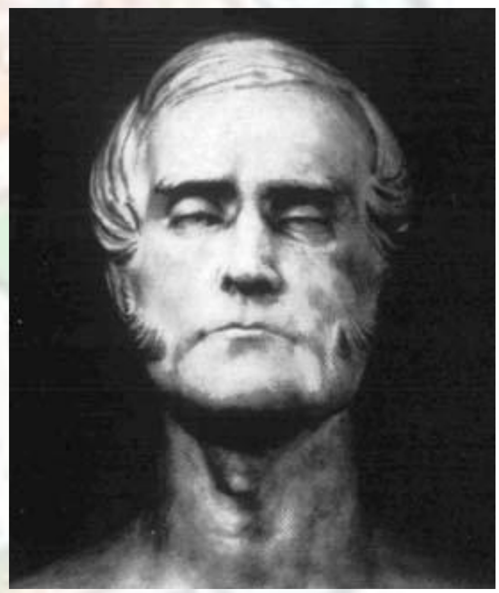


- Hans Christian Oersted was a Danish physicist and chemist who discovered that electric currents create magnetic fields.
- The oersted (Oe), is the cgs unit of magnetic field strength and is named after him.
- He was born in Rudkobing and developed a keen interest in science at a very young age.
- He received most of his early education through self-study at home, going to Copenhagen in 1793 to take entrance exams for the University of Copenhagen.
- By 1796 Oersted had been awarded honors for his papers in both aesthetics and physics.
- He earned his doctorate in 1799 for a dissertation based on the works of Kant entitled "The Architectonics of Natural Metaphysics".
- He became a professor at the University of Copenhagen in 1806 and worked in areas related to electric currents and acoustics.
- In 1800, Alessandro Volta discovered a galvanic battery which inspired Oersted to think about the nature of electricity and to conduct his first electrical experiments.
- Oersted was just 43 when he made the great discovery.

Source: http://en.wikipedia.org/wiki/Hans_Christian_Ørsted

3. Brief life sketch of Jean-Baptiste Biot and Felix Savart


Jean-Baptiste Biot


Felix Savart

a.) Jean-Baptiste Biot (21 April 1774 – 3 February 1862)

- His father was Joseph Biot, whose ancestors were farmers in Lorraine.
- Jean-Baptiste was educated at the college of Louis-le-Grand in Paris, where he specialized in classics. He completed his studies at Louis-le-Grand in 1791 following which, he took private lessons in mathematics at the Collège de France. Joseph Biot then sent his son to Le Havre to become a clerical assistant to a merchant. His job there consisted of copying vast numbers of letters which bored Biot so much that he volunteered for the army.
- He joined the French army in September 1792, and served in the artillery, at the Battle of Hondschoote, wherein, the French defeated the British and Hanoverian soldiers besieging Dunkirk. After the battle Biot, suffered from an illness and decided to leave the army and return to his parents.

- He became Professor of Mathematics at the École Centrale de l'Oise at Beauvais in 1797, where he got a great career support from Laplace.
- Although Biot was a professor of mathematics, he had great interest in experimental physics.
- He was one of the few person to make a balloon flight and also established the reality of meteorites.
- In 1809 Biot was appointed Professor of Physical Astronomy at the Faculty of Sciences. He held this position for over fifty years.
- Biot studied a wide range of mathematical topics, mostly on the applied mathematics side. He made advances in astronomy, elasticity, electricity and magnetism, heat and optics on the applied side while, in pure mathematics, he also did important work in geometry.
- Biot, together with Felix Savart, discovered that the intensity of the magnetic field set up by a current flowing through a wire varies inversely with the distance from the wire.
- Biot had a keen interest in topics related to 'light'. He discovered the laws of rotary polarization by crystalline bodies. Having discovered these laws he used them in analysis of saccharine solutions using an instrument called a polarimeter which he invented. For this work on the polarization of light passing through chemical solutions he was awarded the Rumford Medal of the Royal Society of London in 1840.

b.) Felix Savart (30 June 1791 – 16 March 1841)

- Felix Savart was the son of Gérard Savart, an engineer at the military school of Metz. At the military hospital at Metz, Savart studied medicine and later he went on to continue his studies at the University of Strasbourg, where he received his medical degree in 1816.
- Savart became a professor at Collège de France in 1836 and was the co-originator of the Biot-Savart Law, along with Jean-Baptiste Biot. Together, they worked on the theory of magnetism and electrical currents.
- Félix Savart also studied acoustics and developed the Savart wheel which produces sound at specific graduated frequencies using rotating disks.

Source: http://en.wikipedia.org/wiki/Félix_Savart

4. Brief life sketch of André-Marie Ampère (20 January 1775 – 10 June 1836)

- André-Marie Ampère's father, Jean-Jacques Ampère, was a prosperous man who owned a home in Lyon and a country house in Poleymieux.
- It has been claimed that Ampère had mastered all known mathematics by the age of twelve years
- While still only 13 years old Ampère submitted his first paper to the Académie de Lyon. This work attempted to solve the problem of constructing a line of the same length as an arc of a circle. His method involves the use of infinitesimals. He even composed a treatise on probability.
- He also worked on the wave and corpuscular theory of light light.
- H tried to give a combined theory of electricity and magnetism and gave the circuital law for small closed circuits. He is regarded as one of the main founders of electromagnetism.

Magnetic Field

Definition of Magnetic Field (B)

A layman's or a qualitative definition of magnetic field would be to think of it as the *region of space in which the effects of a magnet, or a current carrying wire/coil can be felt or experienced.* This, of course, is not sufficient. A quantitative definition of magnetic field was provided by the rule that gives the force, experienced by a moving charge in a magnetic field (B). This rule, as we now know, is the rule giving the magnetic force part of what is known as the *Lorentz force*, on a moving charge. For a charge q, moving with a velocity v, the total Lorentz force (F), acting on it, is given by

$\mathbf{F} = q\,(E + \mathbf{v} \times \mathbf{B})$

Where **E** is the electric field and **B** is the magnetic field in the region of space in which the charge is present. The magnetic part of this force is:

$\mathbf{F_m} = q\,(\mathbf{v} \times \mathbf{B})$

Hence,

$|\mathbf{F_m}| = q\, v\, B \sin\theta$

We can use this expression to define the strength of magnetic field.

