Paper: Differential Equations-I

Lesson : Predator-Prey and Epidemic Models

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1. Learning outcomes:

After studying this chapter you should be able to

- Understand the meaning of the term 'Model' and develop the skills of modeling
- How to relate a general life situation to a mathematical model
- How to prepare a mathematical model with the help of differential equations
- Understand the concept of phase-plane and its analysis
- Understand the concept of equilibrium points

2. Introduction

We face many problems in our day to day life. These problems are sometime become very small and sometime become very serious. Everybody wants a better future and mathematics help us to get it. We can model a life situation with mathematics and the results of this model help us to predict the future. To prepare a model, we need to convert the problem into word equation and then associate mathematical equations. In this chapter we use differential equations to develop a model. A single model can be used for many similar situations. Here we develop models for spread of disease, interaction between two species and model for battle.

I.Q. 1

3. An epidemic model for influenza

Here a model is developed to describe the spread of disease in population and apply it to describe the influenza in city. To do so the population is divided into three groups: those susceptible to catching the disease, those infected with disease and capable of spreading it and those who have recovered and are immune from the disease. A system of two coupled differential equations is obtained by modelling these interacting groups.

3.1. Model assumptions

In the case of considering a disease, the population can be categorized into different classes; susceptible $S(t)$ and infectious infectives $I(t)$, where t denotes the time. The population liable to catch the disease is called the susceptibles, while the infectious infectives are those infected with the diseases that are capable to transfer it to a susceptible. The last category is of those who have recovered from the disease and who are now safe from further infection of the disease.

Initially, some assumptions are made to build the model, which are as follows:

- To ignore the random differences between individuals, we assume the populations of susceptibles and infectious infectives are large.
- We assume that the disease is spread by contact only and ignore the births and deaths in this model.
- We set the latent period for the disease equal to zero.
- We assume all those who recover from the disease are then safe (at least within the time period considered).
- At any time, the population is mixed homogenously, i.e. we assume that the susceptibles and infectious infectives are always randomly distributed over the area in which the population lives.

3.2. Formulating the differential equations

The rate of change in the number of susceptibles and infectious infectives describe in word equations with the help of an input-output compartment diagram. This process is illustrated in the following example.

Example 1: Create a compartmental diagram for the model and develop appropriate word equation for the rate of change of susceptibles and infectious infectives.

Solution: Since births are ignored in the model and infectious infectives cannot become susceptibles again i.e. the loss of those who become infected is the only way to change the number of susceptibles. The number of infectives decreases due to those infectives who die, become safe or are isolates and changes due to the susceptibles becoming infected.

The appropriate word equations are

$$\{\text{rate of change in no.of susceptibles}\} = -\{\textit{rate of susceptibles become infected}\}$$

$$\{\text{rate of change in no.of infectives}\} = \begin{Bmatrix} \textit{rate of susceptibles become} \\ \textit{infected} \end{Bmatrix} - \begin{Bmatrix} \textit{rate of infectives} \\ \textit{have recovered} \end{Bmatrix} \quad (1)$$

$$\{\textit{rate of change in no.of recovered}\} = \{\textit{rate of infectives have recovered}\}$$

Figure 1: Compartmental diagram for the epidemic model of influenza in a city, where there is no reinfection.

Let us first consider to model the total rate of susceptibles infected that only a single infective spread the infection in susceptibles. It is clear that the growth in the number of infectives will be greater due to greater the number of susceptibles. Thus, the rate of susceptibles diseased by a single infective will be an increasing function of the number of susceptibles. For ease, let us assume that the rate of susceptibles infected by a single infective is directly proportional to the number of susceptibles. Let $S(t)$ be the number of susceptibles at time t and $I(t)$ be the number of infectives at time t, then

$$\{rate\ of\ \ susceptible\ \inf ected\} \propto S(t)$$
$$\Rightarrow \{rate\ of\ \ susceptible\ \inf ected\} = \lambda S(t)$$

Where, constant λ is called the *transmission coefficient* or infection rate (Proportionality constant).

