Lesson: Limit Theorems for Functions

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1. Learning outcomes:

After studying this chapter you should be able to understand the

- Limit at a point.
- Bounded Function
- Sum, Difference, Product and Quotient of Two Functions
- Algebraic Operations on Limits
- Limit Theorems for Functions

2. Introduction:

Area of mathematics in which various limiting concepts are used in a systematic way is generally known as Mathematical Analysis. The rudimentary notion of a limiting process emerged in the 1680s as Isaac Newton (1642-1727) and Gotfried Leibnitz (1646-1716) struggled with the creation of the calculus. Initially they were unknown to each other's work and their creative insights were quite different. In their work, they realized the need to formulate a notion of function and the idea of quantities being close to one another. In his work Principia in 1687, Newton discussed limits "to which they approach nearer than by any given difference, but never go beyond nor in effect attain to till the quantities are diminished in infinitum and used the word 'fluent' to denote a relationship between variables. Leibnitz invented "infinitesimally small numbers as a way of handling the concept of a limit and introduced the term function to indicate a quantity that depended on a variable. Leibnitz also introduced the term calculus for this new method of calculation. In this lesson we will study the limit theorems on functions.

3. Limit at a Point:

Let X and Y are two non-empty subsets of the real numbers. A real number L is called the limit of the function $f: X \to Y$ at a point $x_0 \in X$ for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x \in X$

$$|f(x) - \ell| < \varepsilon$$

whenever $|x - x_0| < \delta$.

Value Addition: Note
1. Limit of a function $f: X \to Y$ at a point $x_0 \in X$ is denoted by $\lim_{x \to x_0} f(x) = \ell$. 2. If ℓ is the limit of the function $f: X \to Y$ at a point $x_0 \in X$, then the function f(x) is said to converges to ℓ at x_0. 3. If the limit of the function f(x) at x_0 does not exist then the function f(x) is said to diverges at x_0. 4. The value of δ in the definition of limit is usually depends on ε.

4. Bounded Function:

A function $f: X \to Y$ where X and Y are non-empty subsets of R is called bounded if there exists a constant $M \in R$ such that

$|f(x)| \leq M$ for all $x \in X$.

Theorem 1: Let X and Y are two non-empty subsets of R and let $f: X \to Y$. If f(x) has a limit at a point $x_0 \in X$, then f(x) is bounded on some neighborhood of x_0.

Proof: Let $f: X \to Y$ has a limit L at $x_0 \in X$ then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $x \in X$

$$|f(x) - \ell| < \varepsilon \quad (1)$$

whenever $|x - x_0| < \delta$ (2)

from inequality (2) we have

$$|x - x_0| < \delta$$

$\Rightarrow \quad -\delta < x - x_0 < \delta$

$\Rightarrow \quad x_0 - \delta < x < x_0 + \delta$

$\Rightarrow \quad (x_0 - \delta,\ x_0 + \delta)$ is called $\delta -$ neighborhood of x_0.

Now if we take $\varepsilon = 1$ in equation (1), we have

$$|f(x) - \ell| < 1 \quad (3)$$

We know that

$$|f(x)| - |\ell| \leq |f(x) - \ell|$$

$\Rightarrow \quad |f(x)| - |\ell| < 1$

$\Rightarrow \quad |f(x)| < |\ell| + 1$ (4)

Take $M = \sup\{f(x_0), |\ell| + 1\}$

Then we have

$$|f(x)| < M \text{ for each } x \in (x_0 - \delta,\ x_0 + \delta)$$

Thus, f(x) is bounded on the neighborhood of x_0.

5. Sum, Difference, Product and Quotient of Two Functions:

Let A and B are two non-empty subsets of R and let f(x) and g(x) are two functions defined on A and B respectively. Then the domain of the sum, difference, product and quotient of f and g is the set $X = A \cap B$.

5.1. Sum of Two Functions:

The sum of the functions f(x) and g(x) is denoted by $(f+g)(x)$ and defined as follows

$$(f+g)(x) = f(x) + g(x) \text{ for each } x \in X.$$

5.2. Difference of Two Functions:

The difference of the functions f(x) and g(x) is denoted by $(f-g)(x)$ and defined as follows

$$(f-g)(x) = f(x) - g(x) \text{ for each } x \in X$$

5.3. Product of Two Functions:

The product of the functions f(x) and g(x) is denoted by $(f.g)(x)$ and defined as follows

$$(f.g)(x) = f(x).g(x) \text{ for each } x \in X.$$

5.4. Quotient of Two Functions:

The quotient of the functions f(x) and g(x) is denoted by $\left(\frac{f}{g}\right)(x)$ and defined as follows

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ provided } g(x) \neq 0 \text{ for each } x \in X.$$

6. Algebraic Operations on Limits:

Theorem 2: Let X be any non-empty subset of R and let f(x) and g(x) are functions defined on X such that $\lim_{x \to x_0} f(x) = \ell_1$ and $\lim_{x \to x_0} g(x) = \ell_2$ exist. Then

(I) Addition of Limits

$$\lim_{x \to x_0} (f+g)(x) = \ell_1 + \ell_2 = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$$

