Lesson: Method of Undetermined Coefficients and Variation of Parameters

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Table of Contents

1. Learning Outcomes:

After reading this lesson reader will be able to understand the following

- Non-homogeneous linear differential equation
- Non-homogeneous linear differential equation with constant coefficients
- Solutions of Non-homogeneous linear differential equations
- Method of undetermined coefficients
- Rule to find the Particular Solution by Method of Undetermined Coefficients
- Method of Variation of Parameters

2. Introduction:

Method of undetermined coefficients and method of variation of parameters are two well-known methods to solve the non-homogeneous linear differential equations. In this lesson, we will study about these methods and how to solve the non-homogeneous differential equations with the help of these methods.

3. Non-Homogeneous Linear Differential Equations:

A differential equation of the form

$$F(x, y, y', y'', \ldots, y^n) = R(x)$$

or $$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + a_2(x)\frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = R(x); \quad a_0 \neq 0 \qquad (1)$$

Where $a_0(x), a_1(x), a_2(x), \ldots, a_{n-1}(x), a_n(x)$ and $R(x)$ are continuous functions of x only on some open interval I is called non-homogeneous linear differential equation of order n if $R(x) \neq 0$.

If R(x) = 0, then the differential equation of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + a_2(x)\frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0; \quad a_0 \neq 0$$

Where $a_0(x), a_1(x), a_2(x), \ldots, a_{n-1}(x)$ and $a_n(x)$ are continuous functions of x only on some open interval I is called associated homogeneous linear differential equation of order n with non-homogeneous linear differential equation (1).

Value Addition: Second Order Homogeneous and Non-homogeneous Linear Differential Equation
1. Differential equation of the form $$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = R(x)$$ where $a_0(x), a_1(x), a_2(x)$ and $R(x)$ are continuous functions of x only on some open interval I is called second order non-homogeneous linear differential equation if $R(x) \neq 0$. 2. If R(x) = 0, then the differential equation of the form $$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$ where $a_0(x), a_1(x), a_2(x)$ and $R(x)$ are continuous functions of x only on some

open interval I is called associated homogeneous linear differential equation to the non-homogeneous linear differential equation.

3.1. Non-homogeneous Linear Differential Equation with Constant Coefficients:

A differential equation of the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+a_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=R(x);\quad a_0\neq 0$$

Where $a_0, a_1, a_2, \ldots, a_{n-1}$ and a_n are all constants on some open interval I is called non-homogeneous linear differential equation with constant coefficients of order n if $R(x)\neq 0$.

Value Addition: Note

1. The function R(x) in the non-homogeneous differential equation corresponds to some external force on the system.
2. Terms $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^n y}{dx^n}$ in a differential equation may also be denoted by $y', y'', y''', \ldots, y^n$ respectively.

4. Solutions of Non-Homogeneous Differential Equations:

Consider the non-homogeneous linear differential equation of the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+a_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=R(x);\quad a_0\neq 0 \qquad \text{(A)}$$

Where $a_0, a_1, a_2, \ldots, a_{n-1}$ and a_n are all constants on some open interval I is called non-homogeneous linear differential equation with constant coefficients of order n if $R(x)\neq 0$.

Then

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+a_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=0;\quad a_0\neq 0 \qquad \text{(B)}$$

is called the associated homogeneous differential equation of the non-homogeneous differential equation (A).

Then the general solution of the homogeneous differential equation (B) is of the form

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + \ . \ . \ . + c_n y_n(x)$$

where $y_1(x), y_2(x), y_3(x), . . . , y_n(x)$ are linearly independent functions. This solution is also called complementary solution of the non-homogeneous differential equation (A).

Let $y_p(x)$ is the particular solution of non-homogeneous differential equation (A). Then the complete solution of the non-homogeneous equation (A) is

$$y(x) = y_c(x) + y_p(\mathrm{x})$$

for all x in interval I.

Example 1: Let $y_p(x) = 3x$ is a particular solution of the differential equation

$$y'' + 4y = 12x$$

find the complete solution of the differential equation.

Solution: Given differential equation is

$$y'' + 4y = 12x \tag{1}$$

Associated homogeneous differential equation is

$$y'' + 4y = 0$$

Corresponding auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow \quad m = \pm 2i = 0 \pm 2i$$

Hence complementary solution is

$$y_c(x) = e^{0.x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$\Rightarrow \quad y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

Since $y_p(x) = 3x$ is particular solution then the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = c_1 \cos 2x + c_2 \sin 2x + 3x$$

is the required solution.

Example 2: Let $y_p(x) = (x+1)$ is a particular solution of the differential equation

$$y'' - 2y' + 2y = 2x; \quad y(0) = 4,\ y'(0) = 8$$

find the complete solution of the differential equation which satisfy the initial conditions.

Solution: Given differential equation is

$$y'' - 2y' + 2y = 2x; \quad y(0) = 4,\ y'(0) = 8 \tag{1}$$

Associated homogeneous differential equation is

$$y'' - 2y' + 2y = 0$$

corresponding auxiliary equation is

$$m^2 - 2m + 2 = 0$$

$$\Rightarrow \quad m = 1 \pm i = 1 \pm i$$

Hence complementary solution is

$$y_c(x) = e^x (c_1 \cos x + c_2 \sin x)$$

Since $y_p(x) = x + 1$ is particular solution then the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = e^x (c_1 \cos x + c_2 \sin x) + (x+1) \tag{2}$$

on differentiating w.r.t. x we have

$$y'(x) = e^x (c_1 \cos x + c_2 \sin x) + e^x (-c_1 \sin x + c_2 \cos x) + 1$$

using the initial conditions, we have

y(0) = 4

$\Rightarrow \quad e^0(c_1 \cos 0 + c_2 \sin 0) + (0+1) = 4$

$\Rightarrow \quad c_1.1 + c_2.0 + 1 = 4$

$\Rightarrow \quad c_1 = 3$

Now, y'(0) = 8

$\Rightarrow \quad e^0(c_1 \cos 0 + c_2 \sin 0) + e^0(-c_1 \sin 0 + c_2 \cos 0) + 1 =$

$\Rightarrow \quad c_1 + c_2 = 7$

$\Rightarrow \quad 3 + c_2 = 5$

$\Rightarrow \quad c_2 = 4$

putting the values of c_1 *and* c_2 in equation (2) we have

$$y(x) = e^x(3\cos x + 4\sin x) + (x+1)$$

is the required solution.

