Paper: Differential Equations-I

Lesson : Limit points and Bolzano Weierstrass Theorem

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Learning Outcomes

After you have read this chapter, you should be able to:

- Define the topological concepts such as neighborhood of a set, open set and closed set.
- Define the limit point of a set and prove some of its basic properties.
- State and prove Bolzano-Weierstrass Theorem for sets.
- Characterize limit point in term of a closed set and a sequence.
- Define isolated point and discuss various examples.

"It is true that a mathematician who is not also something of a poet will never be a perfect mathematician."

– ***Karl Weierstrass, 1815-1897***

"Even in the realm of things which do not claim actuality, and do not even claim possibility, there exist beyond dispute sets which are infinite"

– ***Bernhard Bolzano, 1781-1848***

2. Introduction

Early results in analysis were implicitly present in the early days of ancient Greek mathematics. For example, an infinite geometric sum is implicit in Zeno's paradox of the dichotomy. Later on Greek mathematicians such as Eudoxus and Archimedes made more explicit, but informal, use of the concepts of limits and convergence when they used the method of exhaustion to compute the area and volume of regions and solids.

Modern mathematics began with two great advances, analytic geometry and the calculus. The former took definite form in the year 1637 while the calculus took the definite shape in 1966. Both Newton and Leibnitz, share the credit for inventing infinitesimal calculus independently in the seventeenth century. Calculus grew with the stimulus of applied work that continued through the 18th century, into analysis topics such as the calculus of variation, differential calculus etc. During this period, calculus techniques were applied to approximate discrete problems by continuous one.

In the 18th century, Euler introduced the notion of mathematical function. Real analysis began to emerge as an independent subject when Bernard Bolzano introduced the modern definition of continuity in 1816, but Bolzano's work did not become widely known until the 1870s. Karl Weierstrass is responsible for the arithmetization of analysis. Undoubtedly, he is known as the father of modern analysis. Among other architects of modern analysis are Cauchy, Reimann, Euler, Cantor, Dedekind, Baire, Banach etc.

Today, calculus and its extension in real analysis which is a part of mathematical analysis are far reaching indeed. Now, every mathematician knows that analysis arose naturally in the nineteenth century out of the calculus, which involves the elementary concepts and techniques of analysis, of the previous two centuries.

We now give the section wise summary of the chapter. The chapter describes the importance of the limit points of a set as open of the most classified topics in both calculus and real analysis. We start the chapter with quotations taken from the literature. The chapter is divided into three sections. In each section, definitions, theorems, remarks etc. are furnished with substantial examples to stimulate the learning process and they are numbered consecutively in each section. We also indicate both the chapter and the section number. Further, important remarks are also added before or after the statements of the definitions, theorems, etc.

The first section is the introduction. One of the most important tools for any course in modern mathematics is the theory of sets, so we begin our study of limit points and related topics with a discussion of the basic notions of set theory. Here we use some elementary set notations and terminology that we use throughout the chapter. The concepts of upper bound, lower bound supremum, infimum etc. are also introduced in this section. The section two deals with some topological concepts, such as, neighborhood of a set, open set, closed set etc. In fact, we noticed that there are sets which are neither open nor closed. Here we study the concept of limit point of a set and establish various results concerning limit points. The section ends with the conclusion that a finite set has no limit point.

Lastly, in the third section we discuss the most famous result known as the Bolzano-Weierstrass Theorem in detail which states that every infinite bounded subset of real numbers has a limit point. Here, also, we characterize limit points in terms of closed sets via the results which states that a subset of real numbers in closed if and only if it contains all its limit points.

Now we define convergent sequences and established a result which connects sequences with limit points of a set. Towards the end of the section, we define an isolated point of a set and discuss various examples of isolated points.

Finally, the chapter ends with a list of exercises (with the answers) and references for further reading.

The concept of a set plays an important role in every branch of modern mathematics. Furthermore, detailed treatments can be found in the references which are mentioned at the end of this chapter.

3. Preliminaries: Set notation

Definition 3.1.: A set is defined as a well-defined collection of objects.

Sets will be denoted by capital letters. The notation $x \in A$ means that the object *x* is in the set *A*, and we write $x \notin A$ to indicate that *x* is not in *A*.

The objects of a set *A* are called elements or the points or the members and these are denoted by small letters $x, y, z, a, b, \ldots\ldots$.

The simplest way of specifying a set is by listing its elements. For instance, we use the notation

$$A = \{\text{sun}, \text{rose}, 0, \sqrt{3}, -\pi\}$$

to represent the set whose elements are sun, rose, the real numbers $0, \sqrt{3}$ and π.

However, this notation is of no use in specifying a set which has an infinite number of elements. Such sets may be specified by naming the property which distinguishes elements of the set from the elements which are not in the set. For example, the notation

$$B = \{x : 0 < x < 1\}$$

denotes the set of all real numbers lying between 0 and 1. A set which contains at least one element is called non empty. The set which has no elements is denoted by the symbol φ and is sometimes called a null or void set or an empty set. A set having only one element is called a singleton.

If *A* and *B* are two sets, than we say that *A* is a subset of *B*, denoting by $A \subseteq B$, if every element of *A* is also an element of *B*.

We note that *A* is itself a subset of *A* and φ is a subset of every set. *A* is called a proper subset of *B*, if A $\subset$ B but $A \neq B$. That is, there exists at least one elements $x \in B$ such that $x \notin A$ and is denoted by $A \subset B$.

Definition 3.2.: Two sets *A* and *B* are said to be equal, if and only if $A \subseteq B$ and $B \subseteq A$.

This is, *A* = *B* if and only if $A \subseteq B$, $B \subseteq A$.