(II) Difference of Limits

$$\lim_{x\to x_0}(f-g)(x)=\ell_1-\ell_2=\lim_{x\to x_0} f(x)-\lim_{x\to x_0} \mathrm{g}(x)$$

(III) Product of Limits

$$\lim_{x\to x_0}(f.g)(x)=\ell_1.\ell_2=\lim_{x\to x_0} f(x).\lim_{x\to x_0} \mathrm{g}(x)$$

(IV) Reciprocal of Limit

$$\lim_{x\to x_0}\left(\frac{1}{g}\right)(x)=\frac{1}{\ell_2}=\frac{1}{\lim\limits_{x\to x_0}\mathrm{g}(x)} \quad \text{provided } \ell_2\neq 0 \ and \ g(x)\neq 0 \ for\, any \ x\in X \ .$$

(V) Quotient of Limits

$$\lim_{x\to x_0}\left(\frac{f}{g}\right)(x)=\frac{\ell_1}{\ell_2}=\frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0}\mathrm{g}(x)} \quad \text{provided } \lim_{x\to x_0}\mathrm{g}(x)=\ell_2\neq 0.$$

(VI) Constant Multiple of Limits

$$\lim_{x\to x_0}(a.f)(x)=a.\ell_1=a.\lim_{x\to x_0} f(x)$$

Proof: Let X be any non-empty subset of R and let

$\lim\limits_{x\to x_0} f(x)=\ell_1$ and $\lim\limits_{x\to x_0}\mathrm{g}(x)=\ell_2$ exist.

(I) Addition of Limits

Since $\lim\limits_{x\to x_0} f(x)=\ell_1$ and $\lim\limits_{x\to x_0}\mathrm{g}(x)=\ell_2$ exist then by the definition of limit for every $\varepsilon>0$ there exist a numbers $\delta_1>0 \ and \ \delta_2>0$ such that

if $0<|x-x_0|<\delta_1 \ then \ |f(x)-\ell_1|<\dfrac{\varepsilon}{2}$ (1)

similarly

if $0<|x-x_0|<\delta_2 \ then \ |\mathrm{g}(x)-\ell_2|<\dfrac{\varepsilon}{2}$ (2)

Now choose $\delta=\min\{\delta_1, \delta_2\}$ and $0<|x-x_0|<\delta$ then

$$|(f+g)(x)-(\ell_1+\ell_2)|=|f(x)+g(x)-(\ell_1+\ell_2)|$$

$$=|(f(x)-\ell_1)+(g(x)-\ell_2)|$$

$$\leq |(f(x)-\ell_1)| + |(g(x)-\ell_2)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[using equation (1) and (2)]}$$

$$\Rightarrow \quad |(f+g)(x)-(\ell_1+\ell_2)| < \varepsilon$$

Thus for every $\varepsilon > 0$ there exist a numbers $\delta > 0$ such that

if $0 < |x - x_0| < \delta$ *then* $|(f+g)(x)-(\ell_1+\ell_2)| < \varepsilon$

Hence

$$\lim_{x\to x_0}(f+g)(x) = \ell_1 + \ell_2 = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x).$$

(II) Difference of Limits

Proof of this part is similar to (I) therefore left as an exercise to the reader.

(III) Multiplication of Limits

Since $\lim_{x\to x_0} f(x) = \ell_1$ exist then by the definition of limit for every $\varepsilon > 0$ there exist a numbers $\delta_1 > 0$ such that

if $0 < |x - x_0| < \delta_1$ *then* $|f(x) - \ell_1| < \dfrac{\varepsilon}{2\ell_2}$ (1)

By the boundedness property there exists a number $\delta_2 > 0$ such that

$|g(x)| \leq \ell_2 \quad$ *for all* $x \in V_{\delta_2}(x_0)$ (2)

Furthermore there exists a numbers $\delta_3 > 0$ such that

if $0 < |x - x_0| < \delta_3$ *then* $|g(x) - \ell_2| < \dfrac{\varepsilon}{2|\ell_1|}$ (1)

Now choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and $0 < |x - x_0| < \delta$ then

$$\begin{aligned}|(f.g)(x)-(\ell_1.\ell_2)| &= |f(x).g(x)-\ell_1 g(x)+\ell_1 g(x)-(\ell_1.\ell_2)| \\ &= \left|g(x)\left(f(x)-\ell_1\right)+\ell_1\left(g(x)-\ell_2\right)\right| \\ &\le \left|g(x)\left(f(x)-\ell_1\right)\right|+\left|\ell_1\left(g(x)-\ell_2\right)\right| \\ &\le |g(x)|\left|\left(f(x)-\ell_1\right)\right|+|\ell_1|\left|\left(g(x)-\ell_2\right)\right| \\ &< \ell_2\frac{\varepsilon}{2\ell_2}+|\ell_1|\frac{\varepsilon}{2|\ell_1|} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

Thus for every $\varepsilon>0$ there exist a numbers $\delta>0$ such that

if $0<|x-x_0|<\delta$ *then* $|(f.g)(x)-(\ell_1.\ell_2)|<\varepsilon$

Hence

$$\lim_{x\to x_0}(f.g)(x)=\ell_1.\ell_2=\lim_{x\to x_0} f(x).\lim_{x\to x_0} g(x).$$

