Lesson: Continuity of Functions

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Continuity of Functions

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After studying this chapter you should be able to understand the

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2. Introduction:

The term "Continuous" has been used since the time of Newton to refer to the motion of bodies or to describe an unbroken curve, but it was not made precise until the nineteenth century. Work of Bernhard Bolzano in 1817 and Augustin Louis Cauchy in 1821 identified continuity as a very significant property of functions and proposed definition but since the concept is tied to that of limit, it was the careful work of Karl Weierstrass in the 1870's that brought proper understanding to the idea of Continuity.

3. Continuity at a Point:

Let X be a non-empty subset of the set of real numbers. Let $f : X \to R$ *and let* $x_0 \in X$. The function f(x) is said to be continuous at x_0 if given any number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for all $x \in X$

$$|f(x) - f(x_0)| < \varepsilon$$

whenever $|x - x_0| < \delta$.

Alternative Definition of Continuity at a Point:

A function f defined on a non-empty subset X of R is said to be continuous at $x_0 \in X$ if and only if $\lim_{x \to x_0} f(x)$ exists and is equal to the value of the function at x_0 *i.e.,* $f(x_0)$.

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Value Addition: Note
We know that a function $f : X \to R$ is said to have a limit at a point $x_0 \in R$ if and only if both left hand limit and right hand limit exists at $x_0 \in R$ and are equal i.e. $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$.

Value Addition: Note
In the definition of limits, we used the inequality $0 < \|x - x_0\| < \delta$, whereas in the definition of continuity, we have used the inequality $\|x - x_0\| < \delta$. The reason is that in case of limits, the function may or may not be defined at x_0 but for the continuity of a function at x_0 the function must be defined on x_0.

Continuity of Functions

I.Q. 1

3.1. Continuity from the Right at a Point:

A function $f:X\to R$ is said to be right-continuous or continuous form the right at a point $x_0\in X$ if the right hand limit of f exists at x_0 and equal to $f(x_0)$ i.e.,

$$\lim_{x\to x_0^+} f(x)=f(x_0).$$

3.2. Continuity from the Left at a Point:

A function $f:X\to R$ is said to be Left-continuous or continuous form the Left at a point $x_0\in X$ if the Left hand limit of f exists at x_0 and equal to $f(x_0)$ i.e.,

$$\lim_{x\to x_0^-} f(x)=f(x_0).$$

4: Continuous Function:

A function $f:X\to R$ is said to be continuous on X if and only if f is continuous at each point of X.

Example 1: Show that the function $f(x)=\frac{1}{x}$ is not continuous at $x=0$.

Solution: Given function is

$$f(x)=\frac{1}{x}$$

since f(x) is not defined for x=0, therefore it cannot be continuous at x=0.

Moreover, we know that

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{1}{x}$$

does not exist in R.

Hence f(x) is not continuous at x=0.

Value Addition: Note
If the limit of a function $f:X\to R$ does not exist at a point $x_0\in X\ in\ R$, then the function $f:X\to R$ cannot be continuous at x_0.

Example 2: Show that the signum function sgn(x) is not continuous at x=0.

Solution: We know that signum function sgn(x) is defined as

$$f(x)=\mathrm{sgn}(x)=\begin{cases}1 & if \quad x>0\\ 0 & if \quad x=0\\ -1 & if \quad x<0\end{cases}$$

Now, Right hand limit at x=0

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+\mathrm{h})$$
$$=\lim_{h\to 0} f(\mathrm{h})$$
$$=\lim_{h\to 0} (1)=1$$

Left hand limit at x=0

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-\mathrm{h})$$
$$=\lim_{h\to 0} f(-\mathrm{h})$$
$$=\lim_{h\to 0} (-1)=-1$$

Since, $\lim_{x\to 0^+} f(x)\neq \lim_{x\to 0^-} f(x)$

Thus, the limit does not exist at 0.

Hence, signum function is not continuous at x=0.

Example 3: Show that the function $f(x)=x^2$ is continuous on R.