Every set *A* is a subset of some fixed set. This fixed set is called the universal set and is denoted by *X*.

The collection of all subsets of a given set A is called the power set of A and is denoted by $\Pi(A)$.

I.Q. 1

Value Addition: Remark
(i) The elements of $\Pi(A)$ are sets. (ii) The number of elements in $\Pi(A)$ is 2^n if a set *A* contains *n* elements.

The Union and intersection are two fundamental operations defined on sets and consequently, we get new set if we apply these operations on any two or more sets.

Definition 3.3.: If *A* and *B* are two sets, then we define

(i) the union of *A* and *B*, denoted by $A \bigcup B$, is defined to be the set

$$A \bigcup B = \{x : x \in A \text{ or } x \in B\}$$

(ii) the intersection of *A* and *B*, denoted by $A \bigcap B$, is defined to be the set

$$A \bigcap B = \{x : x \in A \text{ and } x \in B\}$$

(iii) the set difference of B from A, denoted by $A \setminus B$, id defined to be the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

(iv) the complement of *A*, denoted by A^C, is defined to be the set

$$A^C = X \setminus A = \{x : x \in X \text{ and } x \notin A\}$$

(v) the symmetric difference denoted by $A \Delta B$, is defined to be the set

$$A \Delta B = (A - B) \bigcup (B - A)$$

The set $A \setminus B$ is sometimes called the complement of *B* relative to *A*.

Value Addition: Remark
We note that (i) $A \setminus B = A \bigcap B^c$ (ii) $A \subseteq B$ if and only if $B^c \subseteq A^c$ Two sets *A* and *B* are called **disjoint** if $A \bigcap B = \phi$.

Some basic properties of the operations on sets defined above are mentioned below which can be easily verified.

(i) $A \bigcup B = B \bigcup A$

(ii) $A \bigcap B = B \bigcap A$

(iii) $A \Delta B = B \Delta A$

(iv) $A \cup (B \cup C) = (A \cup B) \cup C$

(v) $A \cap (B \cap C) = (A \cap B) \cap C$

(vi) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

(vii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(viii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ix) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

(x) $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$

(xi) $A \cup A^C = X$

(xii) $A \cap A^C = \phi$

(xiii) $A \cup \phi = A$

(xiv) $A \cup X = X$

(xv) $A \cap \varphi = \phi$

(xvi) $A \cap X = A$

(xvii) $A \cup A = A$

(xviii) $A \cap A = A$

(xix) $(A \cup B)^C = A^C \cap B^C$

(xx) $(A \cap B)^C = A^C \cup B^C$

Value Addition: Remark
The identities (i) – (iii) are known as the commutative laws, (iv) – (vi) are associative law's, (vii) and (viii) are distributive laws and (xix) and (xx) are called De Morgan's law. A set *A* is said to be finite if it is either empty or it has *n* elements for some $n \in \mathbb{N}$. A set *A* is said to be infinite if it is not finite. In this chapter, the universal set is the set of real numbers and is

denoted by $\mathbb{R}$.

Definition 3.4.: Let *A* be a subset of $\mathbb{R}$. We say that

(i) *A* is bounded above if there exists $u \in \mathbb{R}$ such that $x \le u$ for all $x \in A$. Any such *u* is called an upper bound of *A*.

(ii) *A* is bounded below if there exists $\nu \in \mathbb{R}$ such that $\nu \le x$ for all $x \in A$. Any such ν is called a lower bound of *A*.

(iii) *A* is bounded if it is bounded above and bounded below. Thus, *A* is bounded if there exist real numbers *u* and ν such that

$$\nu \le x \le u \text{ for all } x \in A$$

or equivalently, *A* is bounded if there exists a real number *M* > 0 such that

$$|x| \le M \text{ for all } x \in A,$$

that is, $A \subseteq [-M, M]$.

Value Addition: Remark
We note that (i) the maximum of a set is always an upper bound for the set. Likewise, the minimum of a set is always a lower bound for the set. (ii) If a set has one upper bound, then it will have an infinite number of upper bounds. Similarly, a set which is bounded below will have an infinite number of lower bounds. (iii) The sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are not bounded. (iv) The sets $[a,b]$, $[a,b)$, $(a,b]$ and (a,b) are bounded sets where $a, b \in \mathbb{R}$ with *a* < *b*.

In order to discuss the completeness axiom for the set of real numbers, we introduce the following concepts.

Definition 3.5.: Let *A* be any subset of $\mathbb{R}$.

(i) If *A* is bounded above and *A* has a least upper bound (in short, *l.u.b.*), then we define it as the supremum of *A* and is denoted by sup *A*.

(ii) If A is bounded below and A has a greatest lower bound (in short, *g.l.b.*), then we define it as the infimum of A and is denoted by inf A.

Value Addition: Remark
We observe that (i) maximum and minimum element of a set A always belong to the set A but sup A and inf A need not belong to A. (ii) a set can have at the most one maximum, minimum, supremum and infimum. (iii) if the supremum of a set A exists and sup $A \in A$, then sup A becomes the maximum element of A. Likewise, the similar result hold true for infimum of a set.

We now characterize the supremum and infimum of a set as follows:

(i) Assume that the supremum of a subset A of $\mathbb{R}$ exists. Then for every $\varepsilon > 0$, there exists some $x \in A$ such that

$$\sup A - \varepsilon < x \leq \sup A.$$

(ii) Assume that the infimum of a subset A of $\mathbb{R}$ exists. Then for every $\varepsilon > 0$, there exists some $x \in A$ such that

$$\inf A \leq x < \inf A + \varepsilon.$$

I.Q. 2

I.Q. 3

Now, we state the completeness axiom.