(IV) Reciprocal of Limits

Let $\varepsilon>0$ be given. Since $\lim_{x\to x_0} g(x)=\ell_2$ exist then by the definition of limit for every $\varepsilon>0$ there exist a number $\delta_2>0$ such that

if $0<|x-x_0|<\delta_2$ *then* $|g(x)-\ell_2|<\varepsilon$ (1)

Now choose $\varepsilon=\frac{|\ell_2|}{2}$ then from equation (1) we have

$$|g(x)-\ell_2|<\frac{|\ell_2|}{2}$$

$$\Rightarrow \quad \frac{|\ell_2|}{2}<|g(x)|<\frac{3|\ell_2|}{2}$$

$$\Rightarrow \quad |g(x)|>\frac{|\ell_2|}{2}$$

Also there exist a number $\delta_3>0$ such that

if $0<|x-x_0|<\delta_3$ *then* $|g(x)-\ell_2|<\frac{\varepsilon|\ell_2|^2}{2}$ (2)

Now choose $\delta=\min\{\delta_2,\delta_3\}$ and $0<|x-x_0|<\delta$ then

$$\left|\left(\frac{1}{g}\right)(x)-\frac{1}{\ell_2}\right|=\left|\left(\frac{1}{g(x)}\right)-\frac{1}{\ell_2}\right|$$
$$=\left|\frac{\ell_2-g(x)}{g(x).\ell_2}\right|$$
$$\le\frac{|g(x)-\ell_2|}{|\ell_2.g(x)|}$$
$$\le\frac{|g(x)-\ell_2|}{|\ell_2.g(x)|}$$
$$\le\frac{2}{|\ell_2|^2}|g(x)-\ell_2|$$
$$<\frac{2}{|\ell_2|^2}.\frac{\varepsilon.|\ell_2|^2}{2}=\varepsilon$$

Thus for every $\varepsilon>0$ there exist a number $\delta>0$ such that

if $0<|x-x_0|<\delta$ *then* $\left|\left(\frac{1}{g}\right)(x)-\frac{1}{\ell_2}\right|<\varepsilon$

Hence

$$\lim_{x\to x_0}\left(\frac{1}{g}\right)(x)=\frac{1}{\ell_2}=\frac{1}{\lim\limits_{x\to x_0}g(x)}.$$

(V) Reciprocal of Limits

Proof of this part is application of part (III) and part (IV).

(IV) Constant Multiple of Limit

If a = 0 then this case is trivially true as both sides becomes zero.

If $a\neq 0$, then by the definition of limit for every $\varepsilon>0$ there exist a number $\delta>0$ such that

if $0<|x-x_0|<\delta$ *then* $|f(x)-\ell_1|<\frac{\varepsilon}{|a|}$ *where* $a\neq 0$ (1)

Now if $0<|x-x_0|<\delta$ then

$$|(a.f)(x)-a.\ell_1|=|a(f(x)-\ell_1)|$$
$$=|a(f(x)-\ell_1)|$$

$$= |a| \left| \left(f(x) - \ell_1 \right) \right|$$

$$< |a| \frac{\varepsilon}{|a|} = \varepsilon$$

Thus for every $\varepsilon > 0$ there exist a number $\delta > 0$ such that

if $0 < |x - x_0| < \delta$ *then* $|(a.f)(x) - a.\ell_1| < \varepsilon$

Hence

$$\lim_{x \to x_0} (a.f)(x) = a.\ell_1 = a.\lim_{x \to x_0} f(x).$$

Example 1: Determine the limit

$$\lim_{x \to 2} (x^3 - 5), \quad \mathrm{x} \in \mathrm{R}.$$

Solution: We have

$$\lim_{x \to 2} (x^3 - 5) = \lim_{x \to 2} (x^3) - \lim_{x \to 2} (5)$$ [using the theorem of limits]

$$= (2^3) - 5$$

$$= 8 - 5 = 3$$

Hence, $\lim_{x \to 2} (x^3 - 5) = 3, \quad \mathrm{x} \in \mathrm{R}$

Example 2: Determine the limit

$$\lim_{x \to 1} (x + 2)(3x - 1), \quad \mathrm{x} \in \mathrm{R}.$$

Solution: We have

$$\lim_{x \to 1} (x + 2)(3x - 2) = \lim_{x \to 1} (x + 2).\lim_{x \to 1} (3x - 2)$$ [using the theorem of limits]

$$= (1 + 2)(3 - 2)$$

$$= 3$$

Hence, $\lim_{x \to 1} (x + 2)(3x - 1) = 3, \quad \mathrm{x} \in \mathrm{R}$

Example 3: Determine the limit

$$\lim_{x \to 2} \left(\frac{1}{(x + 1)} - \frac{1}{2x} \right), \quad \mathrm{x} > 0.$$

Solution: We have

$$\lim_{x\to 2}\left(\frac{1}{(x+1)}-\frac{1}{2x}\right)=\lim_{x\to 2}\left(\frac{1}{(x+1)}\right)-\lim_{x\to 1}\left(\frac{1}{2x}\right)$$ [using the theorem of limits]

$$=\frac{1}{\lim\limits_{x\to 2}(x+1)}-\frac{1}{\lim\limits_{x\to 2}(2x)}$$ [using the theorem of limits]

$$=\frac{1}{(2+1)}-\frac{1}{2\text{x}2}=\frac{1}{3}-\frac{1}{4}$$

$$=\frac{1}{12}$$

Hence, $\lim\limits_{x\to 2}\left(\frac{1}{(x+1)}-\frac{1}{2x}\right)=\frac{1}{12},\ \ \text{x}>0$.