Solution: Given function is

$$f(x)=x^2$$

Let x_0 be any arbitrary element of R, then

Right hand limit at $x=x_0$

$$\lim_{x\to x_0^+} f(x)=\lim_{h\to 0} f(x_0+\mathrm{h})$$
$$=\lim_{h\to 0}(x_0+h)^2$$
$$=\lim_{h\to 0} (x_0{}^2+h^2+2x_0h)$$
$$=x_0{}^2+0+0$$

$$\lim_{x \to x_0^+} f(x) = x_0^2$$

Left hand limit at $x = x_0$

$$\lim_{x \to x_0^-} f(x) = \lim_{h \to 0} f(x_0 - \mathrm{h})$$

$$= \lim_{h \to 0} (x_0 - h)^2$$

$$= \lim_{h \to 0} (x_0^2 + h^2 - 2x_0 h)$$

$$= x_0^2 + 0 - 0$$

$$\lim_{x \to x_0^+} f(x) = x_0^2$$

Since, $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = x_0^2$

Hence, $\lim_{x \to x_0} f(x) = x_0^2$ (1)

Now, the value of the function at $x = x_0$ is

$$f(x_0) = x_0^2 \quad (2)$$

from equation (1) and (2) we have

$$\lim_{x \to x_0} f(x) = f(x_0) = x_0^2$$

Thus, $f(x) = x^2$ is continuous at $x = x_0$.

Since x_0 is an arbitrary element of R. Therefore f(x) is continuous at each point of R. Hence f(x) is continuous on R.

Example 4: Show that the function f(x) defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & if \;\; x \neq 0 \\ 0 & if \;\; x = 0 \end{cases}$$

is continuous on R.

Solution: Given function is

$$f(x) = \begin{cases} x \sin \frac{1}{x} & if \;\; x \neq 0 \\ 0 & if \;\; x = 0 \end{cases}$$

Now let a be any arbitrary element of R, then there arise two cases:

Case (I) If $a \neq 0$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} x \sin \frac{1}{x}$$
$$= a \sin \frac{1}{a}$$

and $f(a) = a \sin \frac{1}{a}$

Then, $\lim_{x \to a} f(x) = f(a) = a \sin \frac{1}{a}$

Thus, f(x) is continuous at x=a.

Case (II) If $a = 0$ then by the definition of function

$$f(a) = f(0) = 0$$

Now

$$\left| f(x) - f(0) \right| = \left| x \sin \frac{1}{x} - 0 \right|$$
$$= \left| x \sin \frac{1}{x} \right|$$
$$\leq |x| \left| \sin \frac{1}{x} \right|$$
$$\leq |x| < \varepsilon$$

$$\left| f(x) - f(0) \right| < \varepsilon$$

Now, if we choose $\delta = \varepsilon$, then for a given $\varepsilon > 0$, there exists a number $\delta = \varepsilon > 0$ such that

$$\left| f(x) - f(0) \right| < \varepsilon$$

whenever

$$|x - 0| = |x| < \delta$$

Thus, f(x) is continuous at x=0.

Hence, f(x) is continuous on R.

I.Q. 2

I.Q. 3

Theorem 1(Statement Only): A function $f: X \to R$ is said to be continuous at a point $x_0 \in X$ if and only if for given any $\varepsilon - neighborhood\ V_\varepsilon(f(x_0))\ of\ f(x_0)$ there exist a $\delta - neighborhood\ V_\delta(x_0)\ of\ x_0$ such that if x is any point of $X \cap V_\delta(x_0)$, then f(x) belongs to $V_\varepsilon(f(x_0))$, that is $f\left(X \cap V_\delta(x_0)\right) \subseteq V_\varepsilon(f(x_0))$.

5: Discontinuity of a Function at a Point:

Let X be a non-empty subset of R, let $f: X \to R$ and let $x_0 \in X$. Then the function f(x) is said to be discontinuous at x_0 if and only if it is not continuous at x_0.

5.1. Types of Discontinuity:

Let f be a function defined as $f: X \to R$ and let $x_0 \in X$ and let left hand limit, right hand limit, limit of the function and the value of the function at x_0 are denoted by

$$\lim_{x \to x_0^-} f(x),\ \lim_{x \to x_0^+} f(x),\ \lim_{x \to x_0} f(x)\ and\ \ f(x_0)$$

respectively. Then

(I) Removable Discontinuity at x_0

The function $f: X \to R$ is said to have the removable discontinuity at $x_0 \in X$, if the limit of the function exist at x_0 but not equal to the value of the function at x_0 i.e.,

$$\lim_{x \to x_0} f(x) \neq f(x_0).$$

(II) Discontinuity of the First Kind at x_0

The function $f: X \to R$ is said to have a discontinuity of first kind at x_0 if both the left hand limit and right hand limit exist at x_0 but are not equal to each other i.e.

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$$

(II.I) Discontinuity of the First Kind from the Left at x_0

The function $f : X \to R$ is said to have a discontinuity of first kind from the left at x_0 if the left hand limit exist at x_0 but not equal to the value of the function at x_0 i.e.

$$\lim_{x \to x_0^-} f(x) \neq f(x_0)$$

(II.II) Discontinuity of the First Kind from the Right at x_0

The function $f : X \to R$ is said to have a discontinuity of first kind from the right at x_0 if the right hand limit exist at x_0, but not equal to the value of the function at x_0 i.e.

$$\lim_{x \to x_0^+} f(x) \neq f(x_0)$$

(III) Discontinuity of the Second Kind at x_0

The function $f : X \to R$ is said to have a discontinuity of second kind at x_0 if both the left hand limit and right hand limit do not exist at x_0 i.e.