5. Method of Undetermined Coefficients:

The method of undetermined coefficients is applied to find the particular solution of the non-homogeneous differential equation if the function R(x) in the non-homogeneous differential equation is a linear combination of finite products of functions of the following three types:

(i) A polynomial in x

(ii) An exponential function of the form e^{kx}

(iii) A trigonometric function of the form $\cos nx$ *or* $\sin nx$

5.1. Rule to find the Particular Solution by Method of Undetermined Coefficients:

If no term appearing either in R(x) or in any of its derivatives satisfies the homogeneous differential equation associated with the non-homogeneous differential equation (A). Then the particular solution y_p is considered as a linear combination of all linearly independent such terms and their derivatives. Since y_p is a particluar solution of the non-homogeneous differential equation (A). Hence, coefficients of y_p are determined by substituting it into the non-homogeneous equation (A) by comparing the coefficients of like terms of both sides.

Value Addition: Note
Since we determine the coefficients of the particular solution of the differential equation by satisfying the differential equation by putting it into the equation and then comparing the coefficients of like terms. Therefore this method is called the method of undetermined coefficients.

Let R(x) in the non-homogeneous differential equation is of the above three forms then the particular solution y_p is determined as follows:

5.1.1. Case (I): If R(x) is in the form of a Polynomial:

If R(x) is in the form of a polynomial i.e.

$$R(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n$$

Then y_p is considered as follows

$$y_p = (A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n) x^s$$

Where the coefficients $A_0, A_1, A_2, \ldots, A_n \ and \ s$ are to be determined.

Value Addition: Note
If there is no duplication of any part of complementary solution with the function R(x) or with any of its derivative then the particular solution is of the form $A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n$ where the coefficients $A_0, A_1, A_2, \ldots, A_n$ are to be determined.

Example 3: Solve the differential equation by finding the particular solution of the differential equation

$$y'' - y' - 2y = 3x + 4.$$

Solution: Given differential equation is

$$y''-y'-2y=3x+4 \qquad (1)$$

Associated homogeneous differential equation is

$$y''-y'-2y=0$$

Corresponding auxiliary equation is

$$m^2-m-2=0$$

$\Rightarrow \quad (m-2)(m+1)=0$

$\Rightarrow \quad m=-1,\ \ 2$

Thus, complementary solution is

$$y_c(x)=c_1e^{-x}+c_2e^{2x} \qquad (2)$$

Since $R(x)=3x+4$, thus particular solution must be of the form A_0+A_1x then there is no duplication of any term $y_c(x)$ with the particular solution. Then consider

$$y_p(x)=A_0+A_1x \qquad (3)$$

on differentiating w.r.t. x we have

$$y_p'=A_1$$

again differentiating w.r.t. x we have

$$y_p''=0$$

Now putting these values in equation (1) we have

$$0-A_1-(A_0+A_1x)=3x+4$$

$\Rightarrow \quad -A_1-A_0-A_1x=3x+4$

comparing the coefficients of like terms we have

$$A_0=-1 \text{ and } A_1=-3$$

putting the values of $A_0=-1 \text{ and } A_1=-3$ in equation (3) particular solution is

$$y_p(x) = -1 - 3x$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \qquad y(x) = c_1 e^{-x} + c_2 e^{2x} - 3x - 1$$

is the required solution.

Example 4: Solve the differential equation by finding the particular solution of the differential equation

$$y''' + y'' = 4x^2.$$

Solution: Given differential equation is

$$y''' + y'' = 4x^2 \qquad (1)$$

Associated homogeneous differential equation is

$$y''' + y'' = 0$$

Corresponding auxiliary equation is

$$m^3 + m^2 = 0$$

$$\Rightarrow \qquad m^2(m+1) = 0$$

$$\Rightarrow \qquad m = 0,\ 0,\ -1$$

Thus, complementary solution is

$$y_c(x) = (c_1 + c_2 x)e^{0.x} + c_3 e^{-x}$$

$$y_c(x) = c_1 + c_2 x + c_3 e^{-x} \qquad (2)$$

Since $R(x) = 4x^2$, thus particular solution must be of the form $A_0 + A_1 x + A_2 x^2$ then there is a duplication of the term $c_1 + c_2 x$ in $y_c(x)$ with the term $A_0 + A_1 x$ in particular solution. Then consider

$$y_p(x) = (A_0 + A_1 x + A_2 x^2)x^2 \qquad (3)$$

or $\quad y_p(x) = A_0x^2 + A_1x^3 + A_2x^4$

on differentiating w.r.t. x we have

$$y_p' = 2A_0x + 3A_1x^2 + 4A_2x^3$$

again differentiating w.r.t. x we have

$$y_p'' = 2A_0 + 6A_1x + 12A_2x^2$$

on again differentiating w.r.t. x we have

$$y_p''' = 6A_1 + 24A_2x$$

Now putting these values in equation (1) we have

$$(6A_1 + 24A_2x) + (2A_0 + 6A_1x + 12A_2x^2) = 4x^2$$

$$\Rightarrow \quad 2(A_0 + 3A_1) + 6(A_1 + 4A_2)x + 12A_2x^2 = 4x^2$$

comparing the coefficients of like terms we have

$$A_0 = 4,\ A_1 = -\frac{4}{3}\ and\ A_2 = \frac{1}{3}$$

putting the values of $A_0 = 4,\ A_1 = -\frac{4}{3}\ and\ A_2 = \frac{1}{3}$ in equation (3) particular solution is

$$y_p(x) = (4 - \frac{4}{3}x + \frac{1}{3}x^2)\,x^2 = \frac{1}{3}(12\,x^2 - 4x^3 + x^4)$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = c_1 + c_2x + c_3e^{-x} + \frac{1}{3}(x^2 - 4x + 12)\,x^2$$

is the required solution.