Alternative Method:

We have

$$\lim_{x\to 2}\left(\frac{1}{(x+1)}-\frac{1}{2x}\right)=\lim_{x\to 2}\left(\frac{2x-(x+1)}{(x+1)2\,\text{x}}\right)$$

$$=\lim_{x\to 2}\left(\frac{x-1}{(x+1)2\,\text{x}}\right)$$ [using the theorem of limits]

$$=\frac{\lim\limits_{x\to 2}(x-1)}{\lim\limits_{x\to 2}(x+1)2\,\text{x}}$$ [using the theorem of limits]

$$=\frac{\lim\limits_{x\to 2}(x-1)}{\lim\limits_{x\to 2}(x+1)\lim\limits_{x\to 2}(2\,\text{x})}$$ [using the theorem of limits]

$$=\frac{(2-1)}{(2+1)(2\text{x}2)}=\frac{1}{12}$$

Hence, $\lim\limits_{x\to 2}\left(\frac{1}{(x+1)}-\frac{1}{2x}\right)=\frac{1}{12},\ \ \text{x}>0$.

Example 4: Determine the limit

$$\lim_{x\to 1}\left(\frac{x^2+3}{x^2-3}\right),\ \ \text{x}>0.$$

Solution: We have

$$\lim_{x\to 1}\left(\frac{x^2+3}{x^2-3}\right)=\frac{\lim_{x\to 1}(x^2+3)}{\lim_{x\to 1}(x^2-3)}$$ [using the theorem of limits]

$$=\frac{\lim_{x\to 1}(x^2)+\lim_{x\to 1}(3)}{\lim_{x\to 1}(x^2)-\lim_{x\to 1}(3)}$$ [using the theorem of limits]

$$=\frac{1+3}{1-3}$$

$$=\frac{4}{-2}=-2$$

Hence, $\lim_{x\to 1}\left(\frac{x^2+3}{x^2-3}\right)=-2,\ \ x>0$.

Example 5: Determine the limit

$$\lim_{x\to 0}\left(\frac{x-1}{x^2-2}\right).$$

Solution: We have

$$\lim_{x\to 0}\left(\frac{x-1}{x^2-2}\right)=\frac{\lim_{x\to 0}(x-1)}{\lim_{x\to 0}(x^2-2)}$$ [using the theorem of limits]

$$=\frac{\lim_{x\to 0}(x)-\lim_{x\to 0}(1)}{\lim_{x\to 0}(x^2)-\lim_{x\to 0}(2)}$$ [using the theorem of limits]

$$=\frac{0-1}{0-2}$$

$$=\frac{-1}{-2}=\frac{1}{2}$$

Hence, $\lim_{x\to 0}\left(\frac{x-1}{x^2-2}\right)=\frac{1}{2}$.

Example 6: Determine the limit

$$\lim_{x\to 2}\sqrt{\frac{4x+1}{x+2}}.$$

Solution: We have

$$\lim_{x\to 2}\sqrt{\frac{4x+1}{x+2}} = \sqrt{\lim_{x\to 2}\left(\frac{4x+1}{x+2}\right)}$$ [using the theorem of limits]

$$= \sqrt{\frac{\lim_{x\to 2}(4x+1)}{\lim_{x\to 2}(x+2)}}$$ [using the theorem of limits]

$$= \sqrt{\frac{\lim_{x\to 2}(4x)+\lim_{x\to 2}(1)}{\lim_{x\to 2}(x)+\lim_{x\to 2}(2)}}$$

$$= \sqrt{\frac{(4\text{x}2+1)}{(2+2)}}$$

$$= \sqrt{\frac{9}{4}} = \frac{3}{2}$$

Hence, $\lim_{x\to 2}\sqrt{\frac{4x+1}{x+2}} = \frac{3}{2}$.

Example 7: Determine the limit

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right).$$

Solution: Since $\lim_{x\to 3}(x-3)=0$ therefore we cannot apply the limit theorems directly. Thus, we have

$$\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right) = \lim_{x\to 3}\left(\frac{x^2-3^2}{x-3}\right)$$

$$= \lim_{x\to 3}\left(\frac{(x-3)(x+3)}{(x-3)}\right)$$

$$= \lim_{x\to 3}(x+3)$$

$$= (3+3) = 6$$ [using the theorem of limits]

Hence, $\lim_{x\to 0}\left(\frac{x^2-9}{x-3}\right) = 6$.

Example 8: Determine the limit

$$\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right).$$

Solution: Since $\lim_{x\to 1}(x-1)=0$ therefore we cannot apply the limit theorems directly. Thus, we have

$$\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)=\lim_{x\to 1}\left(\frac{(\sqrt{x}-1)}{(x-1)}.\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}\right)$$

$$=\lim_{x\to 1}\left(\frac{(x-1)}{(x-1)}.\frac{1}{(\sqrt{x}+1)}\right)$$

$$=\lim_{x\to 1}\left(\frac{1}{(\sqrt{x}+1)}\right)$$

$$=\frac{1}{\lim_{x\to 1}(\sqrt{x}+1)}$$

$$=\frac{1}{(1+1)}=\frac{1}{2}$$ [using the theorem of limits]

Hence, $\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)=\frac{1}{2}$.