(i) Magnitude of magnetic field:
We note that,

if $q = 1$ unit, $v = 1$m/s and $\sin\theta = \sin \pi/2 = 1$,

⇨ $|\mathbf{F_m}| = B$

Thus the magnitude of the magnetic field, at any point in space, equals the force experienced by a unit charge moving with a unit velocity along a direction that is normal to the direction of magnetic field at that point.

(ii) Direction of the magnetic field at any point:

From, $\mathbf{F_m} = q\,(\mathbf{v} \times \mathbf{B})$, we see that $|\mathbf{F_m}| = 0$ if $v \parallel B$

Thus the direction of the magnetic field at any point is the direction along which a charge moving in that magnetic field experiences no force or zero force.

We thus see that the magnetic part of the Lorentz force expression helps us to define both the direction and the magnitude of the magnetic field at any given point in space.

(iii) The SI Unit of Magnetic Field

The SI unit of magnetic field is the tesla (T), which is is defined as follows:

The magnetic field, at any point in space, equals one tesla if a charge of one coulomb moving with a velocity of 1m/s along a direction normal to the direction of the magnetic field at that point experiences a force of one newton.

It may be kept in mind that tesla is quite a large unit of magnetic field. The magnetic fields often encountered in the laboratory are usually of the order of a few millitesla only.

Did You Know?

1. The unit of magnetic field, defined before the advent of the SI units was the gauss (G). It turns out that,

$1G = 10^{-4}$ T => $1T = 10^{4}$ G

The 'gauss' is thus quite small in comparison to the tesla. It is a unit that has, still, not been completely given up as at present.

2. Magnets and magnetic fields are now a very important and integral part of our life. A very large number of the numerous technological advances and devices that have come up in the recent part make use of magnetic fields. These include, among others, the magnetic resonance imaging (MRI) scan devices in the field of medicine and the computer chips in computer science. Relatively earlier devices like the television and the telephone also made use of magnetic fields.

Magnetic Force between Current Elements

Once it was discovered that current carrying wires can behave like magnets, it was but natural to realize that two current carrying elements would exert forces on each other. Quantitative expressions, for

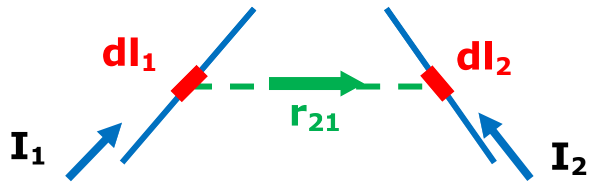
i. the force acting on a current element in a magnetic field , and

ii. for the force between two current elements

are now available. For the present, we just state these expressions.

(i) The force, **dF** (See figure 1), on a current element **dl** (carrying a current I), in a magnetic field, **B**, is given by,

$$\mathbf{dF} = I\, \mathbf{dl} \times \mathbf{B}$$



(Figure 1)

(ii) The force, $\mathbf{dF_{21}}$, on a current element $\mathbf{dl_2}$ (carrying a current I_2), due to another current element $\mathbf{dl_1}$ (carrying a current I_1) is given by

$$\mathbf{dF_{21}} = \frac{\mu 0}{4\pi} I_1 I_2\, \mathbf{dl_2} \times (\mathbf{dl_1} \times \mathbf{r_{21}})/|\mathbf{r_{21}}|^3$$

here, $\mathbf{r_{21}}$ is the position vector of the current element $\mathbf{dl_2}$ with respect to the current element $\mathbf{dl_1}$.

We would have a similar expression for the force $\mathbf{dF_{12}}$, experienced by the current element $\mathbf{dl_1}$, due to the current element $\mathbf{dl_2}$.

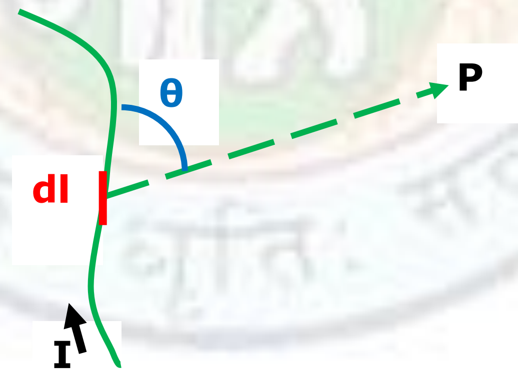
$$\mathbf{dF_{12}} = \frac{\mu 0}{4\pi} I_1 I_2\, \mathbf{dl_1} \times (\mathbf{dl_2} \times \mathbf{r_{12}})/|\mathbf{r_{12}}|^3$$

Here, $\mathbf{r_{12}}$ is the position vector of the current element $\mathbf{dl_1}$ with respect to the current element $\mathbf{dl_2}$.

Biot-Savart Law

As we have already noted, it was the work of two French physicists – Jean Baptiste Biot and Felix Savart that provided us with the basic law for calculating the magnetic field due to a current element.

Through experiments, it was discovered that the magnitude (dB) of the magnetic field, due to a length element (dl) of a current carrying wire, at a field point, defined as shown (in figure 2), is



(Figure 2)

(i) directly proportional to the strength of the current, I, flowing through it.

(ii) directly proportional to the length, dl, of the current element.

(iii) directly proportional to sin θ where θ is the angle between the current element dl and the position vector, r, of the field point P, with respect to the current element dl.

(iv) inversely proportional to the square of the distance, r, of the field point from the current element.

Combining all these results, we get,

$dB \propto I\, dl \sin\theta / r^2$

or $dB = k\, I\, dl \sin\theta / r^2$

In SI units, the constant K is written as $\mu_0 / 4\pi$. Here μ_0 is known as the magnetic permeability of free space and has been assigned a value of $4\pi \times 10^{-7}$ tesla-meter/ampere.

Therefore, $dB = \frac{\mu_0}{4\pi} (I\, dl \sin\theta) / r^2$

The direction of the magnetic field, **dB**, due to the current element dl, at a field point P, (defined by its position vector, r, with respect to the current element dl), is found to be perpendicular to the plane formed by dl and r. It can, therefore, be specified by either of the cross products (dl X r) and (r X dl). Of these two possibilities, the accepted correct direction of dB corresponds to that of the cross product dl X r.

We can, therefore, say that the magnetic field, dB, due to a current element dl, at a field point, defined by its, position vector r with respect to the current element is given by

$$\mathbf{dB} = \frac{\mu_0}{4\pi} I\, \mathbf{dl} \times \mathbf{r} / |\mathbf{r}|^3$$

We refer to this result as the mathematical formulation of the Biot-Savart rule.