$\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ do not exist.

(III.I) Discontinuity of the Second Kind from the Left at x_0

The function $f : X \to R$ is said to have a discontinuity of second kind from the left at x_0 if the left hand limit does not exist at x_0 i.e.

$\lim_{x \to x_0^-} f(x)$ does not exist.

(III.II) Discontinuity of the First Kind from the Right at x_0

The function $f : X \to R$ is said to have a discontinuity of second kind from the right at x_0 if the right hand limit does not exist at x_0 i.e.

$\lim_{x \to x_0^+} f(x)$ does not exist.

Value Addition: Jump of a Function at x_0
If a function $f : X \to R$ has a discontinuity of first kind at x_0 then the function f is said to have a jump at x_0 and the value of the jump is equal to the difference of both the limits at x_0 i.e.

Value of the jump at x_0 = $\lim_{x \to x_0^+} f(x) - \lim_{x \to x_0^-} f(x)$.

Example 5: Show that the function f defined by

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ c \neq 1, & x = 0 \end{cases}$$

has a removable discontinuity.

Solution: We know that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

Since the limit exist at x=0 but

$$\lim_{x \to 0} f(x) \neq f(0)$$

Thus, we conclude that f(x) has a removable discontinuity at x=0.

Now if we redefine $f(0) = 1$, f(x) will be continuous at x=0. Therefore, we find that the removable discontinuity can be removed by properly redefining the function at the given point.

Example 6: Show that $\sin x$ is continuous on R.

Solution: Let $x_0 \in R$ be any arbitrary point, then

$$\sin x - \sin x_0 = 2\cos\left(\frac{x + x_0}{2}\right)\sin\left(\frac{x - x_0}{2}\right)$$

$$\Rightarrow \quad |\sin x - \sin x_0| = \left|2\cos\left(\frac{x + x_0}{2}\right)\sin\left(\frac{x - x_0}{2}\right)\right|$$

$$= 2\left|\cos\left(\frac{x + x_0}{2}\right)\right|\left|\sin\left(\frac{x - x_0}{2}\right)\right|$$

$$\leq 2.1.\left|\sin\left(\frac{x - x_0}{2}\right)\right|$$

$$\leq 2\left|\frac{x - x_0}{2}\right|$$

$$= 2.\frac{1}{2}|x - x_0|$$

$$= |x - x_0|$$

Now, if we let

$$|x - x_0| < \varepsilon$$

and $\delta = \varepsilon$

Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\sin x - \sin x_0| < \varepsilon$$

whenever

$$|x - x_0| < \delta$$

Thus, $\sin x$ is continuous at x_0.

Since x_0 is an arbitrary element of R, therefore $\sin x$ is continuous at each point of R. Hence $\sin x$ is continuous on R.

Example 7: Check the continuity of the function defined on R such that

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & if \ \ x \neq 0 \\ 0 & if \ \ x = 0 \end{cases}$$

Solution: Given function is

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & if \ \ x \neq 0 \\ 0 & if \ \ x = 0 \end{cases}$$

Left hand limit at x = 0 is

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{h \to 0} f(0-h) \\ &= \lim_{h \to 0} f(-h) \\ &= \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}} \\ &= \frac{e^{-\infty}}{1+e^{-\infty}} \\ &= \frac{0}{1+0} \end{aligned}$$

$\Rightarrow \quad \lim_{x \to 0^-} f(x) = 0$.

Right hand limit at x = 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} \frac{e^{1/h}}{1+e^{1/h}}$$
$$= \lim_{h\to 0} \frac{e^{1/h}}{e^{1/h}\left(e^{-1/h}+1\right)}$$
$$= \lim_{h\to 0} \frac{1}{\left(e^{-1/h}+1\right)}$$
$$= \frac{1}{e^{-\infty}+1}$$
$$= \frac{1}{0+1}$$

$$\Rightarrow \quad \lim_{x\to 0^+} f(x) = 1.$$

The value of the function at x=0 is

$$f(0) = 0$$

Thus,

$$\lim_{x\to 0^-} f(x) = f(0) \neq \lim_{x\to 0^+} f(x)$$

Therefore f has a discontinuity of the first kind from the right at x=0.