Example 9: Determine the limit

$$\lim_{x\to 0}\left(\frac{(x+1)^2-1}{x}\right).$$

Solution: Since $\lim_{x\to 0}(x)=0$ therefore we cannot apply the limit theorems directly. Thus, we have

$$\lim_{x\to 0}\left(\frac{(x+1)^2-1}{x}\right)=\lim_{x\to 0}\left(\frac{(x^2+2x+1)-1}{x}\right)$$

$$=\lim_{x\to 0}\left(\frac{x(x+2)}{x}\right)$$

$$=\lim_{x\to 0}(x+2)$$

$$=(0+2)$$ [using the theorem of limits]

$$=2$$

Hence, $\lim_{x\to 0}\left(\frac{(x+1)^2-1}{x}\right)=2$.

Example 10: Determine the limit

$$\lim_{x\to 0}\left(\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}\right).$$

Solution: Since $\lim_{x\to 0}(x+2x^2)=0$ therefore we cannot apply the limit theorems directly. Thus, we have

$$\lim_{x\to 0}\left(\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}\right)=\lim_{x\to 0}\left(\frac{\left(\sqrt{1+2x}-\sqrt{1+3x}\right)}{x+2x^2}.\frac{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{\left((1+2x)-(1+3x)\right)}{(x+2x^2)}.\frac{1}{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{-x}{x(1+2x)}.\frac{1}{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{-1}{(1+2x)}.\frac{1}{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{-1}{(1+2x)}\right)\lim_{x\to 0}\left(\frac{1}{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}\right)$$

$$=\frac{-1}{\lim_{x\to 0}(1+2x)}.\frac{1}{\lim_{x\to 0}\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}$$

$$=\frac{-1}{\lim_{x\to 0}(1+2x)}.\frac{1}{\left(\lim_{x\to 0}\sqrt{1+2x}+\lim_{x\to 0}\sqrt{1+3x}\right)}$$

$$=\frac{-1}{(1+2\lim_{x\to 0}x)}.\frac{1}{\left(\sqrt{\lim_{x\to 0}(1+2x)}+\sqrt{\lim_{x\to 0}(1+3x)}\right)}$$

$$= \frac{-1}{(1+0)} \cdot \frac{1}{\left(\sqrt{(1+0)} + \sqrt{(1+0)}\right)}$$

$$= -1 . \frac{1}{(1+1)}$$

$$= -1 . \frac{1}{2}$$

Hence, $\lim_{x \to 0} \left(\frac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2} \right) = \frac{-1}{2}$.

7. Limit Theorems for Functions:

Theorem 3: If f(x) is a polynomial function, then

$$\lim_{x \to 0} f(x) = f(x_0).$$

Proof: Let f(x) be a polynomial function of degree n in x on R such that

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n \quad (a_0 \neq 0) \tag{1}$$

thus $f(x_0) = a_0 {x_0}^n + a_1 {x_0}^{n-1} + a_2 {x_0}^{n-2} + \ldots + a_{n-1} x_0 + a_n \quad (a_0 \neq 0)$ (2)

We have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n \right), \quad (a_0 \neq 0)$$

$$= \lim_{x \to x_0} \left(a_0 x^n \right) + \lim_{x \to x_0} \left(a_1 x^{n-1} \right) + \lim_{x \to x_0} \left(a_2 x^{n-2} \right) + \ldots + \lim_{x \to x_0} \left(a_{n-1} x \right) + \lim_{x \to x_0} \left(a_n \right)$$

$$= a_0 \lim_{x \to x_0} \left(x^n \right) + a_1 \lim_{x \to x_0} \left(x^{n-1} \right) + a_2 \lim_{x \to x_0} \left(x^{n-2} \right) + \ldots + a_{n-1} \lim_{x \to x_0} (x) + \lim_{x \to x_0} (a_n)$$

$$\Rightarrow \quad \lim_{x \to x_0} f(x) = a_0 {x_0}^n + a_1 {x_0}^{n-1} + a_2 {x_0}^{n-2} + \ldots + a_{n-1} x_0 + a_n \tag{3}$$

Using equation (2) and (3) we have

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Hence for a polynomial f(x) of degree n

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Theorem 4: If f(x) and g(x) are polynomial function on R and if $g(x_0) \neq 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}.$$

Proof: Let f(x) and g(x) are two polynomial functions on R of degrees n and m respectively such that

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n \quad (a_0 \neq 0)$$

and $$\mathrm{g}(x) = b_0 x^m + b_1 x^{m-1} + b_2 x^{m-2} + \ldots + b_{m-1} x + b_m \quad (\mathrm{b}_0 \neq 0)$$