The total magnetic field, B, due to a finite length, l, of a current carrying wire, is given by,

$$\mathbf{B} = \int \frac{\mu_0}{4\pi} I\, \mathbf{dl} \times \mathbf{r} / |\mathbf{r}|^3$$

where the integration limits would be defined by the shape and length of the current carrying wire.

It is easy to realize that the calculation of the magnetic field, due to a general current distribution would be more difficult and challenging than the corresponding calculations for the electrostatic field. This is because of the presence of a vector cross product term as the integral. However, this very cause of difficulty can become a cause of simplicity in some special cases. These are cases of those current distributions for which the plane, defined by (dl X r), remains as one and the same plane for all the current elements of the charge distribution.

Some Simple Applications of the Biot-Savart Law

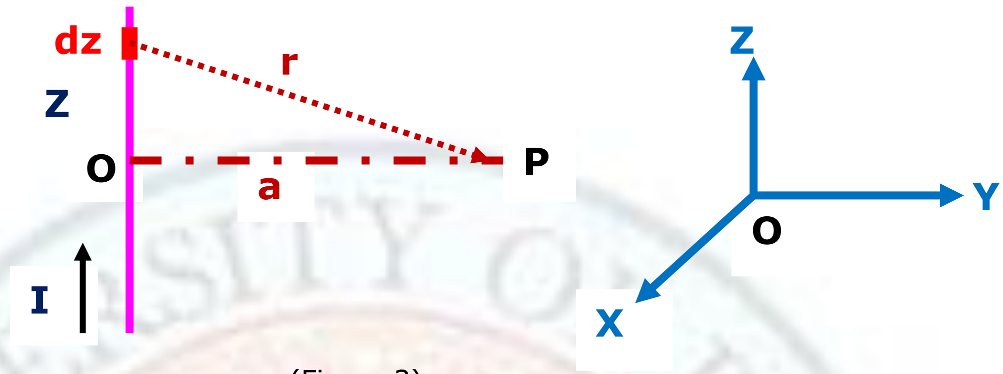
Let us now apply the Biot-Savart law to calculate the magnetic field due to a

(i) Straight current carrying wire

(ii) Circular coil

Straight Wire

Consider a straight wire carrying a current I. We calculate the magnetic field, due to it, at a field point P, distant 'a' from the wire. See figure 3.



(Figure 3)

To apply Biot-Savart law to this case, we choose the co-ordinate system as shown above.

The current element (dl), equal to dz in this case is at a distance z from 0 (the foot of the perpendicular P on the wire, that is being taken as the origin of our co-ordinate system). It can be represented by the vector, **dz** = dz $\hat{k}$

It is located at a point, defined by the vector z $\hat{k}$. The position vector of the field point P, with respect to the origin is a$\hat{j}$. Hence, its position vector with respect to the current element (located around the point z$\hat{k}$, would be given by

$$r = (a\,\hat{j} - z\,\hat{k})$$

We thus see that

$$\mathbf{dz} \times \mathbf{r} = dz\,(\hat{k})\,[a\,\hat{j} - z\,\hat{k}] = -\,(dz)\,a\,\hat{i}$$

By Biot-Savart Law,

$$\mathbf{dB} = \frac{\mu_0}{4\pi} I\,\mathbf{dl} \times \mathbf{r} / |\mathbf{r}|^3$$

Therefore, in this case, $\mathbf{dB} = \frac{\mu_0}{4\pi} I\,(-\,(dz)\,a\,\hat{i}) / r^3$

$$= -\,\frac{\mu_0}{4\pi} I\,a\,\hat{i}\,.\,dz/(a^2 + z^2)^{3/2}$$

Therefore, $\mathbf{B} = -\,\frac{\mu_0}{4\pi} I\,a\,\hat{i} \int dz / (a^2 + z^2)^{3/2}$

It is easy to evaluate the integral by substituting z = a tan θ. We then have,

$$dz/(a^2 + z^2)^{3/2} = a\sec^2\theta d\theta / a^2\,(1 + \tan^2\theta)^{3/2} = 1/a^2\,\sec^2\theta d\theta / \tan^2\theta$$

$$= 1/a^2.\,\mathrm{Cos}\theta\,d\theta$$

Therefore, $\int dz/(a^2 + z^2)^{3/2} = 1/a^2 \int \cos\theta\,d\theta = 1/a^2\,|\sin\theta|$

Since, tanθ = z/a, we have $\sin\theta = z / \sqrt{a^2+z^2}$

The limits for z are clearly, from $z = -l_2$ (corresponding to the distance of the lower end of the wire from 0) to $z = +l_2$ (corresponding to the distance of the upper end of the wire from 0). Thus,

$$1/a^2\ |\sin\theta| = 1/a^2\ |z/(a^2+z^2)^{1/2}|_{-l2}^{l1}$$

$$= 1/a^2\ [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$$

Hence,

$$\mathbf{B} = -\frac{\mu 0}{4\pi} I\, a\, \hat{\imath}\ .\ 1/a^2\ [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$$

$$\Rightarrow\ \mathbf{B} = -\frac{\mu 0}{4\pi a} I\, \hat{\imath}\ [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$$

This is the magnetic field, due to the straight current carrying wire, at the field point P.

Limiting Case: Infinitely long wire

For such a wire, we can take $l_1 >> a$ and $l_2 >> a$

Under this approximation, we would have

$l_1/(a^2+l_1^2)^{1/2} \approx 1$ and $l_2/(a^2+l_2^2)^{1/2} \approx 1$

Therefore, for the limiting case of an infinitely long straight current carrying wire, we have

$$B \approx -\frac{\mu 0}{4\pi a} I\, \hat{\imath}\ [1+1] = -\frac{\mu 0}{2\pi a} I\, \hat{\imath} = \frac{\mu 0 I}{2\pi a} I\ (-\hat{\imath})$$

In this limiting case, $B \approx \frac{\mu 0}{2\pi a} I$

It is this result, for the magnitude of the magnetic field, due to a straight wire, that is quoted and used most often. However, for a finite wire, and for a field point, for which we cannot assume l_1 (or l_2) $>>a$, we cannot use the above approximate result. In such cases, the full expression of the field, namely,

$$\mathbf{B} = -\frac{\mu 0}{4\pi a} I\, \hat{\imath}\ [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$$

must be used.