Example 8: Define a function f on R by the formula

$$f(x) = \begin{cases} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} & if \;\; x \neq 0 \\ 1 & if \;\; x = 0 \end{cases}$$

Discuss the continuity of f at x=0.

Solution: Given function is

$$f(x) = \begin{cases} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} & if \;\; x \neq 0 \\ 1 & if \;\; x = 0 \end{cases}$$

Left hand limit at x = 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$
$$= \lim_{h\to 0} \frac{e^{1/h}\left(e^{-2/h} - 1\right)}{e^{1/h}\left(e^{-2/h} + 1\right)}$$
$$= \lim_{h\to 0} \frac{\left(e^{-2/h} - 1\right)}{\left(e^{-2/h} + 1\right)}$$
$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1}$$
$$= \frac{0-1}{0+1}$$

$$\Rightarrow \quad \lim_{x\to 0^-} f(x) = -1.$$

Right hand limit at x = 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$
$$= \lim_{h\to 0} \frac{e^{1/h}\left(1 - e^{-2/h}\right)}{e^{1/h}\left(1 + e^{-2/h}\right)}$$
$$= \lim_{h\to 0} \frac{\left(1 - e^{-2/h}\right)}{\left(1 + e^{-2/h}\right)}$$
$$= \frac{1 - e^{-\infty}}{1 + e^{-\infty}}$$
$$= \frac{1-0}{1+0}$$

$$\Rightarrow \quad \lim_{x\to 0^+} f(x) = 1.$$

The value of the function at x=0 is

$$f(0) = 1$$

Thus,

$$\lim_{x\to 0^-} f(x) \neq f(0) = \lim_{x\to 0^+} f(x)$$

Therefore f has a discontinuity of the first kind from the left at x=0.

I.Q. 4

Theorem 2: If f(x) is continuous at $x = x_0$ then $|f(x)|$ is also continuous at $x = x_0$. However the converse is not true.

Proof: Let f(x) is continuous at $x = x_0$, then by the definition of continuity for a given $\varepsilon > 0$ there exist a positive number $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta \tag{1}$$

We know that

$$\big||f(x)| - |f(x_0)|\big| \le |f(x) - f(x_0)| \tag{2}$$

From equation (1) and (2), we have

$$\big||f(x)| - |f(x_0)|\big| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta \tag{3}$$

Thus, $|f(x)|$ is continuous at $x = x_0$.

Conversely: Define a function

$$f(x) = \begin{cases} -1 & if \;\; x < x_0 \\ 1 & if \;\; x \ge x_0 \end{cases}$$

Then $|f(x)| = 1 \;\; for \;\; all \;\; x$

and $\lim\limits_{x \to x_0} |f(x)| = 1 = |f(x_0)|$

Thus, $|f(x)|$ is continuous at $x = x_0$.

However

$$\lim_{x \to x_0^+} f(x) = \lim_{h \to 0} f(x_0 + h)$$
$$= \lim_{h \to 0} (1)$$

$\Rightarrow \quad \lim\limits_{x \to x_0^+} f(x) = 1$

Now

$$\lim_{x \to x_0^-} f(x) = \lim_{h \to 0} f(x_0 - h)$$
$$= \lim_{h \to 0} (-1)$$

$$\Rightarrow \quad \lim_{x \to x_0^-} f(x) = -1$$

Thus, $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$

Therefore, $\lim_{x \to x_0} f(x)$ does not exist at $x = x_0$, Thus, f(x) is not continuous at $x = x_0$.

Theorem 3: Show that if f(x) is continuous at x_0 and $f(x) > 0$ *for all* x, then the function $\sqrt{f(x)}$ is also continuous at x_0.

Proof: Let f(x) is continuous at x_0, by the definition of continuity for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta \tag{1}$$

Given that $f(x) \geq 0$ *for all* x, thus $f(x_0) \geq 0$, Now

$$\left|\sqrt{f(x)} - \sqrt{f(x_0)}\right| = \frac{\left(\left|\sqrt{f(x)} - \sqrt{f(x_0)}\right|\right)\left(\left|\sqrt{f(x)} + \sqrt{f(x_0)}\right|\right)}{\left|\sqrt{f(x)} + \sqrt{f(x_0)}\right|}$$

$$\Rightarrow \quad \left|\sqrt{f(x)} - \sqrt{f(x_0)}\right| = \frac{|f(x) - f(x_0)|}{\left|\sqrt{f(x)} + \sqrt{f(x_0)}\right|} \tag{2}$$