5.1.2. Case (II): If R(x) is in the form of sin mx or cosmx:

If R(x) is in the form of

$$R(x) = a\cos mx \ or \ b\sin mx \ or \ a\cos mx + b\sin mx$$

Then y_p is considered as follows

$$y_p = (A\cos mx + B\sin mx)x^s$$

Where the coefficients $A, \ B \ and \ s$ are to be determined.

Value Addition: Note
If there is no duplication of any part of complementary solution with the function R(x) or its derivatives then the particular solution is of the form $A\cos mx + B\sin mx$ where the coefficients $A \ and \ B$ are to be determined.

Example 5: Solve the differential equation by finding the particular solution of the differential equation

$$y'' - 3y' + 2y = 10\cos 3x.$$

Solution: Given differential equation is

$$y'' - 3y' + 2y = 10\cos 3x \qquad (1)$$

Associated homogeneous differential equation is

$$y'' - 3y' + 2y = 0$$

Corresponding auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$\Rightarrow \quad (m-1)(m-2) = 0$

$\Rightarrow \quad m = 1, \ 2$

Thus, complementary solution is

$$y_c(x) = c_1 e^x + c_2 e^{2x} \qquad (2)$$

Since $R(x) = 10\cos 3x$, thus particular solution must be of the form $A\cos 3x + B\sin 3x$ then there is no duplication of the term $y_c(x)$ with the term $A\cos 3x + B\sin 3x$ in particular solution. Then consider

$$y_p(x) = A\cos 3x + B\sin 3x \qquad (3)$$

on differentiating w.r.t. x we have

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

again differentiating w.r.t. x we have

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

Now putting these values in equation (1) we have

$$(-9A\cos 3x - 9B\sin 3x) - 3(-3A\sin 3x + 3B\cos 3x) + 2(A\cos 3x + B\sin 3x) = 10\cos 3x$$

$$\Rightarrow \quad (-7A - 9B)\cos 3\text{x} + (9A - 7B)\sin 3x = 10\cos 3x$$

comparing the coefficients of like terms we have

$$A = -\frac{7}{13} \ and \ B = -\frac{9}{13}$$

putting the values of $A = -\frac{7}{13} \ and \ B = -\frac{9}{13}$ in equation (3) particular solution is

$$y_p(x) = -\frac{7}{13}\cos 3x - \frac{9}{13}\sin 3x = -\frac{1}{13}(7\cos 3x + 9\sin 3\text{x})$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = c_1 e^x + c_2 e^{2x} - \frac{1}{13}(7\cos 3x + 9\sin 3\text{x})$$

is the required solution.

Example 6: Solve the differential equation by finding the particular solution of the differential equation

$$y'' + 4\text{y} = \sin 2x.$$

Solution: Given differential equation is

$$y'' + 4\text{y} = \sin 2x \qquad (1)$$

Associated homogeneous differential equation is

$$y''+4y=0$$

Corresponding auxiliary equation is

$$m^2+4=0$$

$\Rightarrow \quad m=\pm 2i$

Thus, complementary solution is

$$y_c(x)=c_1\cos 2x+c_2\sin 2x \tag{2}$$

Since $R(x)=\sin 2x$, thus particular solution must be of the form $A\cos 2x+B\sin 2x$ then there is a duplication of the term $y_c(x)$ with the term $A\cos 3x+B\sin 3x$ in particular solution. Then consider

$$y_p(x)=(A\cos 3x+B\sin 3x)x \tag{3}$$

on differentiating w.r.t. x we have

$$y_p'=(-2A\sin 2x+2B\cos 2x)x+(A\cos 2x+B\sin 2x)$$

again differentiating w.r.t. x we have

$$y_p''=(-4A\cos 2x-4B\sin 2x)x+(-2A\sin 2x+2B\cos 2x)+(-2A\sin 2x+2B\cos 2x)$$

$\Rightarrow \quad y_p''=(-4A\cos 2x-4B\sin 2x)x+(-4A\sin 2x+4B\cos 2x)$

Now putting these values in equation (1) we have

$$(-4A\cos 2x-4B\sin 2x)x+(-4A\sin 2x+4B\cos 2x)+4(A\cos 2x+B\sin 2x)=\sin 2x$$

$\Rightarrow \quad -4(A\cos 2x+B\sin 2x)x+4(A+B)\cos 2x-4(A-B)\sin 2x=\sin 2x$

comparing the coefficients of like terms we have

$$A+B=0 \quad and \quad -4(A-B)=1$$

on solving these we have

$$A=-\frac{1}{8} \ and \ B=\frac{1}{8}$$

putting the values of $A = -\frac{1}{8}$ *and* $B = \frac{1}{8}$ in equation (3) particular solution is

$$y_p(x) = (-\frac{1}{8}\cos 2x + \frac{1}{8}\sin 2x)\,\mathrm{x} = \frac{1}{8}(\sin 2x - \cos 2\,\mathrm{x})\,\mathrm{x}$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \qquad y(x) = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}(\sin 2x - \cos 2\,\mathrm{x})\,\mathrm{x}$$

is the required solution.

5.1.3. Case (III): If R(x) is in the form of $e^{kx}\cos mx$ *or* $e^{kx}\sin mx$:

If R(x) is in the form of

$$R(x) = e^{kx}\cos mx \;\; or \;\; e^{kx}\sin mx \; or \;\; e^{kx}(\mathrm{a}\cos mx + b\sin mx)$$

Then y_p is considered as follows

$$y_p = e^{kx}(A\cos mx + B\sin mx)x^s$$

Where the coefficients A, B *and* s are to be determined.