We give below some examples where the,

(i) approximate expression, namely, $B \approx \frac{\mu 0}{2\pi a} I$, can be used

(ii) complete expression, namely,

$B = \frac{\mu 0}{4\pi a} I\ [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]\ (-\hat{\imath})$, has to be used.

Example 1:

A straight wire of length 100m carries a current of 5A. Find the magnetic field, due to this wire, at a point distant 0.5m from the wire.

Solution: For the given field point, and for the given length of wire, we can approximate the wire an infinitely long straight line. Hence,

$$B = \mu_0 I/2\pi r \quad = \mu_0/2\pi \times 5/0.5 \text{ T}$$

$$= 5/\pi \times \mu_0 = 5/\pi \times 4 \times 10^{-7} \text{ T}$$

$$= 2/\pi \ \mu\text{T}$$

Example 2:

A field point, P is located midway at a distance of 1 cm from a straight wire of length 10m carrying a current of 1A. Estimate the percentage difference in the values of the magnetic field, at this point, obtained by using the (i) exact method for a finite wire and (ii) approximate result for an infinite wire.

Solution: Since the field point is located midway, the distances of the foot of the perpendicular drawn from it on the wire, from the two ends of the wire are equal to each other.

For a finite wire,

B = $\frac{\mu 0}{4\pi a}$ I [$l_1/ (a^2+l_1^2)^{1/2} + l_2/ (a^2+l_2^2)^{1/2}$]. Here $l_1 = l_2$ = 5m, a = 1cm = 10^{-2}m.

Therefore, $l_1^2 + a^2 = l_2^2 + a^2 = 25 + 10^{-4} = 25 + 0.0001 = 25.0001$

Therefore, $(a^2+l_1^2)^{1/2} = (a^2+l_2^2)^{1/2} \approx 5$

Therefore, $l_1/(a^2+l_1^2)^{1/2} = l_2/(a^2+l_2^2)^{1/2} \approx 1$

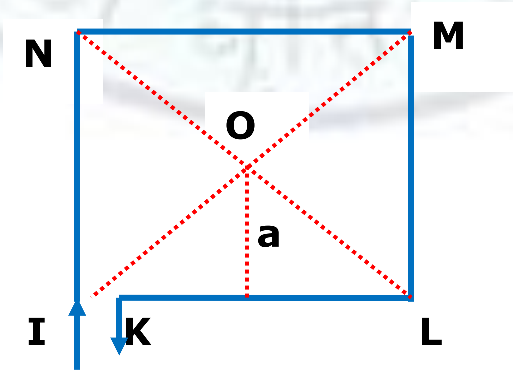
Therefore, B ≈ $\frac{\mu 0}{4\pi a}$ I (1 + 1) ≈ $\frac{\mu 0}{2\pi a}$ I

This is the same as the approximate result for an infinite wire. We can, therefore, say that the percentage difference, in the two values of the magnetic field, calculate here, would be almost negligible.

Example 3:

Find the magnetic field at the centre of a square loop of side l, carrying a current I.

Solution:



Here we need to use the following result for the magnetic field due to a finite straight current carrying wire.

$B = \frac{\mu 0}{4\pi a} I [l_1/ (a^2+l_1^2)^{1/2} + l_2/ (a^2+l_2^2)^{1/2}]$

Here the magnetic fields due to all the four sides of the square, are in the same sense and have equal magnitudes. Hence the net magnetic field, at the centre, would be four times that due to any one side. Also this field, for the sense of current flow shown is directed along the outward directed normal (say $\hat{n}$) to the plane of the page.

For this side KL, say, we have

$l_1 = l_2 = l/2$ and $a = l/2$

Therefore, $l_1^2 + a^2 = l_2^2 + a^2 = l^2/4 + l^2/4 = l^2/2$

Therefore, $l_1/(a^2+l_1^2)^{1/2} = l_2/(a^2+l_2^2)^{1/2} = l/2 . \sqrt{2}/l = 1/\sqrt{2}$

Therefore, $\mathbf{B_{net}} = 4 \times \frac{\mu 0}{4\pi a} I [l_1/ (a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$

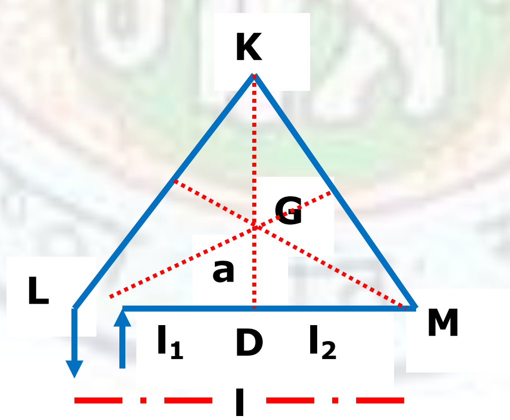
$= 4 \times \frac{\mu 0}{4\pi} I(2/l) [2 . 1/\sqrt{2}] \hat{n}$

$= \frac{\mu 0}{\pi l} I 2\sqrt{2} \hat{n}$

<u>Example 4</u>:

A wire of length 3l is shaped into an equilateral triangle. If the wire were to carry a current I, find the net magnetic field at the centroid of the triangle.

Solution:



$B = \frac{\mu 0}{4\pi a} I [l_1/ (a^2+l_1^2)^{1/2} + l_2/ (a^2+l_2^2)^{1/2}]$

Here we first notice that, at the centroid, the fields due to all the three sides of the equilateral triangle, have equal magnitudes and are in the same direction. Hence the net magnetic field at the centroid is three times that due to one side of the triangle.