Hence, $\lambda S(t)$ will be the rate of susceptibles infected by a single infective and if we multiply $\lambda S(t)$ to the number of infectives, we will get total rate of susceptibles infected by infectives. Hence

$$\{rate\ of\ \ susceptible\ \inf ected\} = \lambda S(t) I(t) \qquad (2).$$

We must also account for those who have recovered from disease. In general, those infectives who died due to disease, those who become protected to the disease and those who become isolated will be counted as removed. The number of infectives removed in the time interval should depend only on the number of infectives, but not upon the number of susceptibles. Let the rate of infectives recovered from the disease is directly proportional to the number of infectives. We write

$$\{rate\ of\ \inf ectives\ recoverd\ from\ the\ disease\} \propto I(t)$$
$$\Rightarrow \{rate\ of\ \inf ectives\ recoverd\ from\ the\ disease\} = \delta I(t) \qquad (3)$$

Where constant δ is called *recovery rate* or the removal rate (constant of proportionality). This rate is a per-capita rate. The residence time in the infective compartment, i.e. the mean that an individual is infectious can be recognized as the reciprocal of the recovery rate i.e. δ^{-1}. Normally the infectious period for influenza is 1-3 days.

Value Addition: Note
❖ The rate at which susceptible converted into infected is proportionate to the number of susceptibles and infectives both, ❖ The rate at which infectives recover and are removed is proportionate to the number of infectives only.

dS/dt is the rate of change in the number of susceptibles with respect to time and the rate of change in the number of infectives with respect to time is given by dI/dt. The rate of change in the number of recovered from the disease i.e. recovered is given by dR/dt. Finally the population word equations 1can be written in mathematical form with the use of equations 2 and 3 .

$$\begin{aligned}\frac{dS}{dt} &= -\lambda SI,\\ \frac{dI}{dt} &= \lambda SI - \delta I, \qquad (4)\\ \frac{dR}{dt} &= \delta I,\end{aligned}$$

with initial condition $S(0) = s_0$, $I(0) = i_0$ and $R(0) = 0$.

Equation 4, a coupled system of nonlinear differential equations, were originally derived by Kermack and McKendrick in 1927 (Kermack and McKendrick, 1927). Since the R variable does appear only in the third differential equation. So the coupled system in (4) without third differential equation can be studied as a system on its own.

I.Q. 2

I.Q. 3

4. Predators and prey

In this section, we develop a simple predator-prey model for omnivores using the evolution of population of small insect pests which interact with another population of beetle predators.

4.1. Background of model

There are several types of predator-prey interactions: that of carnivores which eat animal species, that of herbivores which eat plant species, that of cannibals which eat their own species and that of leeches which lives on or in another species (the host).

4.2. Model assumptions

Initially, few preliminary assumptions are made to build the model, which are as follows:

- To neglect random differences between individuals we assume the populations are sufficiently large.
- Initially, DDT effect is discounted, but later the model is modified to incorporate its influence on the system.
- We assume that the predator and the prey are only two populations, which affect the environment.
- In the absence of a predator, the prey population can grows exponentially.

4.3. Compartmental model

The number of prey and the number of predators are two separates quantities which vary with time. It is better to consider the population density i.e. number per unit area, rather than population size, for population of animal as we do here. The system can be defined in two word equations, one for the rate of change of predator density and one for the rate of change of prey density.

Example 2: Determine a word equation and appropriate compartment diagram for the prey and predator both.

Solution: Births is the only inputs and Deaths is the only outputs for each population. Though, capturing and eating by the predator is the cause for the prey deaths. This is shown in the input-output compartmental diagram of figure 2. Here we consider two reasons for prey; one is natural prey deaths and the other prey deaths due to predators. Similarly we consider and differentiate between natural predator births, taking place in absence of prey, and additional births and that would occur due to the prey being eaten by predators. The input–output diagram for the predator-prey model is