Value Addition: Note
If there is no duplication of any part of complementary solution with the function R(x) or its derivatives then the particular solution is of the form $\mathrm{e}^{kx}(A\cos mx + B\sin mx)$ where the coefficients A *and* B are to be determined.

Example 7: Solve the differential equation by finding the particular solution of the differential equation

$$y'' - 2\,y' + 2\,y = 2e^x\sin x\,.$$

Solution: Given differential equation is

$$y'' - 2\,y' + 2\,y = 2e^x\sin x \qquad (1)$$

Associated homogeneous differential equation is

$$y''-2y'+2y=0$$

Corresponding auxiliary equation is

$$m^2-2m+2=0$$

$\Rightarrow \qquad m=1\pm i$

Thus, complementary solution is

$$y_c(x)=e^x(c_1\cos x+c_2\sin x) \tag{2}$$

Since $R(x)=2e^x\sin x$, therefore particular solution must be of the form $e^x(A\cos x+B\sin x)$ but to remove the duplication of the term $y_c(x)$ with the particular solution. Then consider

$$y_p(x)=e^x(A\cos x+B\sin x)x \tag{3}$$

on differentiating w.r.t. x we have

$$y_p'=e^x(A\cos x+B\sin x)x+e^x(-A\sin x+B\cos x)x+e^x(A\cos x+B\sin x)$$

again differentiating w.r.t. x we have

$$\begin{aligned}y_p''=\;&e^x(A\cos x+B\sin x)x+e^x(-A\sin x+B\cos x)x+e^x(A\cos x+B\sin x)+\\&e^x(-A\sin x+B\cos x)x+e^x(-A\cos x-B\sin x)x+e^x(-A\sin x+B\cos x)\\&+e^x(A\cos x+B\sin x)+e^x(-A\sin x+B\cos x)\end{aligned}$$

$\Rightarrow \qquad y_p''=2e^x(-A\sin x+B\cos x)x+2e^x(A\cos x+B\sin x)+2e^x(-A\sin x+B\cos x)$

Now putting these values in equation (1) we have

$$\begin{aligned}&2e^x(-A\sin x+B\cos x)x+2e^x(A\cos x+B\sin x)+2e^x(-A\sin x+B\cos x)\\&-2e^x(A\cos x+B\sin x)x-2e^x(-A\sin x+B\cos x)x-2e^x(A\cos x+B\sin x)\\&+2e^x(A\cos x+B\sin x)x=2e^x\sin x\end{aligned}$$

$\Rightarrow \qquad -2e^x(A\cos x+B\sin x)x+2e^x(A+B)\cos x+2e^x(-A+B)\sin x=2e^x\sin x$

comparing the coefficients of like terms we have

$$A+B=0 \quad and \quad -A+B=1$$

on solving these we have

$$A = -\frac{1}{2} \text{ and } B = \frac{1}{2}$$

putting the values of $A = -\frac{1}{2}$ *and* $B = \frac{1}{2}$ in equation (3) particular solution is

$$y_p(x) = e^x(-\frac{1}{2}\cos x + \frac{1}{2}\sin x)\,\mathrm{x} = \frac{1}{2}(\sin x - \cos x)xe^x$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = e^x(c_1\cos x + c_2\sin x) + \frac{1}{2}(\sin x - \cos x)xe^x$$

is the required solution.

5.1.4. Case (IV): If R(x) is in the form of $e^{kx}(b_0 + b_1x + b_2x^2 + \ . \ . \ . + b_nx^n)$**:**

If R(x) is in the form of

$$R(x) = e^{kx}(b_0 + b_1x + b_2x^2 + \ . \ . \ . + b_nx^n)$$

Then y_p is considered as follows

$$y_p = e^{kx}(A_0 + A_1x + A_2x^2 + \ . \ . \ . + A_nx^n)x^s$$

Where the coefficients $A_0, A_1, A_2, \ . \ . \ ., A_n$ *and* s are to be determined.

Value Addition: Note
If there is no duplication of any part of complementary solution with the function R(x) or its derivatives then the particular solution is of the form $e^{kx}(A_0 + A_1x + A_2x^2 + \ . \ . \ . + A_nx^n)$ where the coefficients $A_0, A_1, A_2, \ . \ . \ ., A_n$ *and* s are to be determined.

Example 8: Solve the differential equation by finding the particular solution of the differential equation

$$y'' - 4\,\mathrm{y} = xe^x.$$

Solution: Given differential equation is

$$y'' - 4y = xe^{x} \qquad (1)$$

Associated homogeneous differential equation is

$$y'' - 4y = 0$$

Corresponding auxiliary equation is

$$m^2 - 4 = 0$$

$\Rightarrow \qquad m = \pm 2$

Thus, complementary solution is

$$y_c(x) = c_1 e^{-2x} + c_2 e^{2x} \qquad (2)$$

Since $R(x) = xe^{x}$, thus particular solution must be of the form $(A_0 + A_1 x)e^{x}$ thus there is no duplication of terms thus, we consider

$$y_p(x) = (A_0 + A_1 x)e^{x} \qquad (3)$$

on differentiating w.r.t. x we have

$$y_p' = (A_0 + A_1 x)e^{x} + A_1 e^{x}$$

again differentiating w.r.t. x we have

$$y_p'' = (A_0 + A_1 x)e^{x} + 2A_1 e^{x}$$

Now putting these values in equation (1) we have

$$(A_0 + A_1 x)e^{x} + 2A_1 e^{x} - 4(A_0 + A_1 x)e^{x} = xe^{x}$$

comparing the coefficients of like terms we have

$$-3A_1 = 1 \;\; and \;\; (-3A_0 + 2A_1) = 0$$

on solving these we have

$$A_0 = -\frac{2}{9} \; and \; A_1 = -\frac{1}{3}$$

putting the values of $A_0 = -\frac{2}{9}$ *and* $A_1 = -\frac{1}{3}$ in equation (3) particular solution is

$$y_p(x) = (-\frac{2}{9} - \frac{1}{3}x)e^x = -\frac{1}{9}(2+3x)e^x$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$\Rightarrow \quad y(x) = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{9}(2+9x)e^x$

is the required solution.