Here we have $l_1 = l_2 = l / 2$ and $a = GD = 1/3\ KD = 1/3\ l\ \sqrt{3}/2 = l/2\sqrt{3}$

Therefore, $l_1^2 + a^2 = l_2^2 + a^2 = l^2/4 + l^2/12 = l^2/3$

Therefore, $l_1/ (a^2+l_1^2)^{1/2} = l_2/(a^2+l_2^2)^{1/2}$

$= l/2.\ \sqrt{3}/l$

$= \sqrt{3}/2$

Therefore $\mathbf{B}_{\mathbf{net}} = 3 \times \frac{\mu 0}{4\pi a} I\ [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$

$= 3 \times \frac{\mu 0}{4\pi l} I.2\sqrt{3}[2\ \sqrt{3}/2]$

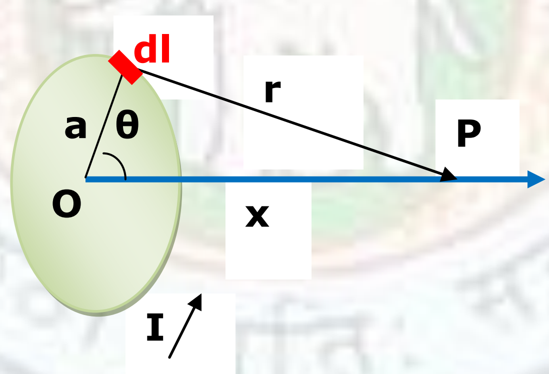
$= 3\times \frac{\mu 0}{4\pi l} I\times 2\sqrt{3} \times \sqrt{3}$

$= 9/2\ \frac{\mu 0}{2\pi l} I\ (-\hat{n})$

Circular Loop

We next consider the case of a circular loop, of radius a, carrying a current I. we use the Biot-Savart Law to calculate the magnetic field, due to this loop, at a field point that lies on the axis of the loop.

We again choose the co-ordinate system as shown in figure 4. The plane of the loop is the Y-Z plane with the origin (O) at the centre of the loop. Let the field point, P, on the axis of the loop, be at a distance x from its centre.



(Figure 4)

A current element dl, situated around some point (y,z) on the loop, can be expressed in terms of its two rectilinear components, as

$$\mathbf{dl} = (dy)\hat{j} + (dz)\ \hat{k}$$

Now, we have

$$y = a \cos\theta \text{ and } z = a \sin\theta$$

Therefore, **dl** = a (-sinθ dθ) $\hat{j}$ + a (cos θ dθ) $\hat{k}$

The position vector of the field point, w.r.t. the origin O, is x $\hat{\imath}$

The current element is located around the point (y,z), i.e., around the point with position vector y $\hat{j}$ + z $\hat{k}$ or (a cos θ $\hat{j}$ + a sin θ $\hat{k}$)

Hence the position vector of the field point, w.r.t. the current element is

r = [x $\hat{\imath}$- ((a cosθ) $\hat{j}$ + (a sin θ) $\hat{k}$)]

= x $\hat{\imath}$ - a (cos θ $\hat{j}$ + sin θ $\hat{k}$)

Therefore, **dl** x **r** = [a (-sinθ dθ $\hat{j}$) + a cosθ dθ $\hat{k}$] x [x $\hat{\imath}$ - a (cos θ $\hat{j}$ + sinθ $\hat{k}$]

= -axsinθdθ (-$\hat{k}$) + $a^2\sin^2\theta d\theta$ $\hat{\imath}$ + axcosθdθ $\hat{j}$ - $a^2\cos^2\theta d\theta$ (-$\hat{\imath}$)

= -axsinθdθ $\hat{k}$ + $a^2(\sin^2\theta + \cos^2\theta)d\theta$ $\hat{\imath}$ + axcosθdθ $\hat{j}$

= axdθ (-sinθ $\hat{k}$ + cosθ $\hat{j}$) + $a^2 d\theta$ $\hat{\imath}$

Therefore, by Biot-Savart law, the field **dB** at point P, due to the current element **dl**, is

$$\mathbf{dB} = \frac{\mu_0}{4\pi} (\mathbf{dl} \times \mathbf{r}/r^3)$$

$$= \frac{\mu_0}{4\pi} \cdot 1/(a^2 + x^2)^{3/2} [axd\theta (-\sin\theta\, \hat{k} + \cos\theta\, \hat{j}) + a^2 d\theta\, \hat{\imath}]$$

(Since $r = (a^2 + x^2)^{1/2}$)

The total field, at P, due to the whole circular loop can be found by integrating the above expression between the limits, θ=0 to θ=2π, of the variable θ. Hence,

$$\mathbf{B} = \frac{\mu_0}{4\pi}(a^2 + x^2)^{3/2} [ax(-\hat{k}) \int_0^{2\pi} \sin\theta d\theta + ax\, \hat{j} \int_0^{2\pi} \cos\theta d\theta + a^2\, \hat{\imath} \int_0^{2\pi} d\theta]$$

The first two of these integrals are zero each. The third integral equals just 2π.

Hence, $\mathbf{B} = \frac{\mu_0}{4\pi} \cdot 1/(a^2 + x^2)^{3/2} . a^2\, \hat{\imath} . 2\pi$

$$\Rightarrow \mathbf{B} = \frac{\mu_0 I a^2}{2(a^2 + x^2)^{3/2}} \hat{\imath}$$

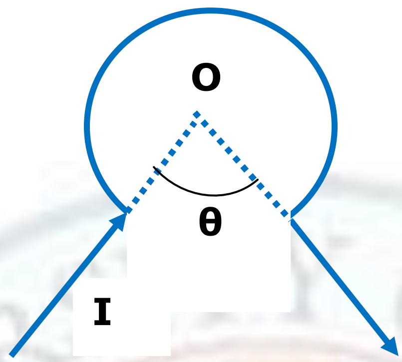
This is the expression for the field, due to a current carrying circular coil, at an axial point P, distant x from the centre.

If we put x=0, we get the field at the centre of the coil. Hence the magnetic field, at centre if the coil is,

$$B_{centre} = \frac{\mu_0 I a^2}{2(a^2)^{3/2}} \hat{\imath} = \frac{\mu_0 I}{2a} \hat{\imath}$$

The magnitude of the field, at the centre of the coil is therefore $\frac{\mu_0 I}{2a}$. The field, as we see, is along the x-axis, i.e., along the normal to the plane of the coil.

Example: Find an expression for the magnetic field, at the centre O, for the circular coil-straight wires combinational set up shown here.



Solution: The straight wire parts do not contribute to the magnetic field at O, as O, is a point on these wires.

Let the magnetic field at O, due t the circular part of the wire, be B. We then have,

B/ $\frac{\mu0\ I}{2r}$ = $(2\pi - \theta)/2\pi$

Therefore, B = (1- $\theta/2\pi$). $\frac{\mu0\ I}{2r}$

Example: The magnetic field at an axial point, distance x from the centre, for a current carrying circular coil of radius R, is 1/n of its value at the centre of the coil. Obtain x in terms of R.