Every non-empty subset of $\mathbb{R}$ that is bounded from above has the least upper bound.

Using the completeness axiom, we can prove the following result:

Every non-empty subset of $\mathbb{R}$ that is bounded from below has the greatest lower bound.

We begin this section with some basic concepts of Real Analysis such as neighborhood of a point, open and closed sets, limit point of a set. We also establish some of their important properties.

Definition 3.6.: A set A is said to be a neighborhood of a point $x \in \mathbb{R}$ if there exists some $\varepsilon > 0$ such that

$x \in (x-\varepsilon, x+\varepsilon) \subseteq A$

Value Addition: Remark	
(i)	The set $\mathbb{N}$ of natural numbers is not a neighborhood of any of its points whereas the set $\mathbb{R}$ of real numbers is a neighborhood of each of the points.
(ii)	Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then ε - neighborhood of x = $N_\varepsilon(x) = \{y \in \mathbb{R} : \lvert y - x \rvert < \varepsilon\}$.
(iii)	If $y \in N_\varepsilon(x)$ for every $\varepsilon > 0$, then $y = x$.
(iv)	If any $x \in A$ satisfies definition 3.2.1, then x is said to be an interior point of A and the set of all interior points is denoted by $\overset{0}{A}$. It is clear that $\overset{0}{A} \subseteq A$.

Now, we define the open subsets of $\mathbb{R}$ in the usual way.

Definition 3.7.: A subset *A* of $\mathbb{R}$ is called open if for each $x \in A$, there exists some $\varepsilon > 0$ such that

$$x \in (x-\varepsilon,\ x+\varepsilon) \subseteq A.$$

We conclude that a subset *A* of $\mathbb{R}$ is open if and only if *A* is a neighborhood of each of its points. In fact, *A* is open if and only if

$$A = \overset{0}{A}.$$

Also, we note that every open interval (a,b) with $a,b \in \mathbb{R}, a < b$, is on open set.

We now mention a result without proof which connects union and intersection of sets with open sets.

I.Q. 4

Theorem 1: In $\mathbb{R}$, the following statements hold:

(i) ϕ and $\mathbb{R}$ are open sets.

(ii) Arbitrary unions of open sets are open sets.

(iii) Finite intersections of closed sets are closed sets.

Definition 3.8.: A subset *A* of $\mathbb{R}$, is called closed if its complement A^C, that is, $X - A$ is an open set.

We not that *A* is closed if and only if for each $x \notin A$, there exists $\varepsilon_x > 0$ such that

$A \bigcap (x-\varepsilon_x, x+\varepsilon_x) = \phi$.

We now state some properties of closed sets.

Theorem 2: In $\mathbb{R}$, the following statements holds:

(i) ϕ and $\mathbb{R}$ are closed sets.

(ii) Arbitrary intersections of closed sets are closed sets.

(iii) Finite unions of closed sets are closed sets.

I.Q. 5

I.Q. 6

Value Addition: Remark	
(i)	We observe that a set which is not open is not necessarily closed and vice-versa. For example, $A = \mathbb{Q}$, the set of rational numbers is neither open nor closed.
(ii)	The condition of finiteness in Theorem 1 (iii) and Theorem 2 (iii) cannot be relaxed. For this, let $G_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right), n \in \mathbb{N}$. Then G_n is open for all $n \in \mathbb{N}$. But intersection $G = \bigcap_{n=1} G_n = \{x\}$ which is not an open subset of $\mathbb{R}$. Also, let $F_n = \left[\frac{1}{n}, 2\right], n \in \mathbb{N}$. Then $F = \bigcup_{n=1}^{\infty} F_n = (0,2]$, which is not a closed set.

Now, we discuss the concept of limit point of a set.

Definition 3.9.: Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then, *x* is called a limit of *A* if every neighborhood of *x* contains a point $y \neq x$ such that $y \in A$.

We deduce that *x* is a limit point of *A* if for each $\varepsilon > 0$, there exists at least point $y \in A$, $y \neq x$ such that $|y - x| < \varepsilon$

That is, $(x-\varepsilon,\ x+\varepsilon) \bigcap A - \{x\} \neq \phi$ for all $\varepsilon > 0$.

Value Addition: Remark	
(i)	We note that *x* need not be an element of *A*.
(ii)	The set of limit points of *A* is called the derived set of *A* and is denoted by A'.
(iii)	Further, a limit point of a set is also called an accumulation point or a cluster point.

(iv) A real number *x* is not a limit point of a subset A of $\mathbb{R}$ if there exists some $\varepsilon > 0$ such that $(x-\varepsilon,\ x+\varepsilon)$ contains no point of *A* different from *x*.
That is, either $(x-\varepsilon,\ x+\varepsilon)\bigcap A=\{x\}$ or $(x-\varepsilon,\ x+\varepsilon)\bigcap A=\phi$.

(v) A set may or may not have a limit point

Examples 1:

(i) Every element of $\mathbb{R}$ is a limit point of $\mathbb{R}$,

For this, let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then,

The open interval $(x-\varepsilon,\ x+\varepsilon)$ contains a point $y \in \mathbb{R}$ such that $y \neq x$. In fact $\mathbb{R}' = \mathbb{R}$.

(ii) The set $\mathbb{N}$ of natural numbers has no limit point

Let $n \in \mathbb{N}$. Then, $\left(n-\frac{1}{2}, n+\frac{1}{2}\right)$ is a neighborhood of *n* which does not contain any point of $\mathbb{N}$ other than *n*.
Thus, $\mathbb{N}' = \phi$.

Indeed, similar arguments can be given in order to show that the set $\mathbb{Z}$ of integers has no limit point and hence, $\mathbb{Z}' = \phi$.