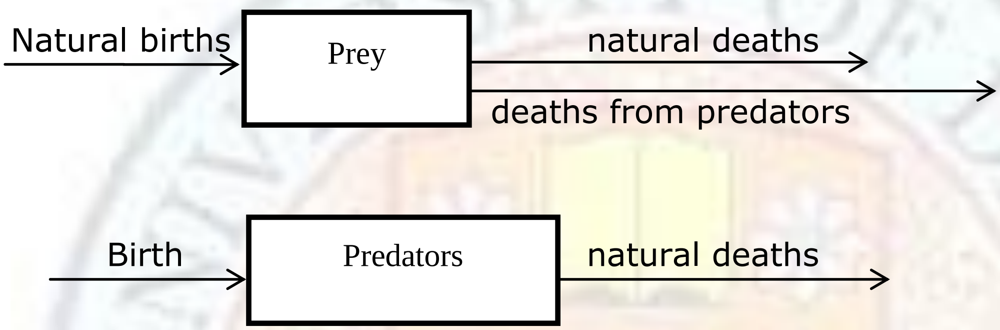

Figure 2: input-output diagram for 2- species predator and prey model.

The applicable word equations for the model are

$$\left\{\begin{matrix} \text{rate of change of} \\ \text{prey} \end{matrix}\right\} = \left\{\begin{matrix} \text{rate of natural prey} \\ \text{births} \end{matrix}\right\} - \left\{\begin{matrix} \text{rate of natural prey} \\ \text{deaths} \end{matrix}\right\} - \left\{\begin{matrix} \text{rate of prey killed} \\ \text{by perdator} \end{matrix}\right\},$$

$$\left\{\begin{matrix} \text{rate of change of} \\ \text{predator} \end{matrix}\right\} = \left\{\begin{matrix} \text{rate of predator} \\ \text{births} \end{matrix}\right\} - \left\{\begin{matrix} \text{rate of natural predator} \\ \text{deaths} \end{matrix}\right\} \qquad (5)$$

The rate of births from an *individual* prey is defined as per-capita birth rate and it does not depend on the density of predators. Let us assume that a constant b_1 is the per-capita birth rate for the prey (the scale insect). Similarly, the natural pre-capita death rate of the scale insect is a constant a_1. On the other hand the per-capita death rate of prey due to being killed by the predators will depend on the predator density and it will be directly proportional to density of predators. For simplicity let us consider this per-capita rate is proportional to the predator density. If the density of predators is more, the probability of an individual prey will be eaten is more. We assume that prey density does not affect a constant per-capita death rate for the predators. It is difficult to get the per-capita birth rate of predator. We assume that the important necessity for the births of the predator are prey, so the per-capita birth-rate for the predators will be the sum of a natural birth rate (which is constant) and a supplementary birth rate which is proportional to the rate of prey killed by predator. It is obvious that if the amount of prey available (food) is more at any time, the per-capita birth rate of predator will increase at that time.

Example 3: Formulate differential equations for the predator and prey density using the above assumptions and word equations 5.

Solution: Let the number of prey per unit area is denoted by $X(t)$ and the number of predators per unit area $Y(t)$. Let us assume that a constant b_1 is the per-capita birth rate for the prey (the scale insect). Similarly, the natural pre-capita death rate of the prey is a constant a_1 and per capita death rate of the predator is given by a_2.

The overall rate can be obtained by multiplying the per-capita rates by the corresponding population densities, we can write,

$$\{rate\ of\ \ prey\ births\} = b_1 X(t),$$

$$\{rate\ of\ \ prey\ natural\ \ deaths\} = a_1 X(t), \qquad (6)$$

$$\{rate\ of\ \ predator\ \ deaths\} = a_2 Y(t)$$

Since deaths of prey (killed) is proportional to the predator density, for the prey deaths, the per-capita death rate is defined as $c_1 Y(t)$, with c_1 as the positive constant of proportionality. Thus the rate at which prey are killed or eaten by predators is given by $c_1 Y(t) X(t)$. The birth rate of predator has a factor which is proportional to this rate of prey killed or eaten by predators, so we write

$$\{rate\ of\ \ prey\ killed\ by\ \ predators\} = c_1 Y(t) X(t),$$
$$\{rate\ of\ \ predator\ births\} = b_2 Y + k c_1 Y(t) X(t) \qquad (7)$$

Where b_2 is per-capita birth is rate of predator and k is positive constant of proportionality.