Since, $f(x) > 0$

$$\Rightarrow \quad \left|\sqrt{f(x)} + \sqrt{f(x_0)}\right| \geq \left|\sqrt{f(x_0)}\right|$$

$$\Rightarrow \quad \frac{1}{\left|\sqrt{f(x)} + \sqrt{f(x_0)}\right|} \leq \frac{1}{\left|\sqrt{f(x_0)}\right|}$$

Thus, from equation (2), we have

$$\left|\sqrt{f(x)} - \sqrt{f(x_0)}\right| \leq \frac{|f(x) - f(x_0)|}{\left|\sqrt{f(x_0)}\right|} \tag{3}$$

Using equation (1) and (3) we have

$$\left|\sqrt{f(x)}-\sqrt{f(x_0)}\right|<\frac{\varepsilon}{\left|\sqrt{f(x_0)}\right|}<\varepsilon \text{ whenever } |x-x_0|<\delta$$

Thus, $g(x)=\sqrt{f(x)}$ is continuous at x_0.

6: Sequential Criterion for Continuity:

Theorem 4: A function $f:X\to R$ is continuous at a point $x_0\in X$ if and only if for every sequence $\langle x_n\rangle$ in X which converges to x_0, the sequence $\langle f(x_n)\rangle$ converges to $f(x_0)$.

Proof: Let the function f(x) is continuous at x_0 and the sequence $\langle x_n\rangle$ in X is convergent such that

$$\lim_{n\to\infty} x_n = x_0$$

Thus, there exist a positive integer m such that

$$|x_n - x_0|<\delta \quad \text{for all } n\geq m \tag{1}$$

Given that f(x) is continuous at x_0, thus for a given $\varepsilon>0$, there exists a number $\delta>0$ such that

$$|f(x)-f(x_0)|<\varepsilon \quad \text{whenever} \quad |x-x_0|<\delta \tag{2}$$

Now replacing x by x_n in equation (2) we have

$$|f(x_n)-f(x_0)|<\varepsilon \quad \text{whenever} \quad |x_n-x_0|<\delta \tag{3}$$

from equation (1) and (3), we have

$$|f(x_n)-f(x_0)|<\varepsilon \quad \text{for all } n\geq m$$

$$\Rightarrow \quad \lim_{n\to\infty} f(x_n)=f(x_0) \tag{4}$$

Hence, the sequence $\langle f(x_n)\rangle$ converges to $f(x_0)$.

Conversely: Let for every sequence $\langle x_n\rangle$ in X which converges to x_0, the sequence $\langle f(x_n)\rangle$ converges to $f(x_0)$, i.e.,

$$\lim_{n\to\infty} f(x_n)=f(x_0)$$

Now suppose f(x) is not continuous at x_0 and let $\varepsilon > 0$ be given then for every $\delta > 0$, there exists an x such that

$$|x - x_0| < \delta \text{ and yet}$$

$$|f(x) - f(x_0)| \geq \varepsilon \qquad (5)$$

By taking $\delta = \frac{1}{n}$, we find that for each positive integer n, there exists $x = x_n$ such that

$$|f(x_n) - f(x_0)| \geq \varepsilon \quad \text{whenever } |x - x_0| < \frac{1}{n}$$

This shows that, the sequence $\langle f(x_n) \rangle$ does not converges to $f(x_0)$.

This contradicts our hypothesis, therefore the function f(x) must be continuous.

7: Sequential Criterion for Discontinuity:

A function $f : X \to R$ is discontinuous at a point $x_0 \in X$ if and only if there exist a sequence $\langle x_n \rangle$ in X such that $\langle x_n \rangle$ converges to x_0 but the sequence $\langle f(x_n) \rangle$ does not converges to $f(x_0)$.

I.Q. 5

I.Q. 6

Example 9: Discuss the continuity of the Riemann function f(x) defined on]0, 1[such as

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x \text{ is rational of the form } \frac{p}{q} : q \neq 0 \end{cases}$$

Solution: Let r be a rational number in]0, 1[of the form $r = \frac{p}{q}\ (q \neq 0)$ where p and q are positive integers having no factor in common, then

$$f(x) = \frac{1}{q}$$

For each positive integer n, choose a positive irrational number x_n such that

$$|x_n - x_0| < \frac{1}{n}$$

Then $\langle x_n \rangle$ is a sequence converging to x_0.

Also $f(x_n) = 0$ for each n, so that

$$\lim_{n\to\infty} f(x_n) = 0 \neq f(\mathrm{r}) \qquad \text{[since } f(r) = \frac{1}{q}\text{]}$$

Thus, by the sequential criterion for discontinuity the function f(x) is discontinuous at r.