(iii) Every real number is a limit point of the set $\mathbb{Q}$ of rational numbers. For this, let $x \in \mathbb{R}$ and ε > 0 be given. Then, the open internal $(x-\varepsilon, x+\varepsilon)$ contains a rational number $q \in \mathbb{Q}$ such that $q \neq x$. Moreover, $\mathbb{Q}' = \mathbb{R}$.

Likewise, every real number is also a limit point of the set of irrational numbers.

(iv) Every element of $[a,b]$ is a limit point of the open internal (a,b). For this, let $x \in [a,b]$ and ε > 0 be given. Then, $(a,b)\bigcap(x-\varepsilon,\ x+\varepsilon)=(c,d)$ where *c* = max $\{x-\varepsilon,\ a\}$, $d=\min\{x+\varepsilon, b\}$.

Thus, the interval (c,d) contains infinitely many elements of (a,b) and in particular, it contain at least one element of (a,b) other than x. Therefore, x is a limit point of (a,b) which concludes that $(a,b)' = [a,b]$.

(v) A finite set has no limit point.

Let $F = \{x_1, x_2, \ldots, x_n\}$ be any finite sub set of $\mathbb{R}$.

There are two cases to discuss.

Case I: Let $x \in \mathbb{R}$ and $x \in F$.

Let us take $\min_{1 \le i \le n} \{ |x_i - x| : x_i \in F - \{x\} \} = \varepsilon$.

It follows that $(x - \varepsilon, x + \varepsilon) \cap F = \{ x \}$ which implies that $x \notin F'$.

Case II: Let $x \in \mathbb{R}$ and $x \notin F$.

We define $\varepsilon = \min_{1 \le i \le n} \{ |x_i - x| \}$

Then, $(x - \varepsilon, x + \varepsilon) \cap F = \phi$.

We conclude, by definition, that $x \notin F'$.

Therefore, a finite set has no limit point, that is, $F' = \phi$.

In particular, the set $A = \{0,1\}$ has no limit points.

We now, discuss a characterization theorem for a limit point of sets.

I.Q. 7

Theorem 3: Let $A \subseteq \mathbb{R}$. Then, a point $x \in \mathbb{R}$ is a limit point of *A* if and only if every neighborhood of *x* contain infinitely many points of *A*.

Proof: We first assume that every neighborhood of *x* contain infinitely many points of *A*.

Then, for each $\varepsilon > 0$, the neighborhood $(x - \varepsilon, x + \varepsilon)$ of *x* contains a point of *A* other than *x* by the hypothesis. Consequently, *x* becomes a limit point of *A*.

Conversely: Let *x* be a limit of *A*. We prove it by contradiction.

Suppose there exists a neighborhood, say *N* of *x* which contain only finitely many points of *A*, that is, $N \cap A$ is finite.

There are two cases to discuss:

Case (i) Let $N \cap A - \{x\} = \phi$

Then, *x* is not a limit of *A* which is a contradiction to the fact that *x* is given to be a limit point of *A*.

Case (ii) Let $N \cap A - \{x\} = \{x_1, x_2, \ldots x_n\}$.

Let us take $\varepsilon = \min\{|x_1 - x|,\ |x_2 - x|, \ldots, |x_n - x|\}$.

Now, we select $M = N \cap (x - \varepsilon,\ x + \varepsilon)$.

Then, *M* is a neighborhood of *x* as it is the intersection of two neighborhood of *x*.

Also, $M \cap A - \{x\} = N \cap (x - \varepsilon,\ x + \varepsilon) \cap S - \{x\} = \phi$.

Thus, there exists a neighborhood say *M* of *x* which contain no point of *A* other than *x*.

Therefore, *x* is not a limit point of *A* which is again a contradiction.

Thus, our assumption was wrong.

Hence, it follows that every neighborhood of *x* contain infinitely many points of *A*.

Value Addition: Remark	
(i)	In particular, the above theorem concludes that a set cannot have a limit point unless it is an infinite set. However, the converse is not true in general. For example the set $\mathbb{N}$ is an infinite set with no limit points.
(ii)	We conclude, from the above theorems that the empty set ϕ and a finite set has no limit point.

The following theorem tells some important properties of a limit point.

Theorem 4: Let *A* and *B* be any subsets of $\mathbb{R}$. Then the following statements hold:

(i) $\phi' = \phi$

(ii) If $A \subseteq B$, then $A' \subseteq B'$

(iii) $(A \cup B)' = A' \cup B'$

(iv) $(A \cap B)' = A' \cap B'$

(v) $A'' \subseteq A'$

(vi) If $x \in A'$, then $x \in (A - \{x\})'$

Proof. Let xÎ ¡ .

Since ¡ ⊄= ¡ , therefore ¡ is a neighborhood of *x* such that ¡ I f = f .

Consequently, $x \notin \phi'$ which shows that no point of ¡ can belong to ϕ'. Hence $\phi = \phi'$.

(ii) Let $x \in A'$.

Then, $(x-\varepsilon,\ x+\varepsilon) \cap A - \{x\} \neq \phi$ for all ε > 0.

Since A⊂B, therefore, $(x-\varepsilon,\ x+\varepsilon) \cap A - \{x\} \subset (x-\varepsilon,\ x+\varepsilon) \cap B - \{x\}$.

This gives us that

$$(x-\varepsilon,\ x+\varepsilon) \cap B - \{x\} \neq \phi \text{ for all } \in > 0.$$

We conclude that $x \in B'$ which implies that $A' \subset B'$.

(iii) We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

Therefore by the above result (ii), we obtain.

$$A' \subseteq (A \cup B)' \text{ and } B' \subseteq (A \cup B)'.$$

Thus, $A' \cup B' \subseteq (A \cup B)'$.