Now we change the word equation (5) into the pair of differential equations with the help of the equations (6-7).

$$\frac{dX}{dt} = b_1 X - a_1 X - c_1 XY \Rightarrow \frac{dX}{dt} = (b_1 - a_1)X - c_1 XY,$$

$$\frac{dY}{dt} = b_2 Y + fc_1 XY - a_2 Y \Rightarrow \frac{dY}{dt} = (b_2 - a_2)Y + fc_1 XY.$$

Let $\lambda_1 = b_1 - a_1$, $-\lambda_2 = b_2 - a_2$ and $c_2 = fc_1$, then

$$\frac{dX}{dt} = \lambda_1 X - c_1 XY, \qquad \frac{dY}{dt} = c_2 XY - \lambda_2 Y \qquad (8)$$

Where, $\lambda_1, \lambda_2, c_1$ and c_2 are all positive constants.

This system of equation is called the *Lotka-Volterra predator-prey system*. The constraints c_1 and c_2 are known as interaction parameters. Since on the right hand side of each differential equation we have positive and negative terms, we can expect the growth or decline in population. Further, the differential equations in (8) are coupled as solution of one equation will be used to solve other differential equation. These differential equations are nonlinear as they have the product XY. The product XY can be interpreted as it is proportional to the rate of contacts (encounters) between the two species i.e. predator and prey.

Example 4: Check the Predator-Prey model in the restrictive cases of prey without predator, and predator without prey.

Solution: Suppose there are no predator i.e number of prey is zero so that $Y = 0$. The equations then reduce to

$$\frac{dX}{dt} = \lambda_1 X$$

$$\Rightarrow \quad \frac{dX}{X} = \lambda_1 dt$$

$$\Rightarrow \quad X(t) = e^{\lambda_1 t}$$

Hence the prey grows exponentially in the absence of predators.

Similarly, If there are no prey then $X = 0$ and the equation reduce to

$$\frac{dY}{dt} = -\lambda_2 Y \Rightarrow Y(t) = e^{-\lambda_2 t}$$

That is, the predator population will decay exponential and die out in the absence of prey.

I.Q. 4

I.Q. 5

5. Model of a Battle

Now we study an original type of population interaction: battle between two contrasting groups or a destructive struggle. These may be fights between two aggressive insect groups, human armies or athletic teams. We will develop the model for the battle of two human armies. Many other example can modeled after generalizing of the model.

5.1. Background

Since the ancient times we have seen/heard about the battles between armies. In ancient times battles were mostly fought hand-to-hand. After the development of many disasters weapons, the battle has been fought with weapons like gun machine etc. Although there are many reasons to influence the battle outcome but numerical superiority and superior military training are crucial. F. W. Lanchester who was famous for his contributions to the theory of fight first developed this model in 1920s.

We want to develop a simple model to predict the soldier's strength in each army at any given time, provided we know the initial number of soldiers in each army.

5.2. Model assumptions

Initially few basic assumptions are made and then develop the model based on these.

- To neglect the random differences between armies, we assume the number of soldiers is sufficiently large.
- There are no backups and no functioning loses (i.e. due to desertion or disease).

Few assumptions can be easily relaxed at a later stage in case of inadequate model. We take an example of army to develop this model.

Value Addition: We develop the model in following steps
❖ First we develop compartmental diagram and then word equations based on input-output principal of balance law.   ❖ Then we develop a system of two coupled differential equation with the help of word equations.

Example 5: Let us suppose that green army and yellow army are two opposite groups or populations. Draw the suitable compartment diagram and linked word equations for the number of soldiers in both the green and yellow armies.

Solution: Since there are backups or operational losses, the number of soldiers who are injured by the other army can change each population size. So we can prepare an input-output diagram of figure 3.