Now let s is any irrational number in]0, 1[and let $\varepsilon > 0$ be given, now choose a positive integer n such that

$$\frac{1}{n} < \varepsilon$$

Now there are only finitely many rational numbers $\frac{p}{q}$ in]0, 1[having the denominator less than n.

We can therefore, find a $\delta > 0$ such that no rational number in $]s-\delta, s+\delta[\subset]0,1[$ has its denominator less than n.

Then

Case (I) If x is an irrational number then

$$|x - s| < \delta$$

$$\Rightarrow \quad |f(x) - f(s)| = |f(x) - 0| = |f(x)|$$

$$\Rightarrow \quad |f(x) - f(s)| = 0 \qquad (1)$$

Case (II) If x is a rational number then

$$|x - s| < \delta$$

$$\Rightarrow \quad |f(x) - f(s)| = |f(x) - 0| = |f(x)| \leq \frac{1}{n} \qquad \text{[since x is a rational number]}$$

$\Rightarrow \quad |f(x)-f(s)|<\varepsilon$ (2)

From equation (1) and (2) together we have

$|x-s|<\delta$

$\Rightarrow \quad |f(x)-f(s)|<\varepsilon$ for all x

Thus, f(x) is continuous at s.

Hence f(x) is continuous at each irrational number in]0, 1[and discontinuous at each rational number in]0, 1[.

Example 10 (Dirichlet's Function): A function f on R is defined by

$$f(x)=\begin{cases} 1 & if\ \ x\ is\ rational \\ -1 & if\ \ x\ is\ irrational \end{cases}$$

Show that f(x) is discontinuous at every point of R.

Solution: Let r be a rational number. For each positive integer n let s_n be a sequence of irrational numbers such that

$$|s_n-r|<\frac{1}{n}$$

Thus, the sequence $\langle s_n \rangle$ converges to r, but

$$f(s_n)=-1 \quad for\ \ all\ \ n$$

$\Rightarrow \quad \lim_{n\to\infty} f(s_n)=-1\neq f(r)$

This shows that f(x) is discontinuous at r.

Now let s be any irrational number. For each positive integer n, let r_n be a sequence of rational number such that

$$|r_n-s|<\frac{1}{n}$$

Thus the sequence $\langle r_n \rangle$ converges to s, but

$$f(\mathrm{r}_n)=1 \ \ for\ all\ n$$

$\Rightarrow \quad \lim_{n\to\infty} f(\mathrm{r}_n)=1\neq f(s)$

Thus, by the sequential criterion for discontinuity the function f(x) is discontinuous at s.

Therefore f(x) is discontinuous at every point of R.

I.Q. 7

Example 11: Show that the function f(x) defined on R by

$$f(x)=\begin{cases} x & \text{if } x \text{ is irrational} \\ -x & \text{if } x \text{ is rational} \end{cases}$$

is continuous only at x=0.

Solution: Given function is

$$f(x)=\begin{cases} x & \text{if } x \text{ is irrational} \\ -x & \text{if } x \text{ is rational} \end{cases} \qquad (1)$$

Case (I): To check the continuity at a non-zero rational number r, then

$$f(r)=-r$$

We know that in every interval however small there exist an infinite number of rational and irrational numbers,

Thus for each positive integer n, let s_n be a sequence of irrational numbers such that

$$|s_n - r| < \frac{1}{n} \qquad (2)$$

Thus, the sequence $\langle s_n \rangle$ converges to r.

Now, using equation (1), we have

$$\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty}(s_n) = r$$

and $f\left(\lim_{n\to\infty}(s_n)\right) = f(r) = -r$

$$\Rightarrow \quad \lim_{n\to\infty} f(s_n) \neq f\left(\lim_{n\to\infty}(s_n)\right) \qquad [\text{since } r \neq 0]$$

Thus, the function f(x) is discontinuous at every non-zero rational number.

Case (II): To check the continuity at irrational number s, then

$$f(s) = s$$

For each positive integer n, let r_n be a sequence of rational numbers such that

$$|r_n - s| < \frac{1}{n} \quad (2)$$

Thus, the sequence $\langle r_n \rangle$ converges to s.

Now, using equation (1), we have

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty}(-r_n) = -s$$

and $f\left(\lim_{n\to\infty}(r_n)\right) = f(s) = s$

$$\Rightarrow \quad \lim_{n\to\infty} f(r_n) \neq f\left(\lim_{n\to\infty}(r_n)\right) \quad \text{[since } r \neq 0 \text{]}$$

Thus, the function f(x) is discontinuous at every irrational number.