For the reverse part, we have two cases to discuss.

Case I. When $(A \cup B)' = \phi$, then $(A \cup B)' \subseteq A' \cup B'$ and the part (iii) follows obviously.

Case II. When $(A \cup B)' \neq \phi$. So let $x \in (A \cup B)'$.

Then, $(x-\varepsilon,\ x+\varepsilon) \cap (A \cup B) - \{x\} \neq \phi$ for all $\in$ > 0.

that is, $[(x-\varepsilon, x+\varepsilon) \cap A - \{x\}] \cup [(x-\varepsilon,\ x+\varepsilon) \cap B - \{x\}] \neq \phi$ for all ε>0.

that is, $(x-\varepsilon,\ x+\varepsilon)\cap A-\{x\}$ or $(x-\varepsilon,\ x+\varepsilon)\cap B-\{x\}$

is non-empty for each $\varepsilon > 0$.

Therefore, $x \in A'$ or $x \in B'$ which implies that $x \in A' \cup B'$.

It follows that $(A\cup B)' = A' \cup B'$.

(iv) We know that

$$(A\cap B)\subset A \text{ and } (A\cap B)\subset B.$$

By applying part (ii), we find that

$$(A\cap B)' \subset A' \text{ and } (A\cap B)' \subset B'.$$

Thus, $(A\cap B)' \subset A'\cap B'$.

(v) If $A'' = \phi$, then obviously we have $A'' \subset A'$.

If $A'' \neq \phi$, then let $x \in A''$. Then, by definition, we get

$$(x-\varepsilon,\ x+\varepsilon)\cap A'-\{x\} \neq \phi \text{ for all } \varepsilon > 0.$$

So that $y \in (x-\varepsilon,\ x+\varepsilon)\cap A',\ y \neq x$.

It implies that $(x-\varepsilon,\ x+\varepsilon)$ is a neighborhood of *y* which is a limit point of *A*.

Then, $(x-\varepsilon,\ x+\varepsilon)\cap A-\{x\} \neq \phi$ for all $\varepsilon > 0$. and consequently $x \in A'$.

We conclude that $(A')' \subseteq A'$.

(vi) We note that

$$N\cap A-\{x\} = \left[N\cap(A-\{x\})\right]-\{x\}$$

Thus, we obtain that if $x \in A'$ then $x \in (A-\{x\})'$.

Value Addition: Remark
The reverse inclusion, i.e., $A'\cap B' \subset (A'\cap B)'$, does not hold in part (iv) of the theorem 4. For example, let $A=$ ¤ , the set of rational numbers and $B=$P, the set of irrational numbers. Then, A¢= ¡ ,B¢= ¡ , by using part (iii) of

the Example 1.

So that $A'\cap B' = \mathbb{R}$ but $(A\cap B)' = \phi' = \phi$. (by using Theorem 4 (i))

Therefore, $A'\cap B' \not\subset (A\cap B)'$.

Example 2: Find the derived set of each of the following sets:

(i) $\{1+2^{-n} : n\in \mathbb{N}\}$

(ii) $\left\{2-\frac{3}{n} : n\in \mathbb{N}\right\}$

(iii) $\left\{\frac{n+1}{n} : n\in \mathbb{N}\right\}$

(iv) $(-\infty, a)$

(v) $\left\{\frac{1}{n}+\frac{1}{m} : m,n\in \mathbb{N}\right\}$.

Solution: (i) Let $A=\{1+2^{-n} : n\in \mathbb{N}\}$, then $A'=\{1\}$

(ii) Let $A=\left\{2-\frac{3}{n} : n\in \mathbb{N}\right\}$, then $A'=\{2\}$

(iii) Let $A=\left\{\frac{n+1}{n} : n\in \mathbb{N}\right\}$, then $A'=\{1\}$

(iv) Let $A=(-\infty, a)$, then $A'=(-\infty, a]$

(v) Let $A=\left\{\frac{1}{n}+\frac{1}{m} : m,n\in \mathbb{N}\right\}$,

For *n* = 1, the elements of *A* are $1+1, 1\frac{1}{2}, 1\frac{1}{3}, ...$

For *n* = 2, the elements of *A* are $\frac{1}{2}+1, \frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{3}, ...$ and so on.

Therefore, $1, \frac{1}{2}, \frac{1}{3}, ...$ are all limit points of *A*.

Also, every nbd of zero contains infinitely many points of *A*. Thus, $A' = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\} \cup \{0\}$.

I.Q. 8

I.Q. 9

Value Addition: Note
We have established that a finite set has no limit point. Furthermore, even an infinite set, for example, the set of natural numbers, has no limit point. We, now, discuss a well-known important result named after two great mathematician namely Bernhard Bolzano and Karl Weierstrass of the eighteenth century which guarantees the existence of a limit point for a special class of subsets of $\mathbb{R}$. This vital result is known as the Bolzano-Weierstrass theorem.

Theorem 5: (Bolzano – Weierstrass Theorem) Every infinite bounded subset of real numbers has a limit point.

Proof: Let *A* be an infinite bounded subset of $\mathbb{R}$.

We show that *A* has a limit point. Since *A* is bounded, there exists some $x \in \mathbb{R}$ such that

$$A \subseteq [-x, x].$$

Now, at least one of the subintervals $[-x, 0]$ or $[0, x]$ contains an infinite subset of *A* which is given to be infinite. Let us denote one such subinterval by $[x_1, y_1]$. Again, bisect $[x_1, y_1]$ and obtain a sub internal, say, $[x_2, y_2]$ which contain an infinite subset of *A*.