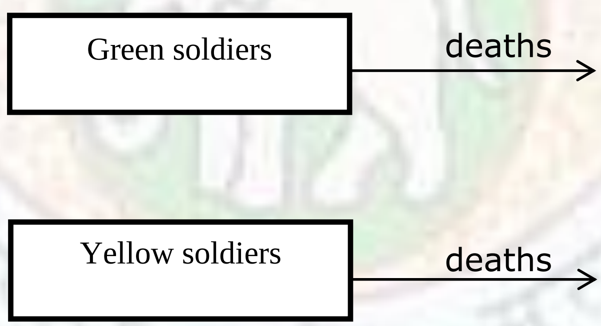

Figure 3: Compartment diagram for the simple battle model.

Thus, the word equations for the battle model at any time,

$$\{rate\ of\ change\ of\ green\ soldiers\} = -\{rate\ red\ soldiers\ wounded\ by\ yellow\ army\}$$
$$\{rate\ of\ change\ of\ yellow\ soldiers\} = -\{rate\ blue\ soldiers\ wounded\ by\ green\ army\} \quad (9)$$

In a real battle situation there is a combination of shots; (a) one fired into an area where the chances that the enemy will be hidden are more and

(b) one fired directly at a soldier of the opposite army. The method of firing can dominate some battles. We consider these two ideologies of shots as *target fire* and *arbitrary fire.* For both the armies we assume only targeted fire in the model.

In the targeted fire ideology, we consider all targets are visible to army persons firing at them. If the yellow army used targeted fire on the green army, then each time an individual green soldier is targeted by a yellow soldier. The rate of injured green army soldiers gets affected only by the number of yellow soldiers firing at them but not on the number of green soldiers. For arbitrary fire a soldier firing a gun on hidden target, into a region where opponent soldiers are known to be hidden. So in arbitrary fire we consider the rate of enemy soldiers wounded will depend on both the army strength i.e. number of soldiers firing and the soldiers being fired at.

5.3. Formulating the differential equations

Let the number of soldiers of the green army is denoted by $G(t)$ and the number of soldiers of the yellow army is denoted by $Y(t)$. We assume that both ate armies fired on visible target.

After the above discussion we can make the following assumptions:

- The rate at which the soldiers are wounded is directly proportional to the number of enemy/opponent soldiers only for targeted fire.
- The rate of soldiers wounded is directly proportional to both number of soldiers in arbitrary fire.

These assumptions can be expressed mathematically by writing

$$\{\textit{rate green soldiers wounded by yellow army}\} = \lambda_1 Y(t),$$

$$\{\textit{rate yellow soldiers wounded by green army}\} = \lambda_2 G(t) \qquad (10)$$

Where λ_1 and λ_2 are positive constant of proportionality, and are called *attrition coefficients*. They measure the effectiveness of yellow army and green army respectively.

We also assume that attrition rates depend only on the firing rates and are a measure of the success of each firing.

Now we put equation (10) into the word equation (9), where dG/dt denotes the rate of change in the number of green soldiers and it is dY/dt f represent the change in the number of yellow soldiers. So the two simultaneous differential equations are

$$\frac{dG}{dt} = -\lambda_1 Y, \qquad \frac{dY}{dt} = -\lambda_2 G \qquad (11)$$

I.Q. 6

6. Interpretation of parameters

Now to refine the model we try to express the parameters λ_1 and λ_2 in terms of possible quantities which could be measured. The soldiers are wounded at a rate which depends on both the firing rate and probability of a shot hitting a target.

Now again from equations (10). Consider a single yellow soldier firing at the green army. Let f_y be a constant rate at which each yellow soldier fires (rate of bullet fired) . Then

$$\left\{\begin{matrix} \text{rate of green soldiers} \\ \text{wounded by single} \\ \text{yellow soldier} \end{matrix}\right\} = \left\{\begin{matrix} \text{rate of bullets} \\ \text{fired in time} \\ \text{interval} \end{matrix}\right\} \times \left\{\begin{matrix} \text{probability of} \\ \text{a single bullet} \\ \text{hitting target} \end{matrix}\right\} = f_y p_y$$

Where p_y is the probability (constant) that a green soldier is wounded by a single bullet from the yellow soldiers. Hence the green soldiers wounded by the entire yellow army (per unit time) can be counted by multiply by the number of yellow soldiers. This gives

$$\{\text{rate of green soldiers wounded by yellow army}\} = f_y p_y Y(t) \qquad (12)$$

Similarly,

$$\{\text{rate of yellow soldiers wounded by green army}\} = f_g p_g G(t)$$

Equating equation (10) and equation (12) ,we get the attrition rates, or coefficients, λ_1 and λ_2 as

$$\lambda_1 == f_y p_y, \qquad \lambda_2 = f_g p_g \qquad (13)$$

Where f_g denotes the firing rate by the single green soldier and the probability that a single green bullet hits its target is denoted by p_g.