Example 9: Solve the differential equation by finding the particular solution of the differential equation

$$y''' - 6y'' + 12y' - 8y = x^2 e^{2x}.$$

Solution: Given differential equation is

$$y''' - 6y'' + 12y' - 8y = x^2 e^{2x} \tag{1}$$

Associated homogeneous differential equation is

$$y''' - 6y'' + 12y' - 8y = 0$$

Corresponding auxiliary equation is

$$m^3 - 6m^2 + 12m - 8 = 0$$
$$(m-2)^3 = 0$$

$\Rightarrow \quad m = 2, 2, 2$

Thus, complementary solution is

$$y_c(x) = (c_1 + c_2 x + c_3 x)e^{2x} \tag{2}$$

Since $R(x) = x^2 e^{2x}$, thus particular solution must be of the form $(A_0 + A_1 x + A_2 x^2)e^{2x}$ but to remove the duplication of terms, we consider

$$y_p(x) = (A_0 x^3 + A_1 x^4 + A_2 x^5)e^{2x} \tag{3}$$

on differentiating w.r.t. x we have

$$y_p' = 2(A_0x^3 + A_1x^4 + A_2x^5)e^{2x} + (3A_0x^2 + 4A_1x^3 + 5A_2x^4)e^{2x}$$

again differentiating w.r.t. x we have

$$y_p'' = 4(A_0x^3 + A_1x^4 + A_2x^5)e^{2x} + 4(3A_0x^2 + 4A_1x^3 + 5A_2x^4)e^{2x}$$
$$+ (6A_0x + 12A_1x^2 + 20A_2x^3)e^{2x}$$

On again differentiating w.r.t. x we have

$$y_p''' = 8(A_0x^3 + A_1x^4 + A_2x^5)e^{2x} + 12(3A_0x^2 + 4A_1x^3 + 5A_2x^4)e^{2x} +$$
$$6(6A_0x + 12A_1x^2 + 20A_2x^3)e^{2x} + (6A_0 + 24A_1x + 60A_2x^2)e^{2x}$$

Now putting these values in equation (1) we have

$$(6A_0 + 24A_1x + 60A_2x^2)e^{2x} = x^2e^{2x}$$

comparing the coefficients of like terms we have

$$A_0 = 0,\ A_1 = 0 \ \ and \ \ A_2 = \frac{1}{60}$$

putting the values of $A_0 = 0,\ A_1 = 0 \ and \ A_2 = \frac{1}{60}$ in equation (3) particular solution is

$$y_p(x) = \frac{1}{60}x^5e^{2x}$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \qquad y(x) = (c_1 + c_2x + c_3x^2)e^{2x} + \frac{1}{60}x^5e^{2x}$$

is the required solution.

Example 10: Solve the differential equation by finding the particular solution of the differential equation

$$4y'' + 4y' + y = 3xe^x.$$

Solution: Given differential equation is

$$4y''+4y'+y=3xe^{x} \quad (1)$$

Associated homogeneous differential equation is

$$4y''+4y'+y=0$$

Corresponding auxiliary equation is

$$4m^2+4m+1=0$$
$$(2m+1)^2=0$$

$$\Rightarrow \quad m=-\frac{1}{2},-\frac{1}{2}$$

Thus, complementary solution is

$$y_c(x)=(c_1+c_2x)e^{-\frac{1}{2}x} \quad (2)$$

Since $R(x)=3xe^{x}$, thus particular solution must be of the form $(A_0+A_1x)e^{x}$ as there is no duplication of terms, we consider

$$y_p(x)=(A_0+A_1x)e^{x} \quad (3)$$

on differentiating w.r.t. x we have

$$y_p'=(A_0+A_1x)e^{x}+A_1e^{x}$$

again differentiating w.r.t. x we have

$$y_p''=(A_0+A_1x)e^{x}+2A_1e^{x}$$

Now putting these values in equation (1) we have

$$4(A_0+A_1x)e^{x}+8A_1e^{x}+4(A_0+A_1x)e^{x}+4A_1e^{x}+(A_0+A_1x)e^{x}=3xe^{x}$$
$$9(A_0+A_1x)e^{x}+12A_1e^{x}=3xe^{x}$$

$$9(A_0+12A_1)e^{x}+9A_1xe^{x}=3xe^{x}$$

comparing the coefficients of like terms we have

$$9A_1 = 3 \quad and \quad (9A_0 + 12A_1) = 0$$

on solving these we have

$$A_0 = -\frac{4}{9} \; and \; A_1 = \frac{1}{3}$$

putting the values of $A_0 = -\frac{4}{9} \; and \; A_1 = \frac{1}{3}$ in equation (3) particular solution is

$$y_p(x) = (-\frac{4}{9} + \frac{1}{3}x)e^x = \frac{1}{9}(3x-4)e^x$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \qquad y(x) = (c_1 + c_2 x)e^{-\frac{1}{2}x} + \frac{1}{9}(3x-4)e^x$$

is the required solution.

5.1.5. Case (V): If R(x) is in the form of $(b_0 + b_1 x + b_2 x^2 + \ . \ . \ . + b_n x^n)(a\cos mx + b\sin mx)$**:**

If R(x) is in the form of

$$R(x) = (b_0 + b_1 x + b_2 x^2 + \ . \ . \ . + b_n x^n)(a\cos mx + b\sin mx)$$

Then y_p is considered as follows

$$y_p = \left((A_0 + A_1 x + A_2 x^2 + \ . \ . \ . + A_n x^n)\cos mx + (B_0 + B_1 x + B_2 x^2 + \ . \ . \ . + B_n x^n)\sin mx\right)x^s$$

Where the coefficients $A_0, A_1, A_2, \ . \ . \ ., A_n, B_0, B_1, B_2, \ . \ . \ . \ , B_n \; and \; s$ are to be determined.