Case (III): To check the continuity at 0, then

$$f(0) = 0 \ .$$

Now there may arise two sub cases

if $x = r$ are rational numbers, then

$$|r - 0| < \delta$$

$$\Rightarrow \quad |f(r) - f(0)| = |-r - 0| = |-r| = |r - 0| < \varepsilon \quad (3)$$

if $x = s$ are irrational numbers, then

$$|s - 0| < \delta$$

$$\Rightarrow \quad |f(s) - f(0)| = |s - 0| < \varepsilon \quad (3)$$

from equation (3) and (4), for all x we have

$$|x - 0| < \delta$$

$$\Rightarrow \quad |f(x) - f(0)| < \varepsilon$$

Thus, f(x) is continuous at x=0.

I.Q. 8

Example 12: Suppose that f(x) and g(x) are continuous functions on $]a, b[$ and $f(x) = g(x) \text{ for all } x \in Q \cap]a,b[$. Prove that $f(x) = g(x) \text{ for all } x \in]a,b[$

Solution: Given that f(x) and g(x) are continuous functions on]a, b[and

$$f(x) = g(x) \text{ for all } x \in Q \cap]a,b[$$

Let $x_0 \in]a, b[$ be an arbitrary element.

Then there is a sequence of rational numbers r_n in]a, b[such that

$$r_n \to x_0 \text{ in }]a, b[$$

By hypothesis

$$f(r_n) = g(r_n) \qquad [\text{since } r_n \in Q \cap]a,b[\,]$$

From the continuity of f(x) and g(x) in]a, b[we get

$$f(r_n) \to f(x_0)$$

since $f(r_n) \to g(r_n)$

$$\Rightarrow \quad g(r_n) \to g(x_0) \qquad (1)$$

Because $x_0 \in]a,b[$ is arbitrary, equation (1) yields that

$$f(x) = g(x) \text{ for every } x \in]a, b[\,.$$

Example 13: A continuous function f(x) satisfies the functional equation

$$f(x+y) = f(x) + f(y)$$

Show that f(x) = kx, where k is a constant.

Solution: Given that

$$f(x+y) = f(x) + f(y) \qquad (1)$$

putting x=y=0 in equation (1), we have

$$f(0) = f(0) + f(0)$$

$$\Rightarrow \quad f(0) = 0$$

Now putting y=-x in equation (1), we have

$$f(0) = f(x) + f(-x)$$

$\Rightarrow \quad f(-x) = -f(x)$

$\Rightarrow$ f(x) is an odd function. (9)

If x is a positive integer, then repeated use of equation (1) provides

$$f(x) = f(\underbrace{1 + 1 + \;.\;.\;. + 1}_{x\ times})$$

$$= \underbrace{f(1) + f(1) + \;.\;.\;. + f(1)}_{x\ times}$$

$$= x\, f(1)$$

Let f(1)=k, then

$\Rightarrow \quad f(x) = kx$ (2)

If x is a negative integer, then

$f(x) = -f(-x)$ [since f(x) is odd function]

$\Rightarrow \quad = -k(-x)$ [using equation (2)]

$\Rightarrow \quad f(x) = kx$

Now, if $x = \frac{p}{q}, (q > 0)$ is a rational number, then using equation (1) repeatedly, we have

$$f(p) = f\left(\frac{p}{q}.q\right)$$

$$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \;.\;.\;. + f\left(\frac{p}{q}\right) (q\ times)$$

$$= q.f\left(\frac{p}{q}\right)$$

$\Rightarrow \quad f\left(\frac{p}{q}\right) = \frac{1}{q} f(p)$

$\Rightarrow \quad f(x) = kx,\ $ where x is rational (3)

Now let x be any real number and let $\langle r_n \rangle$ be a sequence of rational numbers such that

$$\lim_{n\to\infty} r_n = x$$

from equation (3)

$$f(r_n) = k\,r_n$$

Let $n \to \infty$, since f(x) is continuous function

$$\lim f(r_n) = k(\lim r_n)$$

$\Rightarrow \quad f(\lim r_n) = k(\lim r_n)$

$\Rightarrow \quad f(x) = kx$

Thus, f(x) = kx for all values of x.

Example 14: Let f(x) be a function defined on [-1, 1] by

$$f(x) = \begin{cases} x & if\ \ x\ \ is\ irrational \\ 0 & if\ \ x\ is\ rational \end{cases}$$

Show that f(x) is continuous only at 0.