The probability of a single bullet wounding a soldier cannot be constant for arbitrary fire. It will fluctuate and depend on the number of target soldiers actually occurs within a targeted area. Thus, this probability will get affected by the number of target soldiers and the area into which the opposite army fired both.

7. Equilibrium points

Let us consider the linear differential equations (coupled) of first order

$$\frac{dX}{dt} = Y, \qquad \frac{dY}{dt} = -X.$$

A point where the solutions of a coupled system of differential equations are constant is known as *equilibrium points*, i.e. where $dX/dt = 0$ and $dY/dt = 0$, simultaneously.

Therefore, from the given differential equations, we get

$$Y = 0, \qquad X = 0$$

So $(X,Y) = (0,0)$ is the equilibrium solution.

I.Q. 7

I.Q. 8

8. Trajectories and phase-plane diagram

Let us consider the (X,Y)-plane: known as the phase-plane. Dividing the plane into four quadrants as shown in the figure below, in the first quadrant we have $X > 0$ and $Y > 0$, i.e. $dX/dt = Y > 0$ *and* $dY/dt = -X < 0$. Hence $X(t)$ is increasing and $Y(t)$ is decreasing and for any solution in that quadrant, we get a direction vector, given by the arrow in figure below. Similarly we can consider each quadrant. Hence we can conclude that the phase-plane trajectories moving in a clockwise direction are the solutions.

8.1. Using the chain rule

Since on the right hand side, none of the differential equation has time variable t i.e. time variable t does not explicitly used in any differential equation. But the derivatives on the left hand side are with respect to time so the solutions will be time dependent. Hence we find an relation between X and Y, independent of t. In other word, we may express Y as a function of X. That is, we are making X the independent variable while previously it was a time t dependent variable.

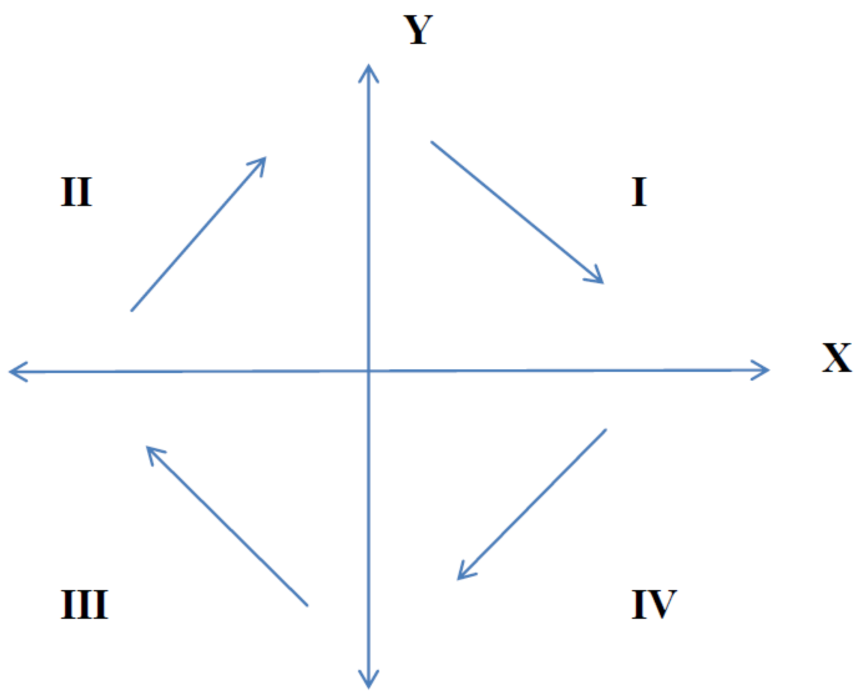

Fig: Direction vector for the trajectories in phase plane of above equations

An expression for the chain rule is

$$\frac{dY}{dt} = \frac{dY}{dX}\frac{dX}{dt}$$

which gives the derivative of Y with respect to t in terms of the derivative of Y with respect to X and the derivative of X with respect to t. Dividing by $\frac{dX}{dt}$ both side, we get