Value Addition: Note
If there is no duplication of any part of complementary solution with the function R(x) or its derivatives then the particular solution is of the form $(A_0 + A_1 x + A_2 x^2 + \ . \ . \ . + A_n x^n)\cos mx + (B_0 + B_1 x + B_2 x^2 + \ . \ . \ . + B_n x^n)\sin mx$ where the coefficients $A_0, A_1, A_2, \ . \ . \ ., A_n, B_0, B_1, B_2, \ . \ . \ . \ , B_n \; and \; s$ are to be determined.

Example 11: Solve the differential equation

$$(D^2+9)\,y = x\sin 3x\,.$$

Solution: Given differential equation is

$$(D^2+9)\,y = x\sin 3x$$
$$D^2y+9\,y = x\sin 3x$$

$$y''+9\,y = x\sin 3x \qquad (1)$$

Associated homogeneous differential equation is

$$y''+9y=0$$

Corresponding auxiliary equation is

$$m^2+9=0$$

$$\Rightarrow \qquad m=\pm 3i$$

Thus, complementary solution is

$$y_c(x) = (c_1\cos 3x + c_2\sin 3x) \qquad (2)$$

Since $R(x) = x\sin 3x$, thus particular solution must be of the form $(A_0+A_1x)\cos 3x + (A_2+A_3x)\sin 3x$. Since there is a duplication in the complementary solution and the particular solution. Thus to eliminate the duplication of terms, we consider

$$y_p(x) = (A_0x + A_1x^2)\cos 3x + (A_2x + A_3x^2)\sin 3x \qquad (3)$$

on differentiating w.r.t. x we have

$$y_p' = -3(A_0x + A_1x^2)\sin 3x + (A_0+2A_1x)\cos 3x + 3(A_2x + A_3x^2)\cos 3x + (A_2+2A_3x)\sin 3x$$
$$y_p' = (A_2+2A_3x-3A_0x-3A_1x^2)\sin 3x + (A_0+2A_1x+3A_2x+3A_3x^2)\cos 3x$$

again differentiating w.r.t. x we have

$$y_p'' = 3(A_2+2A_3x-3A_0x-3A_1x^2)\cos 3x + (2A_3-3A_0-6A_1x)\sin 3x$$
$$-3(A_0+2A_1x+3A_2x+3A_3x^2)\sin 3x + (2A_1+3A_2+6A_3x)\cos 3x$$

$$y_p'' = (2A_1 + 6A_2 - 9A_0 x + 12A_3 x - 9A_1 x^2)\cos 3x$$
$$+(-6A_0 + 2A_3 - 12A_1 x - 9A_2 x - 9A_3 x^2)\sin 3x$$

Now putting these values in equation (1) we have

$$(-6A_0 + 2A_3)\sin 3x + (2A_1 + 6A_2)\cos 3x + 12A_3 x\cos 3x - 12A_1 x\sin 3x = x\sin 3x$$

comparing the coefficients of like terms we have

$$-12A_1 = 1;\ 12A_3 = 0;\ -6A_0 + 2A_3 = 0 \ \ and\ \ 2A_1 + 6A_2 = 0$$

on solving these we have

$$A_0 = 0,\ A_1 = -\frac{1}{12},\ A_2 = \frac{1}{36}\ and\ A_3 = 0$$

putting the values of $A_0 = 0,\ A_1 = -\frac{1}{12},\ A_2 = \frac{1}{36}\ and\ A_3 = 0$ in equation (3) particular solution is

$$y_p(x) = -\frac{1}{12}x^2\cos 3x + \frac{1}{36}x\sin 3x$$
$$y_p(x) = \frac{1}{36}x(\sin 3x - 3x\cos 3x)$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \qquad y(x) = c_1\cos 3x + c_2\sin 3x + \frac{1}{36}x(\sin 3x - 3x\cos 3x)$$

is the required solution.

6. Method of Variation of Parameters:

Consider the second order non-homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

Where p(x) and q(x) are continuous functions on an open interval I. Then the complementary solution is of the form

$$y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

Where $y_1(x)$ and $y_2(x)$ are linearly independent functions.

Then the particular solution of the equation (1) is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x).r(x)}{W(y_1,\, y_2)}dx + y_2(x)\int \frac{y_1(x).r(x)}{W(y_1,\, y_2)}dx$$

Where $W(y_1,\, y_2)$ is the Wronskian of two independent solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous equation of the non-homogeneous equation given by (1).

Example 12: Using the method of variation of parameters solve the differential equation

$$y'' + 9y = 2\sec 3x \ .$$

Solution: Given differential equation is

$$y'' + 9y = 2\sec 3x \qquad (1)$$

Associated homogeneous differential equation is

$$y'' + 9y = 0$$

Corresponding auxiliary equation is

$$m^2 + 9 = 0$$

$$\Rightarrow \qquad m = \pm 3i$$

Thus, the complementary solution is

$$y_c(x) = c_1 \cos 3x + c_2 \sin 3x \qquad (2)$$

On comparing equation (2) with

$$y_c(x) = c_1 y_1 + c_2 y_2$$

$$\Rightarrow \qquad y_1(x) = \cos 3x \quad and \quad y_2(x) = \sin 3x$$

Then wronskian of y_1 *and* y_2 is

$$W(y_1,\ y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1,\ y_2)=\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$=3\cos^2 3x+3\sin^2 3x$$

$$=3(\cos^2 3x+\sin^2 3x)$$

$$W(y_1,\ y_2)=3$$

Given $r(x)=2\sec 3x$

Using the method of variation of parameter we have

$$y_p(x)=-y_1(x)\int \frac{y_2(x).r(x)}{W(y_1,\ y_2)}dx+y_2(x)\int \frac{y_1(x).r(x)}{W(y_1,\ y_2)}dx$$