Then

$$f(x_0) = a_0 x_0{}^n + a_1 x_0{}^{n-1} + a_2 x_0{}^{n-2} + \ldots + a_{n-1} x_0 + a_n \quad (a_0 \neq 0) \qquad (1)$$

and $$\mathrm{g}(x_0) = b_0 x_0{}^m + b_1 x_0{}^{m-1} + b_2 x_0{}^{m-2} + \ldots + b_{m-1} x_0 + b_m \quad (\mathrm{b}_0 \neq 0) \qquad (2)$$

Since g(x) is a polynomial function of degree m, then by the theorem in algebra there are at most m distinct numbers $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$ exist such that

$$g(\beta_1) = g(\beta_2) = g(\beta_3) = \ldots = g(\beta_m) = 0$$

Now, for any $x \notin (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)$ and $g(x) \neq 0$

Hence if $x \notin (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)$ we can define

$$r(x) = \frac{f(x)}{g(x)} \qquad (3)$$

Given that $g(x_0) \neq 0$ this implies that

$$x_0 \notin (\beta_1, \beta_2, \beta_3, \ldots, \beta_m)$$

Thus applying the theorem 3 on equation (3), we have

$$\lim_{x \to x_0} r(x) = r(x_0)$$

$$\Rightarrow \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

$$= \frac{f(x_0)}{g(x_0)}$$

Hence, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}$.

Value Addition: Note
Let X be a non-empty subset of R and let $f_1, f_2, f_3, \ldots, f_n$ are functions defined on X on R such that $\lim_{x\to x_0} f_k(x)$ for k = 1, 2, 3, . . . , n exists such that $\lim_{x\to x_0} f_k(x) = \ell_k \;\; for \;\; k = 1, 2, \ldots, n$. Then $\lim_{x\to x_0} \left(f_1(x) + f_2(x) + \ldots + f_n(x) \right) = \ell_1 + \ell_2 + \ldots + \ell_n$ And $\lim_{x\to x_0} \left(f_1(x).f_2(x)\ldots f_n(x) \right) = \ell_1.\ell_2\ldots.\ell_n$ In particular, if $\lim_{x\to x_0} f(x) = \ell \;\; and \;\; n \in N$, Then $\lim_{x\to x_0} [f(x)]^n = \ell^n \;\; for \; all \; n \in N$.

Theorem 5: Let X be a non-empty subset of R and let $f : X \to R$. If $a \le f(x) \le b \;\; for \; all \;\; x \in X, \, x \ne x_0$ and if $\lim_{x\to x_0} f(x)$ exists. Then $a \le \lim_{x\to x_0} f(x) \le b$.

Proof: Let $\lim_{x\to x_0} f(x)$ exists and let

$$\lim_{x\to x_0} f(x) = \ell \qquad (1)$$

Then by the sequential criteria it follows that if $< x_n >$ is any sequence of real numbers such that $x_0 \ne x_n \in X \;\; for \; all \;\; n \in N$ and if the sequence $< x_n >$ converges to x_0, then the sequence $f(x_n)$ converges to ℓ.

Since

$$a \le f(x_n) \le b \;\; for \; all \;\; n \in N$$

Then using the theorem that if $< x_n >$ is a convergent sequence and if $a \le x_n \le b \;\; for \; all \;\; n \in N, then \; a \le \lim(x_n) \le b$.

$$\Rightarrow \quad a \le \lim f(x_n) \le b$$

or we can say that

$$a \le \lim_{x \to x_0} f(x) \le b \,.$$

Hence Proved.

Value Addition: Sequential Criterion:
Let $f: X \to R$ then the statement $\lim_{x \to x_0} f(x) = \ell$ is equivalent to the statement that for every sequence $< x_n >$ in X that converges to x_0 such that $x_n \ne x_0$ *for all* $n \in N$, the sequence $f(x_n)$ converges to ℓ.

Theorem 6 (Squeeze Theorem): Let X be a non-emoty subset of R and f(x), g(x) and h(x) are three functions defined on X, such that $f(x) \le g(x) \le h(x)$ *for all* $x \in X$.

And if

$$\lim_{x \to x_0} f(x) = \ell = \lim_{x \to x_0} h(x)$$

Then,

$$\lim_{x \to x_0} g(x) = \ell \,.$$

Proof: Given that

$$f(x) \le g(x) \le h(x) \quad \text{for all } x \in X \tag{1}$$

and $$\lim_{x \to x_0} f(x) = \ell \tag{2}$$

$$\lim_{x \to x_0} h(x) = \ell \tag{3}$$

By the definition of limit, for a given $\varepsilon > 0$ there exist positive numbers δ_1 and δ_2 such that if

$$0 < |x - x_0| < \delta_1$$

$\Rightarrow$ $$|f(x) - \ell| < \varepsilon$$

$\Rightarrow$ $$-\varepsilon < f(x) - \ell < \varepsilon$$

$\Rightarrow$ $$\ell - \varepsilon < f(x) < \ell + \varepsilon \tag{4}$$

and $0 < |x - x_0| < \delta_2$

$\Rightarrow \quad |h(x) - \ell| < \varepsilon$

$\Rightarrow \quad -\varepsilon < h(x) - \ell < \varepsilon$

$\Rightarrow \quad \ell - \varepsilon < h(x) < \ell + \varepsilon \qquad (5)$

Now let $\delta = \min\{\delta_1, \delta_2\}$

Then if

$$0 < |x - x_0| < \delta$$

Then using equation (1), (4) and (5), we have

$$\ell - \varepsilon < f(x) \le g(x) \le h(x) < \ell + \varepsilon$$

$\Leftrightarrow \quad \ell - \varepsilon < g(x) < \ell + \varepsilon$

$\Leftrightarrow \quad |g(x) - \ell| < \varepsilon$

Therefore

$$\lim_{x \to x_0} g(x) = \ell.$$

Theorem 7: Let X be a non-empty subset of R. Suppose that $f(x) \le g(x) \le h(x)$ in a deleted neighborhood of x_0 and if