Solution: The magnetic field, at the axial point, is given by

$$B = \frac{\mu0\ I\ R2}{2(R2+\ x2)3/2}$$

We therefore, have,

$$\frac{\mu0\ I\ R2}{2(R2+\ x2)3/2} = 1/n.\ \frac{\mu0\ I}{2R}$$

Therefore, $\frac{R2}{(R2+\ x2)3/2}$ = 1/nR

Or $nR^3 = R^2+x^2$

Or $n^{2/3}R^2-R^2 = x^2$

Therefore, $x^2 = R^2\ (n^{2/3}-1)$

Or $x = R\ (n^{2/3}-1)^{1/2}$

A Recall-The Axial Electric Field of an Electric Dipole:

For an electric dipole of dipole moment, p (=q2a), the electric field, at an axial point, distant x, from the centre of the dipole, is given by

$E = \frac{1}{4\pi\epsilon 0}$

For far-off points (or for a small dipole), we can assume that x>>a. We then have,

$E \approx \frac{1}{4\pi\epsilon 0} 2p/x^3$

This field is directed along the axis of the dipole, which also defines the direction of its dipole moment. We can therefore, say:

For a small electric dipole (or for far-off points where x>>a) the axial electric field is given by:

$\mathbf{E} \approx \frac{1}{4\pi\epsilon 0} 2\mathbf{p}/x^3$

Circular Current Loop as Magnetic Dipole

The expression, for the magnetic field at an axial point, for a circular loop, gives us an interesting result. We notice that the axial magnetic field is given by (refer to figure 4),

$B = \frac{\mu 0\, I\, a^2}{2(a^2 + x^2)3/2}$

For a far-off point (or for a small loop), we can take x >>a. thus the far off field is given by,

$B \approx \frac{\mu 0\, I\, a^2}{2(x^2)3/2} \approx \frac{\mu 0\, I\, a^2}{2\, x3}$

We can rewrite it as,

$B \approx \frac{2\, \mu 0\, I\, (\pi a^2)}{4\pi\, x^3} = \frac{\mu 0}{4\pi\, x^3} . 2\mu_m$

Where, $\mu_m = I.\, \pi a^2$

$= I.$ area of the circular loop

This expression, for the far-off field of a circular loop, has a striking resemblance to the far-off field of an electric dipole. Here we need to think of $\mu_m = I.\, \pi\, a^2$ as the magnetic dipole moment of a magnetic dipole. We can, therefore, say:

At far-off points, a current carrying circular loop can be regarded as equivalent to a magnetic dipole whose dipole moment has a magnitude equal to the product of the current through the coil and the area of the coil.

To be consistent with the direction of the field, we need to assign the direction to this dipole moment in accordance with the 'right hand screw rule'. We can say that the direction of the magnetic dipole moment, associated with a current carrying circular loop, is perpendicular to the plane of the loop and is in the sense of advance of a right handed screw rotated along the sense of current flow.

We thus observe that (through analogy with the electric dipole) a small circular loop (a<<x) can be considered as equivalent to a magnetic dipole. The magnetic dipole of the dipole moment, of this equivalent dipole, equals the product of the current through the loop and the area of the loop. The direction of the dipole moment is normal to the plane of the loop

and is along the direction of advance of a right handed screw, rotated in the sense of the current flow.

Generalization of the above Result

The above discussion shows that we can think of a small current carrying circular loop (or any circular loop for far-off points) as equivalent to a magnetic dipole. The magnitude and direction of the dipole moment of this equivalent dipole can be determined as per the rule outlined above.

It can be shown that the equivalence noted above, is not restricted only to a circular loop. This equivalence turns out to be a general one valid for a small loop (or for any arbitrary shaped loop at far-off points). We can thus say:

Any current carrying small loop (or any loop at far-off points), can be thought of as being equivalent to a magnetic dipole.

The magnitude of the magnetic dipole moment μ_m, (of equivalent dipole) equal the product of the current (I) through the loop and the area (A) of the loop. ($\mu_m = IA$)

The direction of this equivalent magnetic dipole moment, is normal too the plane of the loop and is in the sense of advancement of a right handed screw rotated in the sense of current flow.

This general equivalence between a current carrying small loop and a magnetic dipole is of considerable significance and proves quite useful in very many practical solutions.

Example:

Show that the orbital magnetic moment, associated with the orbital motion of the electron, in the (Bohr) hydrogen atom is an integral multiple of a minimum value of this magnetic moment.

Solution: Consider the electron, in the (Bohr) hydrogen atom, to be orbiting in the n^{th} permitted orbit of radius r_n. Let v_n be the speed of the electron in this orbit.

The time period, T_n, of the electron in the nth orbit is,

$$T_n = 2\pi r_n/v_n$$

Therefore, the frequency of the electron in the nth orbit is,

$$\upsilon_n = v_n/2\pi r_n$$

This orbiting electron is equivalent to a current I_n, where $In = e.\ \upsilon_n = e.v_n/2\pi r_n$

The nth orbit is a circular loop of radius r_n. Hence, its area, A_n is πr^2.

The magnetic dipole moment associated with the orbital motion of the electron, in the nth permitted orbit, is therefore,

$$M_n = I_n.\ \pi a^2 = \mu_m = e.v_n/2\pi r_n.\ \pi r_n^2 = e.v_n r_n/2$$

According to Bohr's theory,

$$mv_n r_n = n.h/2\pi$$

Or $v_n r_n = nh/2\pi m$

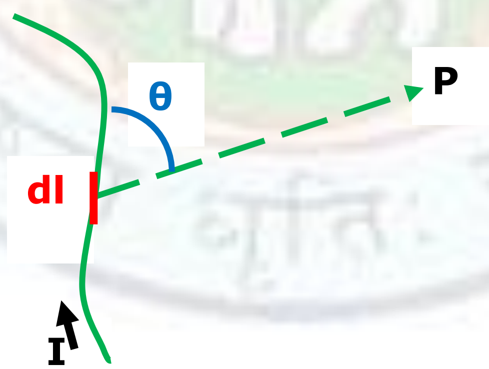
Therefore, $\mu_n = e/2 \,.\, nh/2\pi m = n.\,(eh/4\pi m)$

$= n\,\mu_B$ where $\mu_B = eh/4\pi m$

The associated orbital magnetic dipole moment is therefore, an integral multiple of (a minimum value equal to μ_B. the quantity $\mu_B = eh/4\pi m$ has been given the name Bohr magnetism.