$$\frac{dY}{dX} = \frac{dY/dt}{dX/dt}$$

2

We substitute the value from the equations (1) into (2) which give

$$\frac{dY}{dX} = -\frac{Y}{X}. \qquad 3$$

Hence we get a first-order differential equation with Y a function of X .

Now we can easily solve the differential equation (3) by variable separable mothod, we can write

$$Y\frac{dY}{dX} = -X. \qquad 4$$

Solution: Integrate both sides with respect to the independent variable X we get

$$\int Y\frac{dY}{dX}\,dX = \int -X\,dX.$$

$$\Rightarrow \quad \int Y\,dY = \int -X\,dX.$$

Which gives

$$\frac{1}{2}Y^2 = -\frac{1}{2}X^2 + A$$

Where A is the integration constant. If we multiplying both side by 2, we have t

$$X^2 + Y^2 = B$$

where $B = 2A$. With the help of initial condition we can obtain the value of B .

This solution is the equation of a circle. It defines the paths drawn out by the (X, Y) pair over time, depending on starting conditions or the initial values. This will be the exact solutions to the phase-plane trajectories.

8.2. Interpretation of the phase-plane

After the analysis or interpretation of these trajectories, we found that the system start at the point (x_0, y_0) in the phase-plane if a system has its initial values x_0 and y_0, as time changes, it gives the trajectories curve (circle in above discussed case) in clockwise direction as sketched in above figure. The value of $X(t)$ and $Y(t)$ will be the coordinates of this trajectory at any consequent time. In the case of closed trajectory, the motion will be repeated continuously as in above discussed case of circle.

Generally we need an exact solution of the original coupled equations to check how the system varies with time. Sometimes it is not possible then we use the chain rule to gather valuable information about the system.

Phase-plane analysis is an easy technique to understand some common feature of the system which can be done by drawing a phase-plane diagram together with the phase-plane trajectories. If the differential equations are adequately simple, we may get an exact expression which relates the two dependent variables and describes the trajectory path by eliminating the time variable with chain rule.

The behavior of solutions for a variety of initial conditions can be easily understood with the phase-plane diagram. In the above example, we saw that all solutions of the differential equations have phase-trajectories as circles. We see here, as we move along the trajectory, both the variable X and Y shall return to their original values so the plot for both variables as functions of time should be oscillations. Also, the amplitude of the oscillation is reduced as the initial point approaches the equilibrium point, with the equilibrium point itself corresponding to a solution which is constant in time.

Note: To get a single first-order equation by reducing the coupled differential equations, we have to pay some price and we lost information about time in this procedure

I.Q. 9

Skill developed:

- Understand the theory of equilibrium solution.
- Create the direction of trajectories.
- Able to draw phase-plane based on the information on equilibrium points and trajectory direction.
- Use of chain rule to eliminate time variable and to reduce a coupled pair of differential equation into a single differential equation.

result of the model we obtain a pair of coupled differential equations, where the numbers of soldiers in the green and yellow army is denoted by $G(t)$ and $Y(t)$ denote, respectively. We supposed, both armies used only aimed fire. And we obtain a pair of the differential equations.

$$\frac{dG}{dt} = -\lambda_1 Y, \qquad \frac{dY}{dt} = -\lambda_2 G, \tag{15}$$

Where λ_1 and λ_2 are attrition coefficients (positive constant).

Applying the chain rule to eliminate the time variable t

We can write

$$\frac{dY}{dG} = \frac{dY/dt}{dG/dt} = \frac{\lambda_2}{\lambda_1}\frac{G}{Y}.$$

$$\Rightarrow \quad \frac{dY}{dG} = \lambda \frac{G}{Y} \tag{16}$$

Thus we get a single first order differential equation independent of time variable t, which relates Y and G.