By repeating the above process, we obtain countable collection of intervals, the n^{th} internal being $[x_n, y_n]$ with length $y_n - x_n = \frac{x}{2^{n-1}}$.

We note that the supremum of the left end points x_n and the infimum of the right end points y_n must be equal and we denote it by ℓ.

Let $\varepsilon > 0$ be given.

For large *n*, it follows that the internal $[x_n, y_n]$ will be contained in the internal $(\ell - \varepsilon, \ell + \varepsilon)$ so that

$$y_n - x_n < \frac{\varepsilon}{2}$$

Thus, the interval $(\ell - \varepsilon,\ \ell + \varepsilon)$ contain a point of *A* different from ℓ which shows that ℓ is a limit point of *A*.

Value Addition: Remarks

(i) Bolzano– Weierstrass theorem establishes the existence of a limit point for an infinite bounded set. There may exist infinite unbounded sets having no limit point, e.g., ¥ (the set of natural numbers). Also, as discussed earlier, no finite set has a limit point.
Therefore, the condition of the B – W theorem cannot be dropped.

(ii) We note that an infinite set having a limit point need not be bounded, for example, ¤ -the set of rational number has limit points and is obviously not bounded, that is, unbounded.

(iii) We observed that B – W Theorem is contradicted if an infinite bounded set has no limit point.

Example 3: Give an example of an infinite bounded set which has a limit point.

Solution. Let us consider the set

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots.\right\}$$

Then, $A \subseteq (0,1]$ which shows that *A* is a bounded set. Also, *A* is infinite.

Thus, by B–W theorem 5, the set *A* has a limit point. We prove that 0 is a limit point of *A*. Let $\varepsilon > 0$ be given.

Then, by using the Archimedean property, there exists a natural number *m* such that $\frac{1}{\varepsilon} < m$.

Therefore, the ε–neighborhood of $(-\varepsilon,\ \varepsilon)$, contain a point of *A* other than 0, namely, 1/*m*.

Since $\varepsilon > 0$ is arbitrarily chosen, therefore, we conclude that 0 is a limit point of *A*

Example 4: Does there exists an infinite unbounded set having a limit point.

Solution: Let us consider the set

$$A=\left\{1,\ 2,\ \frac{1}{2},\ 3,\ \frac{1}{3},\ 4,\ \frac{1}{4},.....\right\}$$

Note that the set is infinite and unbounded.

However, *A* has a limit point, viz 0. (See example 3)

The next theorem establishes a close connection between a closed set and a limit point. In fact, it characterizes closed sets in terms of its limit point.

Theorem 6: A subset *A* of ¡ is closed if and only if it contain all of its limit points.

Proof: We first assume that *x* is a limit point of a closed set *A*. Then

$(x-\varepsilon,\ x+\varepsilon)\bigcap A-\{x\}\neq\phi$ for all ε > 0

Or $N\bigcap A-\{x\}\neq\phi$ for all neighborhoods ¥ of *x*.

It follows that there cannot be a neighborhood of *x* which is entirely contained in A^C. That is, *x* is not an interior point of *A*.

Since A is closed, therefore A^c is open. We conclude that $x\notin A^C$, that is, $x\in A$.

Since *x* was chosen arbitrarily, it shows that *A* contain all its limit point.

Conversely. Let $A'\subset A$.

We show that *A* is a closed set, that is A^C is open.

Let $x\in A^C$. Then $x\notin A'$.

This implies that

$(x-\varepsilon,\ x+\varepsilon)\bigcap A-\{x\}=\phi$ or $(x-\varepsilon,\ x+\varepsilon)\bigcap A=\{x\}$ for some ε>0.

We conclude that $x\in(x-\varepsilon, x+\varepsilon)\subset A$ for some ε > 0.

It follows that x is an interior point of A^C. But *x* was taken arbitrarily, therefore every point of A^C is an interior point.

By definition, A^C is an open set. Thus, *A* is closed which completes the proof of the theorem.

Corollary 1: A non-empty bounded closed subset A of $\mathbb{R}$ contains its supremum as well as its infimum.

Proof. We are given that A is non-empty, bounded closed subset of $\mathbb{R}$.

If A is finite, then its greatest elements is its supremum and its least element is its infimum. In this case, both supremum and infimum belong to A.

Now, let us assume that A is infinite.

Since A is bounded and non-empty, therefore by completeness axiom it has the supremum, say x.

We need to show that x is a limit point of A.

Let $\varepsilon > 0$ be given. Since x is the supremum of A, therefore, by its characterization property, there exists some $t \in A$ such that

$$x - \varepsilon < t$$

Then, $x-\varepsilon<t<x<x+\varepsilon,\ \ \varepsilon>0$, that is $(x-\varepsilon, x+\varepsilon)\cap A-\{x\}\neq\phi$ for all $\varepsilon > 0$.

It follows that x is a limit point of A and consequently $x \in A$ by using Theorem 3.3.5.

Similar arguments can be given for the infimum case.

I.Q. 10

I.Q. 11

Value Addition: Remark	
(i)	Every finite set is closed. By using Example 1, a finite set has no limit points which mean that its derived set is empty. Since every set contain an empty set, therefore, every finite set is closed (by using Theorem 6)
(ii)	Every singleton set is closed. Since every singleton set is a finite set, therefore by using part (i) are obtain that it is closed.
(iii)	The set $\mathbb{Z}$ of integers is closed. We know that $\mathbb{Z}' = \phi \subseteq \mathbb{Z}$. By using Theorem 6, it follows that $\mathbb{Z}$ is a closed set. Likewise, we can show that $\mathbb{N}$, the set of natural numbers, is closed.