$$\lim_{x \to x_0} f(x) = \ell = \lim_{x \to x_0} h(x)$$

Then,

$$\lim_{x \to x_0} g(x) = \ell.$$

Proof: Let δ is a positive number, then consider

$$S = (x_0 - \delta, \ x_0 + \delta)$$

$\Rightarrow \quad$ S is a neighborhood of x_0.

Given that

$f(x) \le g(x) \le h(x)$ in deleted neighborhood of x_0

$\Rightarrow \quad f(x) \le g(x) \le h(x) \quad for\ all\ \ x \in A\ and\ \ x \ne x_0$

Now to show that

$$\lim_{x \to x_0} g(x) = \ell$$

We have to show that if x_n is any sequence which converges to x_0, then $g(x_n)$ is a sequence which converges to ℓ.

Let $< x_n >$ is a sequence such that

$$x_n \to x_0 \quad and \quad x_n \ne x_0 \ \ for\ all\ n$$

Then

$$f(x_n) \to \ell$$

and $\quad h(x_n) \to \ell$

Since

$$f(x_n) \le g(x_n) \le h(x_n) \quad for\ all\ n.$$

Thus, applying the squeeze theorem for sequence we conclude that

$$g(x_n) \to \ell$$

$\Rightarrow \quad \lim_{x \to x_0} g(x) = \ell$.

Example 11: Show that $\lim_{x \to x_0} x^{3/2} = 0, \quad (x > 0)$.

Solution: Let $f(x) = x^{3/2} \quad for \ \ x > 0$

Now we know that the inequality

$$x \le x^{1/2} \le 1 \ \ holds \ \ for \ 0 < x \le 1$$

$\Rightarrow \quad x^2 \le x^{3/2} \le x \quad for \ 0 < x \le 1$

Since

$$\lim_{x \to 0} x^2 = 0$$

and $\quad \lim_{x \to 0} x = 0$

Thus applying the squeeze theorem on equation (1), we have

$$\lim_{x\to 0} x^{3/2} = 0.$$

Example 12: Show that $\lim_{x\to 0} \sin x = 0$.

Solution: We know that

$$-1 \le \cos x \le 1 \quad \textit{for all } x \ge 0$$

on integrating both sides w.r.t. x, between the limits 0 and x.

$$\Rightarrow \quad -\int_0^x dx \le \int_0^x \cos dx \le \int_0^x dx$$

$$\Rightarrow \quad -x \le \sin x \le x \quad \textit{for all } x \ge 0$$

since

$$\lim_{x\to 0}(-x) = -\lim_{x\to 0}(x) = 0$$

and $\lim_{x\to 0}(x) = 0$

thus applying the squeeze theorem we have

$$\lim_{x\to 0}(\sin x) = 0.$$

Example 13: Show that $\lim_{x\to 0} \cos x = 1$.

Solution: We know that

$$-1 \le \cos x \le 1 \quad \textit{for all } x \ge 0$$

on integrating both sides w.r.t. x, between the limits 0 and x.

$$\Rightarrow \quad -\int_0^x dx \le \int_0^x \cos dx \le \int_0^x dx$$

$$\Rightarrow \quad -x \le \sin x \le x \quad \textit{for all } x \ge 0$$

on again integrating both sides w.r.t. x, between the limits 0 and x.

$$-\int_0^x x dx \le \int_0^x \sin dx \le \int_0^x x dx$$

$$\Rightarrow \quad -\frac{x^2}{2} \le [-\cos x]_0^x \le \frac{x^2}{2}$$

$\Rightarrow \quad -\frac{x^2}{2} \le -\cos x + 1 \le \frac{x^2}{2}$

$\Rightarrow \quad -1 - \frac{x^2}{2} \le -\cos x \le -1 + \frac{x^2}{2}$

$\Rightarrow \quad 1 + \frac{x^2}{2} \ge \cos x \ge 1 - \frac{x^2}{2}$

$\Rightarrow \quad 1 - \frac{x^2}{2} \le \cos x \le 1 + \frac{x^2}{2}$

Since

$$\lim_{x \to 0}(1 - \frac{x^2}{2}) = 1 - \frac{1}{2}\lim_{x \to 0}(x^2) = 1 - 0 = 1$$

and $\lim_{x \to 0}(1 + \frac{x^2}{2}) = 1 + \frac{1}{2}\lim_{x \to 0}(x^2) = 1 + 0 = 1$

thus applying the squeeze theorem we have

$$\lim_{x \to 0}(\cos x) = 1.$$

Example 14: Show that $\lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1$.

Solution: We know that

$$-1 \le \cos x \le 1 \quad \textit{for all } x \ge 0 \tag{1}$$

on integrating both sides w.r.t. x, between the limits 0 and x.