$$y_p(x)=-\cos 3x\int \frac{\sin 3x.2\sec 3x}{3}dx+\sin 3x\int \frac{\cos 3x.2\sec 3x}{3}dx$$

$$y_p(x)=-\frac{2}{3}\cos 3x\int \tan 3x dx+\frac{2}{3}\sin 3x\int dx$$

$$y_p(x)=-\frac{2}{3}\cos 3x.\frac{1}{3}\ln\sec 3x+\frac{2}{3}\sin 3x.x$$

$$y_p(x)=-\frac{2}{9}\cos 3x.\ln\sec 3x+\frac{x}{2}\sin 3x$$

Hence the general solution is

$$y(x)=y_c(x)+y_p(x)$$

$$y(x)=c_1\cos 3x+c_2\sin 3x-\frac{2}{9}\cos 3x.\ln\sec 3x+\frac{x}{2}\sin 3x$$

Example 13: Using the method of variation of parameters solve the differential equation

$$y''+y=12x^2\sin x\ .$$

Solution: Given differential equation is

$$y''+y=12x^2\sin x \qquad (1)$$

Associated homogeneous differential equation is

$$y'' + y = 0$$

Corresponding auxiliary equation is

$$m^2 + 1 = 0$$

$\Rightarrow \quad m = \pm i$

Thus, the complementary solution is

$$y_c(x) = c_1 \cos x + c_2 \sin x \qquad (2)$$

On comparing equation (2) with

$$y_c(x) = c_1 y_1 + c_2 y_2$$

$\Rightarrow \quad y_1(x) = \cos x \;\; and \;\; y_2(x) = \sin x$

Then wronskian of y_1 *and* y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

$\Rightarrow \quad W(y_1, y_2) = 1$

Given $r(x) = 12x^2 \sin x$

Using the method of variation of parameter we have

$$y_p(x) = -y_1(x)\int \frac{y_2(x).r(x)}{W(y_1, y_2)}dx + y_2(x)\int \frac{y_1(x).r(x)}{W(y_1, y_2)}dx$$

$$y_p(x) = -\cos x \int \frac{\sin x.12x^2 \sin x}{1}dx + \sin x \int \frac{\cos x.12x^2 \sin x}{1}dx$$

$$y_p(x) = -12\cos x \int x^2 \sin^2 x dx + 12 \sin x \int x^2 \sin x \cos x dx$$

$$y_p(x) = -6\cos x \int x^2(1-\cos 2x)dx + 6\sin x \int x^2 \sin 2x dx$$

On solving it we have

$$y_p(x) = -6\cos x\left[\frac{x^3}{3} - \frac{1}{2}x^2\sin x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right] +$$
$$6\sin x\left[-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]$$

Thus, the complete solution is

$$y(x) = y_c + y_p$$
$$= c_1\cos x + c_2\sin x - 6\cos x\left[\frac{x^3}{3} - \frac{1}{2}x^2\sin x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]$$
$$+ 6\sin x\left[-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]$$

Example 14: Using the method of undetermined coefficients solve the initial value problem

$$y'' + 9y = \sin 2x; \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: Given differential equation is

$$y'' + 9y = \sin 2x; \quad y(0) = 1, \quad y'(0) = 0 \tag{1}$$

Associated homogeneous differential equation is

$$y'' + 9y = 0$$

Corresponding auxiliary equation is

$$m^2 + 9 = 0$$

$$\Rightarrow \quad m = \pm 3i$$

Thus, the complementary solution is

$$y_c(x) = c_1\cos 3x + c_2\sin 3x \tag{2}$$

Since $R(x) = \sin 2x$, thus particular solution must be of the form $A\cos 2x + B\sin 2x$ then there is no duplication of the terms $y_c(x)$ with the term $A\cos 2x + B\sin 2x$ in particular solution Thus we consider

$$y_p(x) = A\cos 2x + B\sin 2x \tag{3}$$

on differentiating w.r.t. x we have

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

again differentiating w.r.t. x we have

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

Now putting these values in equation (1) we have

$$(-4A\cos 2x - 4B\sin 2x) + 9(A\cos 2x + B\sin 2x) = \sin 2x$$

$$\Rightarrow \quad 5A\cos 2x + 5B\sin 2x = \sin 2x$$

comparing the coefficients of like terms we have

$$A = 0 \text{ and } B = \frac{1}{5}$$

putting the values of $A = 0 \text{ and } B = \frac{1}{5}$ in equation (3), particular solution is

$$y_p(x) = \frac{1}{5}\sin 2x$$

Thus the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$\Rightarrow \quad y(x) = c_1\cos 3x + c_2\sin 3x + \frac{1}{5}\sin 2x \qquad (4)$$

On differentiating w.r.t. x we have

$$y'(x) = -3c_1\sin 3x + 3c_2\cos 3x + \frac{2}{5}\cos 2x$$

Putting the initial values we have

$$y(0) = 1$$

$$\Rightarrow \quad c_1\cos 0 + c_2\sin 0 + \frac{1}{5}\sin 0 = 1$$

$$\Rightarrow \quad c_1 = 1$$

Now, $y'(0)=0$

$\Rightarrow \quad -3c_1\sin 0+3c_2\cos 0+\frac{2}{5}\cos 0=0$

$\Rightarrow \quad c_2=-\frac{2}{15}$

Putting these values of c_1 *and* c_2 in equation (4), we have

$$y(x)=\cos 3x-\frac{2}{5}\sin 3x+\frac{1}{5}\sin 2x$$

is the required solution.