Theorem 7: The derived set of any set is always closed.

Proof: Let $A \subseteq \mathbb{R}$. In view of Theorem 6, we need to show that A' contains all its limit points.

Assume that *x* is a limit point of A'.

Then, $(x-\varepsilon, x+\varepsilon) \cap A' - \{x\} \neq \phi$ for each $\varepsilon > 0$.

So, let $y \in (x-\varepsilon,\ x+\varepsilon) \cap A' - \{x\}$.

This implies that $y \in A'$ and $y \neq x$, i.e., *y* is a limit point of *A*.

Then, every neighborhood of *y* contains a point of *A* different from *y*.

In particular, $(x-\varepsilon, x+\varepsilon)$ which is a neighborhood of *y* contains a point *z* of *A* such that $y \neq z$.

It follows that $(x-\varepsilon,\ x+\varepsilon)$ contains a point $z \in A$ where $z \neq x$. (since $x \neq y$ and $y \neq z$).

We conclude that *x* is a limit point of *A*.

Thus, $x \in A'$ and consequently A' contains all its limit points.

Therefore, A' is a closed set.

Now, we study a special class of functions namely sequences.

I.Q. 12

I.Q. 13

Definition 3.10.: A sequence in $\mathbb{R}$ is a real-valued function f whose domain is the set $\mathbb{N}$ of natural numbers.

The element $x_n = f(n), n \in \mathbb{N}$ of range of the function *f* is called the n-th term of the given sequence.

A sequence may be regarded as a list of numbers $x_1, x_2, x_3, \ldots\ldots$. Usually, we shall denote sequences by $\langle x_n \rangle$, $\langle y_n \rangle$ and so on.

Here, we mention some examples of sequences.

Examples 5: (i) $\langle n \rangle$ is the sequence (1, 2, 3, …)

(ii) $\langle (-1)^n \rangle$ is the sequence $(-1,1-1,1,\ldots.)$

(iii) $\langle c \rangle$ is the sequence (c, c, c, …)

(iv) $\left\langle \frac{1}{3^n} \right\rangle$ is the sequence $\left(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \right)$

(v) $\left\langle 1 + \frac{(-1)^n}{n} \right\rangle$ is the sequence $\left(0, \frac{3}{2}, \frac{2}{3}, \ldots \right)$

One of the main notion associated with a sequence is that of convergence. Now, we introduce the concept of converging sequence.

Definition 3.11.: A sequence $\langle x_n \rangle$ is said to converge to a limit $x \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a natural number *m* (depending on ε only) such that for all $n \geq m$, we have

$$|x_n - x| < \varepsilon$$

i.e., $x - \varepsilon < x_n < x + \varepsilon$ for all $n \geq m$,

We write $x_n \to x$ as $n \to \infty$ or simply $\lim x_n = x$.

From the above definition, the convergence of a sequence require the values to be close to the limit for all large values of *n*.

Example 6: Show that $\lim \frac{1}{n} = 0$.

Solution: Let $\varepsilon > 0$ be given.

Consider $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon \text{ iff } n > \frac{1}{\varepsilon}$.

We set $m \in \mathbb{N}$ such that $m > \frac{1}{\varepsilon}$. Then, for all $n \geq m$, we obtain

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{m} < \varepsilon, \text{ that is, } \left| \frac{1}{n} - 0 \right| < \varepsilon \text{ for all } n \geq m.$$

By definition, $\lim \frac{1}{n} = 0$.

Value Addition: Remark
We notice that not every sequence has a limit. For example, the sequence $\langle (-1)^n \rangle$ does not have a limit because $x_n = 1$ for large even values of *n* and $x_n = -1$ for large odd values of *n*.

Definition 3.12.: A sequence $\langle x_n \rangle$ is said to be a divergent sequence if it does not converge to some real number.

I.Q. 14

Value Addition: Remark	
(i)	In order to show $x_n \not\to x$ as $n \to \infty$, we need to find some value of $\varepsilon > 0$, and we can find natural number *m* with $n \geq m$ (any prescribed or winning integer) for which $\lvert x_n - x \rvert \not< \varepsilon$ i.e., $\lvert x_n - x \rvert \geq \varepsilon$.
(ii)	A convergent sequence can have at most one limit.
(iii)	Every convergent sequence is bounded but the converse is not true.

Now we discuss a result which connects limit point of a set with a sequence in the set.

Theorem 8: Let $A \subseteq \mathbb{R}$. A number $x \in \mathbb{R}$ is a limit point of *A* if and only if there exists a sequence $\langle x_n \rangle$ in *A* such that $\lim x_n = x$ and $x_n \neq x$ for all $n \in \mathbb{N}$.

Proof: Firstly, we assume that *x* is a limit point of *A* . Then,

$x \in (x-\varepsilon,\ x+\varepsilon) \cap A - \{x\} \neq \phi$ for all $\varepsilon > 0$, or

equivalently, $\left(x - \frac{1}{n},\ x + \frac{1}{n}\right) \cap A - \{x\} \neq \phi$ for all $n \in \mathbb{N}$. (by using Archimedean property of real numbers)

Let us pick $x_n \in \left(x - \frac{1}{n},\ x + \frac{1}{n}\right) \cap A - \{x\}$ for any $n \in \mathbb{N}$.

Then, $\langle x_n \rangle \in A$ with $x_n \neq x$ for all $n \in \mathbb{N}$.

Also, it follows that $\lvert x_n - x \rvert < \frac{1}{n}$ for all $n \in \mathbb{N}$ which shows that

$\lim x_n = x$.

Thus, if *x* is a limit point of *A*, then there exists a sequence $\langle x_n \rangle$ in *A* with $x_n \neq x$ for all $n \in \mathbb{N}$ and $\lim x_n = x$.