Example 6: Find the solution of the differential equation 16.

Solution: Using separation of variable and then integrate both side with respect to independent variable G. It gives

$$\int Y \frac{dY}{dG} dG = \int \frac{\lambda_2}{\lambda_1} G\, dG.$$

$$\Rightarrow \quad \int Y\, dY = \int \frac{\lambda_2}{\lambda_1} G\, dG.$$

Integrate both side to get the equation

$$\frac{1}{2}Y^2 = \frac{\lambda_2}{2\lambda_1} G^2 + A,$$

Where A is constant of integration. Multiplying both sides of the equation by 2 we get

$$Y^2 = \frac{\lambda_2}{\lambda_1} G^2 + B$$

Where $B = 2A$ is also an arbitrary constant.

Relating the initial condition $G(0) = g_0$ *and* $Y(0) = y_0$ we get

$$y_0{}^2 = \frac{\lambda_2}{\lambda_1} g_0^2 + B$$

So that

$$B = y_0{}^2 - \frac{\lambda_2}{\lambda_1} g_0^2$$

Example 7: Find the equilibrium [points of the differential equations $\frac{dX}{dt} = \lambda_1 X - c_1 XY$ *and* $\frac{dY}{dt} = c_2 XY - \lambda_2 Y$

Solution: we set $dX/dt = 0$ *and* $dY/dt = 0$ to get the equations

$$\lambda_1 X - c_1 XY = 0, \qquad -\lambda_2 Y + c_2 XY = 0$$

Or in factor form we can write

$$X(\lambda_1 - c_1 Y) = 0, \qquad (17)$$

$$Y(-\lambda_2 + c_2 X) = 0. \qquad (18)$$

Two possible solutions arise from (17) which are: $X = 0$ *or* $\lambda_1 - c_1 Y = 0$. Each case is necessary to consider. All the parameters $\lambda_1, \lambda_2, c_1$ *and* c_2 are positive (non-zero) constants.

If $X = 0$, then putting this into (18) gives $-\lambda_2 Y = 0$ hence $Y = 0$. so $(X, Y) = (0,0)$ is one possible solutions of both equations.

If , $\lambda_1 - c_1 Y = 0$, then $Y = \lambda_1 / c_1$. Put this into (18) gives $-\lambda_2 + c_2 X = 0$ or $X = \lambda_2 / c_2$. Hence the second solution of both equations is $(X, Y) = (\lambda_2 / c_2, \lambda_1 / c_1)$.

I.Q. 10

Exercise:

1. Develop the differential equations for the epidemic model. Use the parameter λ=0.004 and δ=0.4 , and assume that initially there are

550 susceptible and only one infective. Calculate how many susceptibles never get infective.

2. Consider the system of differential equations

$$X(\beta_1 - c_1 Y - d_1 X) = 0$$
$$Y(\beta_2 - c_2 X - d_2 Y) = 0$$

Find all the equilibrium points.

3. Prove that the direction trajectory for the predator-prey model is closed.

4. Consider the predator-prey model with density dependent growth of the prey and explain why it is differ from standard Lotka-Voltera system.

5. Find equilibrium points for the following equations

$$(i)\; x' = 2x - 3xy,\; y' = xy - 2y,$$
$$(ii)\; x' = 3x - xy,\; y' = y - 2xy$$

6. Apply chain rule to find relation between X and Y for the differential equations

$$\frac{dx}{dt} = -2xy, \quad \frac{dy}{dt} = -3y$$

Summary:

In this lesson we have emphasized on the following

- Meaning of the term 'Model' and develop the skills of modeling
- How to relate a general life situation to a mathematical model
- How to prepare a mathematical model with the help of differential equations
- Concept of phase-plane and its analysis
- Concept of equilibrium points

References:

Predator-Prey and Epidemic Models

1. Belinda Barnes and Glenn Robert Fulford, Mathematical modeling with case studies: A differential equation approach using Maple and Matlab, 2011

2. http://en.wikipedia.org/wiki/mathematical_model
3. http://www.egyankosh.ac.in