Summary

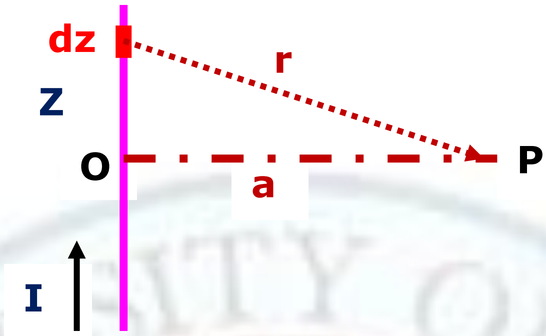
- Magnetism was probably discovered by a Cretan shepherd named Magnus.
- The Chinese discovered and used to properties of magnets in the 12^{th} century.
- The inability to get isolated magnetic poles and Oersted's discovery of the magnetic effects of a current carrying wire led to a revision of the then existing theories of magnetism.
- The magnetic field, at any point in space, equal the force experienced by a unit charge, moving with a unit velocity, along a direction that is normal to the direction of the magnetic field at that point.
- The SI unit of magnetic field is the tesla (T). It equals 10^4 gauss, where gauss is an earlier unit of magnetic field.
- Two current elements exert magnetic forces on each other. These forces follow the qualitative rule:
  'Like currents attract each other while unlike currents repel each other'
- The Biot-Savart law provides us with the basic rule for calculating the magnetic field due to a current element. This law is expressed through the equation:
  $\mathbf{dB} = \frac{\mu 0}{4\pi} I\,\mathbf{dl} \times \mathbf{r} / |\mathbf{r}|^3$



- We can use Biot-Savart law to calculate the magnetic field for different current distributions.

- The magnetic field, due to a straight current carrying wire, is given by

$$B = \frac{\mu_0}{4\pi a} I \, [l_1/(a^2+l_1^2)^{1/2} + l_2/(a^2+l_2^2)^{1/2}]$$



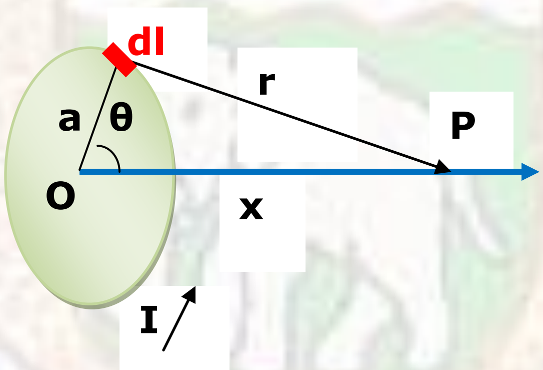
- For an infinitely long wire, i.e., where l1>>a and l2>>a, we have

  $B \approx \frac{\mu_0 I}{2\pi a}$

- The direction of the magnetic field, due to a straight current carrying wire, is normal to the plane formed by **dl** and r and is in the sense of advance of a right handed screw rotated from **dl** to **r**.

- The axial magnetic field, due to a current carrying circular loop, is given by

  $B = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{3/2}}$



- The field at the centre of the loop (x-0), equals $\frac{\mu_0 I}{2R}$

- The direction of the axial field, due to the circular loop, is normal to the plane of the loop, i.e. along the axis and is in the sense of advance of a right handed screw rotated along the sense of current flow.

- The similarity of expression for the far-off axial field of a circular loop, with the axial field of a small electric dipole, enables us to think of a current loop as a magnetic dipole.

- The dipole moment of a current loop equals the product of the current through the loop and the area of the loop.

- The direction of the magnetic dipole moment associated with a current loop, is normal to the plane of the loop and is in the sense of advance of a right handed screw rotated along the sense of current flow.

Exercise

Fill in the Blanks

(a) It was _____ who discovered that a compass needle gets deflected when it is in the vicinity of a current carrying wire.

(b) The SI unit of magnetic field is the _____ and it equals _____ gauss.

(c) In the expression,

$$\mathbf{dB} = \frac{\mu 0}{4\pi} \text{ I } \mathbf{dl} \text{ X } \mathbf{r} \text{ / } |\mathbf{r}|^3$$

the vector **r** stands for the position vector of the _____ _____ with respect to the _____ _____.

(d) The expression, B = $\frac{\mu 0 \text{I}}{2\pi a}$ gives the magnetic field due to an _____ _____ _____ _____ straight wire at a field point distant 'a' from it.

(e) The dipole moment, associated with a current loop, depends on the _____ through the loop and the _____ of the loop.

Answers:

(a) Oersted

(b) tesla ; 10^4

(c) field point ; current element

(d) infinitely long; current carrying

(e) current; area

<u>True or False</u>

State whether the following statement are true or false:

(a) A moving charge would always experience a force in a magnetic field.

(b) The general expression for the force on a current element due to another current element, involves a vector triple product.

(c) The magnetic field, (B), due to any current carrying straight wire, at any field point, is given by B = $\frac{\mu 0}{2\pi r}$I where, r is the distance of the field point from the wire.

(d) The axial magnetic field, due to any current carrying circular loop, has its maximum value at the centre of the loop.

(e) The magnetic dipole moment, associated with a current loop, has a magnitude equal to quotient of the current through the loop to the area of the loop.

Answers:

(a) False

The force is zero when the moving charge moves along or parallel to the direction of the magnetic field.

(b) True

The general expression involves a vector triple product.

(c) False

The given result is true only for an infinitely long straight wire or for field points for which a <<l_1 and a <<l_2.

(d) True

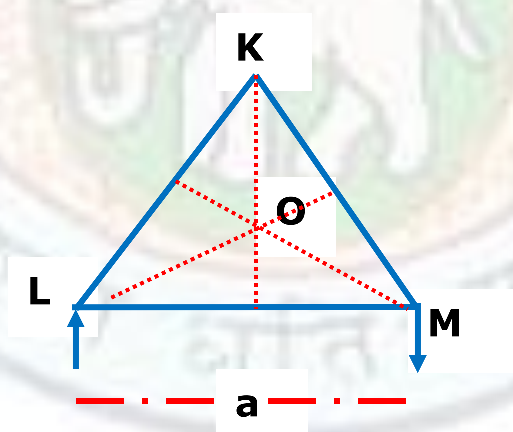
(e) False

The magnitude equals the product and not the quotient of the area of the loop and the current through the loop.

Multiple Choice Questions:

In each of the following questions, a statement is followed by four choices, of which only one is a logically valid, or correct, choice. You have to select the choice in each case.