Conversely: Suppose that there exists a sequence, namely, $\langle x_n \rangle$ in *A* with $\lim x_n = x$ and $x_n \neq x$ for all $n \in \mathbb{N}$.

Since $\lim x_n = x$, therefore for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \text{ for all } n \geq m,$$

i.e., $x - \varepsilon < x_n < x + \varepsilon$ for all $n \geq m$

i.e., $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq m$

i.e., $x_n \in (x - \varepsilon, x + \varepsilon) \cap A - \{x\}$

(because $x_n \in A$ for all $n \in \mathbb{N}$ and $x_n \neq x$ for all $n \in \mathbb{N}$)

Thus, $(x - \varepsilon, x + \varepsilon) \cap A - \{x\} \neq \phi$ for all $\varepsilon > 0$.

It follows that *x* is a limit point of *A* which completes the proof of the theorem.

Value Addition: Remark
The above theorem, sometimes, known as the characterization theorem for limit points in terms of sequences.

Definition 3.13: Let $A \subseteq \mathbb{R}$. Then $x \in \mathbb{R}$ is called an isolated point of if $x \in A$ and *x* is not a limit point of *A*.

Examples 7: (i) Consider the set $\mathbb{N}$ of natural numbers.

Let $n \in \mathbb{N}$. Then, $\left(n - \frac{1}{2}, n + \frac{1}{2}\right) \cap \mathbb{N} = \{n\}$.

This implies that *n* is not a limit point of $\mathbb{N}$.

We conclude that every point of $\mathbb{N}$ is an isolated point.

(ii) The set of real numbers $\mathbb{R}$ has no isolated points. We note that $\mathbb{R}' = \mathbb{R}$ (see Example 1(i)). Thus, every element of $\mathbb{R}$ is a limit point of $\mathbb{R}$ and consequently $\mathbb{R}$ has no isolated point.

(iii) Consider the set $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$.

Then, every element of A is an isolated point except 0.

(iv) The set of rational numbers has no isolated point. In fact,

$$\mathbb{Q}' = \mathbb{R} \quad \text{(See Example 1(iii))}$$

(v) The interval [*a, b*] has no isolated point.

I.Q. 15

I.Q. 16

Theorem 1.3.20: Let $A \subseteq \mathbb{R}$ be a non-empty finite set. Then every point of *A* is an isolated point in A.

Proof: See Example 1(v).

Exercises:

1. Show that the set $\left\{-1, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \ldots\right\}$ is neither open nor closed.
2. Give examples of sets A and A′ such that A ⊂ A′.
3. Show that the set of integers has no limit point.
4. State Bolzano-Weierstrass Theorem. Give an example of an infinite bounded set which has a limit point.
5. Verify Bolzano-Weierstrass Theorem for the set

   A = {–1/2, 2/3, –3/4, 4/5, –5/6, 6/7, ...}
6. Find the derived set of the set $\left\{3^n + \frac{1}{3^n} : n \in \mathbb{N}\right\}$.
7. Give an example of a set whose derived set is an infinite bounded set.
8. Let $A \subseteq \mathbb{R}$ and B be an open subset of $\mathbb{R}$. Show that if $B \cap A = \phi$, then $B \cap A' = \phi$.
9. Construct a set with only $\sqrt{2}$ as its limit point.
10. Fill in the blanks:
    - (i) The set of natural numbers has _________limit points.
    - (ii) The set of rational numbers has _________ limit points.
    - (iii) The set irrational numbers has _________ isolated points.
    - (iv) The finite set has _________ isolated points.
    - (v) The derived set of any set is always _________.
11. Give an example of each of the following :

(i) a bounded set of real numbers having no limit point.

(ii) an unbounded set of real numbers having no limit point.

(iii) an infinite set of real numbers having a limit point.

(iv) an infinite set of real numbers having no limit point.

(v) an unbounded infinite set of real numbers having a limit point.

Solutions:

1. Hint: 0 is the limit point of the given set and 0 does not belong to the set.

2. Hint: Let A = (a, b). Then A′ = [a, b].

4. Hint: Consider the set A = (1, 2).

   Then, A has a limit point. Infact, A′ = [1, 2].

5. We note that A is an infinite and bounded set because $-1 < x < 1$ for all $x \in A$.

   Thus, by Bolzano-Weierstrass Theorem it must have a limit point. Infact, 0 is the limit point of A.

6. Hint: The derived set of the given set is empty.

7. Hint: Let $A = \left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N} \right\}$.

   Then, A′ = {0, 1, 1/2, 1/3, ...} which is infinite and bounded.

8. Hint: Prove it by contradiction, i.e., let, if possible, $x \in B \cap A'$.

9. Hint: Consider the set $\left\{ \sqrt{2} + \frac{1}{n} : n \in \mathbb{N} \right\}$.

10. (i) No

    (ii) Infinitely many. Infact $\mathbb{Q}' = \mathbb{R}$.

    (iii) No

    (iv) Finitely many. Infact, every point of a finite set is an isolated point.

    (v) Closed set.

11. (i) Let A = {a}, $a \in \mathbb{R}$.

(ii) Let A = the set of natural numbers.

(iii) Let A = [0, 1]

(iv) Let B = the set of integers.

(v) Let A = the set of rational numbers

Summary:

In this chapter, we have emphasized on the followings:

- definition of the topological concepts such as neighborhood of a set, open set and closed set.
- definition of a limit point of a set and proofs of some of its basic properties.
- statement of Bolzano-Weierstrass Theorem and its proof.
- characterization of a limit points in terms of a closed set and a sequence.
- definition and examples of an isolated point.

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