Solution: Given that

$$f(x) = \begin{cases} x & if\ \ x\ \ is\ irrational \\ 0 & if\ \ x\ is\ rational \end{cases}$$

Let x_0 be any point of [-1, 1]. For each positive integer n choose a rational number r_n and an irrational number s_n both in [-1, 1] such that

$$|r_n - x_0| < \frac{1}{n}$$

and $\quad |s_n - x_0| < \frac{1}{n}$

Then $\lim\limits_{n\to\infty} r_n = x_0 = \lim\limits_{n\to\infty} s_n$

Now, by the definition of f(x), we have

$$f(r_n) = 0 \quad for\ \ all\ \ n$$

and $\quad f(s_n) = s_n \quad for\ \ all\ \ n$

Therefore, we must have

$$0 = f(x_0) = \lim_{n\to\infty} s_n$$

$\Rightarrow \qquad 0 = f(x_0) = x_0$ (1)

from (1) it is clear that 0 is the only possible point of continuity.

Now given any $\varepsilon > 0$, if we choose $\delta = \frac{1}{2}\varepsilon$,

If x is rational then

$$|x| < \delta \Rightarrow |f(x) - f(0)| = 0 \qquad (2)$$

If x is irrational then

$$|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \delta < \varepsilon \qquad (3)$$

Thus from equation (2) and (3) we have

$$|x| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$$

Hence f(x) is continuous at 0.

I.Q. 9

I.Q. 10

Exercise:

1. Investigate the continuity at the indicated points

(I) $f(x) = \begin{cases} 2x & if \;\; 0 \le x < 1 \\ 3 & if \;\; x = 1 \\ 4x & if \;\; 1 < x \le 2 \end{cases}$ at x = 0, 1 and 2.

(II) $f(x) = \begin{cases} -x^2 & if \;\; x \le 0 \\ 5x - 4 & if \;\; 0 < x \le 1 \\ 4x^2 - 3x & if \;\; 1 < x < 2 \\ 3x + 4 & if \;\; x \ge 2 \end{cases}$ $at \;\; x = 0, \, 1 \;\; and \;\; 2$

(III) $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}} & if \;\; x \ne 0 \\ 1 & if \;\; x = 0 \end{cases}$ $at \;\; x = 0$

(IV) $f(x) = \begin{cases} (1+x)^{1/x} & if \;\; x \ne 0 \\ 1 & if \;\; x = 0 \end{cases}$ $at \;\; x = 0$

(V) $f(x) = \begin{cases} \dfrac{xe^{1/x}}{1 + e^{1/x}} & if \;\; x \ne 0 \\ 0 & if \;\; x = 0 \end{cases}$ $at \;\; x = 0$

2. Let f(x) be defined for all $x \in R,\ x \neq 2$ *by*

$$f(x) = \frac{x^2 + x - 6}{x - 2}$$

can f(x) be defined at x=2 in such a way that f(x) is continuous at this point?

3. Show that the absolute value function $f(x) = |x|$ is continuous at every point $x_0 \in R$.

4. Determine the points of continuity of the following functions, where $[x]$ denotes the greatest integer function.

(I) $[x]$

(II) $x[x]$

5. Let f(x) be the function defined on R by setting

$$f(x) = \sqrt{(x - [x])} \quad for\ \ all\ \ x \in R$$

Show that f(x) is discontinuous at the points x = n, where n is any integer and is continuous at all other points.

6. Let f(x) be defined on R by

$$f(x) = \begin{cases} x & if\ \ x\ \ is\ \ rational \\ 1 - x & if\ \ x\ is\ irrational \end{cases}$$

Show that f(x) is continuous at $x = \frac{1}{2}$ and is discontinuous at every other point.

7. Let f(x) be the function defined on R by

$$f(x) = \begin{cases} \dfrac{1}{1 - e^{1/x}} & if\ \ x \neq 0 \\ 1 & if\ \ x = 0 \end{cases}$$

Examine the continuity of f(x) at points of R.

8. Let f(x) be a function defined on R as follows

$$f(x) = \begin{cases} e^{1/x} \sin(1/x) & if\ \ x \neq 0 \\ 0 & if\ \ x = 0 \end{cases}$$

Examine the points of discontinuity.

9. Let $f:[0, 1] \to R$ is given by

$$f(x)=\begin{cases} x^2 & if \quad x \in Q \\ 0 & if \quad x \in R-Q \end{cases}$$

Show that f(x) is continuous only at x=0.

10. Let $f:]a,b[\to R$ be continuous and f(r)=0 for every rational number $r \in]a,b[$. Prove that $f(x)=0 \;\; \forall \;\; x \in]a,b[$.

Summary:

In this lesson we have emphasized on the followings

- Continuity at a Point
- Continuity from the Right at a Point
- Continuity from the Left at a Point
- Continuous Function
- Discontinuity of a Function at a Point
- Types of Discontinuity
- Sequential Criterion for Continuity
- Sequential Criterion for Discontinuity

References:

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