Example 15: Using the method of variation of parameters solve the initial value problem

$$y''+y=\cos x;\quad y(0)=1,\ y'(0)=-1.$$

Solution: Given differential equation is

$$y''+y=\cos x;\quad y(0)=1,\ y'(0)=-1 \tag{1}$$

Associated homogeneous differential equation is

$$y''+y=0$$

Corresponding auxiliary equation is

$$m^2+1=0$$

$\Rightarrow \quad m=\pm i$

Thus, the complementary solution is

$$y_c(x)=c_1\cos x+c_2\sin x \tag{2}$$

On comparing equation (2) with

$$y_c(x)=c_1y_1+c_2y_2$$

$\Rightarrow \quad y_1(x)=\cos x \quad and \quad y_2(x)=\sin x$

Then wronskian of y_1 *and* y_2 is

$$W(y_1,\, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1,\, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$W(y_1,\, y_2) = 1$$

Given $r(x) = \cos x$

Using the method of variation of parameter we have

$$y_p(x) = -y_1(x)\int \frac{y_2(x).r(x)}{W(y_1,\, y_2)}dx + y_2(x)\int \frac{y_1(x).r(x)}{W(y_1,\, y_2)}dx$$

$$y_p(x) = -\cos x\int \frac{\text{sinx}.\text{cosx}}{1}dx + \sin x\int \frac{\cos x.\cos x}{1}dx$$

$$y_p(x) = -\cos x\int \sin x.\cos x dx + \sin x\int \cos^2 x dx$$

$$y_p(x) = -\cos x\int \sin x.\cos x dx + \frac{\sin x}{2}\int (1+\cos 2x)dx$$

On solving it we have

$$y_p(x) = -\frac{1}{2}\cos x\sin^2 x + \frac{1}{2}x\cos x + \frac{1}{4}\cos x\sin 2x$$

Hence the complete solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1\cos x + c_2\sin x - \frac{1}{2}\cos x\sin^2 x + \frac{1}{2}x\cos x + \frac{1}{4}\cos x\sin 2x \qquad (4)$$

On differentiating w.r.t. x we have

$$y'(x) = -c_1\sin x + c_2\cos x + \frac{1}{2}\sin^3 x - \cos^2 x\sin x + \frac{1}{2}\cos x - \frac{1}{2}x\sin x$$

$$+\frac{1}{2}\cos x\cos 2x - \frac{1}{4}\sin x\sin 2x$$

Using the initial values, we have

$$\text{y}(0) = 1$$

$\Rightarrow \quad c_1.1 + c_2.0 - \frac{1}{2}.0 + \frac{1}{2}.0 + \frac{1}{4}.0 = 1$

$\Rightarrow \quad c_1 = 1$

And $\quad y'(0) = -1$

$\Rightarrow \quad -c_1.0 + c_2.1 + \frac{1}{2}.0 - 0 + \frac{1}{2}.1 - \frac{1}{2}.0 + \frac{1}{2}.1 - \frac{1}{4}.0 = -1$

$\Rightarrow \quad c_2 = -2$

On putting the values of c_1 *and* c_2 in equation (4) we have

$$y(x) = \cos x - 2\sin x - \frac{1}{2}\cos x \sin^2 x + \frac{1}{2} x\cos x + \frac{1}{4}\cos x \sin 2x$$

is the required solution.

Exercise:

1. Using the method of undetermined coefficients solve the following differential equations

I. $y'' - 4y' + 3y = 1$
II. $y'' - 4y' = 5$
III. $y''' - 4y' = x$
IV. $y'' - 6y' + 9y = e^{2x}$
V. $y'' + y' - 2y = 2(1 + x - x^2)$
VI. $y'' - y = 4xe^x$
VII. $y'' - y = \sin^2 x$
VIII. $y'' - 3y' + 2y = e^{-x}\sin x$
IX. $y'' + y = \sin x + x\cos x$
X. $y'' + 9y = 2x^2 e^{3x} + 5$
XI. $y'' + 4y = 3x\cos 2x$
XII. $y'' + 3y' + 2y = x(e^{-x} - e^{-2x})$
XIII. $y'' - 6y' + 13y = xe^{3x}\sin 2x$

2. Using the method of undetermined coefficients solve the following initial value problem

I. $y''+4y=2x; \quad y(0)=1, \; y'(0)=2$

II. $y''+2y'+2y=\sin 3x; \quad y(0)=2, \; y'(0)=0$

III. $y''+y=\cos x; \quad y(0)=1, \; y'(0)=-1$

IV. $y''+y=\cos x; \quad y(0)=1, \; y'(0)=-1$

V. $y''+3y'+2y=e^{x}; \quad y(0)=0, \; y'(0)=3$

3. Using the method of variation of parameter solve the following differential equations

I. $y''+4y=\sin^2 x$

II. $y''+y=\operatorname{cosec}^2 x$

III. $y''+9y=x\cos x$

IV. $y''+4y=2\cos x\cos 3x$

V. $y''+y'+y=\sin x\sin 3x$

VI. $y''+y=x\cos^3 x$

VII. $y''-2y'-8y=3e^{-2x}$

VIII. $y''+3y'+2y=x(e^{-x}-e^{-2x})$

IX. $y''+y=\tan x$

X. $y''-2y'+2y=4xe^{x}\sin x$

XI. $y''+y=4x\sin x$

XII. $y''+6y'+13y=e^{-3x}\cos 2x$

XIII. $y''-3y'+2y=3e^{-2x}-10\cos 3x$

4. Using the method of variation of parameters solve the following initial value problems

I. $y''-2y'+2y=x+1; \quad y(0)=3, \; y'(0)=0$

II. $y''+3y'+2y=e^{x}; \quad y(0)=0, \; y'(0)=3$

III. $y''+y=\cos x; \quad y(0)=1, \; y'(0)=-1$

IV. $y''+4y=2x; \quad y(0)=1, \; y'(0)=2$

Summary:

In this lesson we have defined and emphasized on the following

- Non-homogeneous linear differential equation
- Non-homogeneous linear differential equation with constant coefficients
- Solutions of Non-homogeneous linear differential equations
- Method of undetermined coefficients
- Rule to find the Particular Solution by Method of Undetermined Coefficients
- Method of Variation of Parameter

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