(a) The set up shown here, has been made from a uniform wire. The total magnetic field, due to this set up at the centroid of the equilateral triangle is:



(i) $\frac{\mu 0i}{\sqrt{3}\pi a}$

(ii) $\frac{2\mu 0i}{\sqrt{3}\pi a}$

(iii) $\frac{3\mu 0i}{\sqrt{3}\pi a}$

(iv) zero

Answer: (iv)

Justification for choice (iv)

The wire LM will carry a current 2I/3 while each of the remaining two wires will carry a current of I/3 each. The field, at O, due to the wire LM (carrying double the current as compared to the wires LK and KM) will cancel the sum of the fields due to the other two wires. Hence the resultant field at O is zero. Hence the choices (i), (ii), (iii) are incorrect and choice (iv) is correct.

(b) The electron, in the hydrogen atom is orbiting in its nth permitted (Bohr) orbit (radius=r_n, speed=v_n). The magnetic field, at the centre, would then equal

(i) $\mu_0 e v_n/4\pi r^2_n$

(ii) $\mu_0 e v_n/2\pi r_n$

(iii) $\mu_0 2\pi r_n/ e v_n$

(iv) $\mu_0 4\pi r^2_n / e v_n$

Answer: (i)

Justification for choice (i)

The equivalent current is $I= e.\,\nu_n = e.v_n/2\pi r_n$

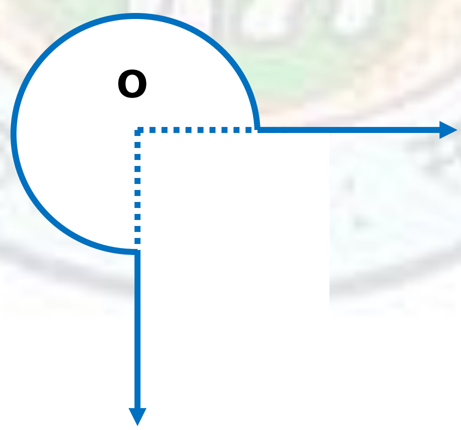
Therefore, the magnetic field, at the centre is,

$B= \mu_0 I/2r_n = \mu_0 e v_n/2\pi r_n(2r_n)$

$= \mu_0 e v_n/4\pi r^2_n$

Hence choice (i) is correct.

(c) For a wire bent as shown, the magnetic field at the centre, would be



(i) $\mu_0 I /8R$

(ii) $2\mu_0 I /8R$

(iii)3 $\mu_0 I / 8R$

(iv)4 $\mu_0 I / 8R$

Answer: (iii)

Justification for choice (iii)

The magnetic field at the centre is due to 3/4[th] of the complete circular loop. Since the centre, O, is a point on each of the two straight wire portions, their contribution, to the field at O, is zero. Hence the net magnetic field at O, is:

$B = \frac{3}{4} \mu_0 I / 2R$

$= 3 \mu_0 I / 8R$

Hence choice (iii) is correct.

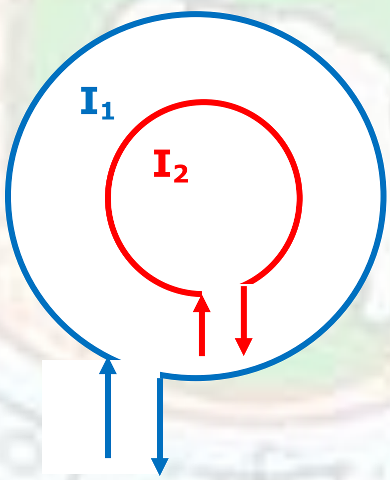
(d) The net magnetic field, at the centre, due to the two concentric co-planar circular coils, carrying current I_1 and I_2, as shown in the diagram, would be:

(i) $\mu_0/2 [I_2/R_2 - I_1/R_1]$

(ii) $\mu_0/4 [I_1/R_1 - I_2/R_2]$

(iii) $\mu_0/2 [I_1/R_1 + I_2/R_2]$

(iv) $\mu_0/4 [I_2/R_2 + I_1/R_1]$



Answer: (iii)

Justification for choice (iii)

The two coils carry currents in the same sense. Hence their magnetic fields add together to produce the resultant field. Hence the resultant field, at O, is

$B = \mu_0/2 [I_1/R_1 + I_2/R_2]$

Hence choice (iii) is correct and others are incorrect.

(e) A magnet, of dipole moment μ_m, experiences a torque τ, in a magnetic field B. A coil of area A, carrying a current I $(=|\mu_m I|/A)$ is next put in the same magnetic field. The torque experienced by the coil in the same magnetic field,

(i) will be necessarily equal to τ

(ii) will be necessarily greater than τ

(iii) will be necessarily smaller than τ

(iv) can be equal to, greater to or smaller than τ

Answer: (iv)

Justification for choice (iv)

The magnitude of the dipole moment of the coil (= A I = A μ_m/A = μ_m) is equal to the magnitude of the dipole moment of the magnet.

However, the torque experience depends both on the magnitude of the dipole moment and the angle between the direction of the dipole moment and the magnetic field. Since the orientation of the coil has not been specified, this angle could have any value. Hence the torque experienced, by the coil, may be equal to, greater than or smaller than, the torque experienced by the magnet. Hence choice (iv) is correct and others are incorrect.

4. Write brief answers to the following:

(a) Define the SI unit of magnetic field.

(b) How can one use the Biot-Savart rule to arrive at a general expression for calculating the magnetic field due to any given current distribution?

(c) State the expressions for the magnetic field due to a finite straight current carrying wire. Obtain its limiting form when the wire can be assumed to be infinitely long.

(d) Use Biot-Savart law to calculate the magnetic field at the centre of a current carrying circular coil.

(e) How do we find the (i) magnitude and (ii) direction of the magnetic dipole moment associated with a planar current carrying coil?

5. Write answers to the following:

(a) Use Biot-Savart law to obtain an expression for the magnetic field due to a finite straight current carrying wire.

(b) A given length of wire is bent into a rectangle of length l and breadth b. if the wire were to carry a current I, find the magnetic field at the centre of the rectangle.

(c) A given length, L, of the wire is bent into (i) a square and (ii) a circular shape. Compare the magnetic fields, at the centre, in each of the two cases when the same current I, flows through each shape.

(d) How do we use (i) the analogy of the electric dipole and (ii) the expression for the axial magnetic field of a circular coil, to find the magnetic dipole moment to be associated with a current loop?

(e) Obtain general expressions for the

(i) magnetic field at the centre

(ii) orbital magnetic moment

associated with the orbital motion of the electron in t

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