$\Rightarrow \quad -\int_0^x dx \le \int_0^x \cos dx \le \int_0^x dx$

$\Rightarrow \quad -x \le \sin x \le x \quad \textit{for all } x \ge 0$

on again integrating both sides w.r.t. x, between the limits 0 and x.

$$-\int_0^x x dx \le \int_0^x \sin dx \le \int_0^x x dx$$

$\Rightarrow \quad -\frac{x^2}{2} \le -\cos x + 1 \le \frac{x^2}{2}$

$\Rightarrow \qquad 1-\frac{x^2}{2} \le \cos x \le 1+\frac{x^2}{2}$ (2)

using equation (1) and (2), we have

$$1-\frac{x^2}{2} \le \cos x \le 1 \qquad (3)$$

On integrating equation (3) w.r.t. x, between the limits o and x, we have

$$x-\frac{x^3}{6} \le \sin x \le x \quad \textit{for all } x \ge 0 \qquad (4)$$

and $\quad x \le \sin x \le x-\frac{x^3}{6} \quad \textit{for all } x < 0$ (5)

Thus from (4) and (5) we conclude that

$$1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad \textit{for all } x \ne 0$$

Since

$$\lim_{x\to 0}(1-\frac{x^2}{6}) = 1-\frac{1}{6}\lim_{x\to 0}(x^2) = 1-0 = 1$$

and $\lim_{x\to 0}(1) = 1$

thus applying the squeeze theorem we conclude that

$$\lim_{x\to 0}(\frac{\sin x}{x}) = 1.$$

Example 15: Show that $\lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right) = 0$.

Solution: We know that

$$-1 \le \cos x \le 1 \quad \textit{for all } x \ge 0 \qquad (1)$$

$\Rightarrow \qquad -2 \le \cos x - 1 \le 0 \quad \textit{for all } x \ge 0$ (2)

on integrating both sides of inequality (1) w.r.t. x, between the limits 0 and x.

$\Rightarrow \qquad -\int_0^x dx \le \int_0^x \cos dx \le \int_0^x dx$

$$\Rightarrow \quad -x \le \sin x \le x \quad for\ all\ x \ge 0$$

on again integrating both sides w.r.t. x, between the limits 0 and x.

$$-\int_0^x x dx \le \int_0^x \sin dx \le \int_0^x x dx$$

$$\Rightarrow \quad -\frac{x^2}{2} \le -\cos x + 1 \le \frac{x^2}{2}$$

$$\Rightarrow \quad \frac{x^2}{2} \ge \cos x - 1 \ge -\frac{x^2}{2}$$

$$\Rightarrow \quad -\frac{x^2}{2} \le \cos x - 1 \le \frac{x^2}{2} \qquad (3)$$

Using equation (2) and (3), we have

$$-\frac{x^2}{2} \le \cos x - 1 \le 0$$

$$\Rightarrow \quad -\frac{x}{2} \le \frac{\cos x - 1}{x} \le 0 \quad for\ all\ x > 0$$

and $-\frac{x}{2} \ge \frac{\cos x - 1}{x} \ge 0 \quad for\ all\ x < 0$

$$\Rightarrow \quad 0 \le \frac{\cos x - 1}{x} \le -\frac{x}{2} \quad for\ all\ x < 0$$

Now let

$$f(x) = \begin{cases} -\frac{x}{2} & for\ x \ge 0 \\ 0 & for\ x < 0 \end{cases}$$

and $h(x) = \begin{cases} 0 & for\ x \ge 0 \\ -\frac{x}{2} & for\ x < 0 \end{cases}$

Then, we have

$$f(x) \le \frac{\cos x - 1}{x} \le h(x) \quad for\ all\ x \ne 0$$

Since

$$\lim_{x \to 0} f(x) = 0$$

and $\lim_{x \to 0} h(x) = 0$

Thus, using the squeeze theorem, we have

$$\lim_{x \to 0} \left(\frac{\cos x - 1}{x} \right) = 0.$$

Example 16: Show that $\lim_{x \to 0} \left(x \cos \frac{1}{x} \right) = 0$.

Solution: We know that

$$|\cos z| \le 1 \quad \text{for all } z \in R$$

Let $z = \frac{1}{x}$ *for* $x \in R$ *and* $x \ne 0$

$$\Rightarrow \quad \left| \cos \frac{1}{x} \right| \le 1 \quad \text{for all } x \in R,\ x \ne 0$$

$$\Rightarrow \quad |x| \left| \cos \frac{1}{x} \right| \le |x| \quad \text{for all } x \in R,\ x \ne 0$$

$$\Rightarrow \quad \left| x \cos \frac{1}{x} \right| \le |x| \quad \text{for all } x \in R,\ x \ne 0$$

$$\Rightarrow \quad -|x| \le \left| x \cos \frac{1}{x} \right| \le |x| \quad \text{for all } x \in R,\ x \ne 0$$

Since

$$\lim_{x \to 0} |x| = 0$$

Thus, using the squeeze theorem, we have

$$\lim_{x \to 0} \left(x \cos \frac{1}{x} \right) = 0.$$

Example 16: Evaluate $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x}$.

Solution: We have

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\left(\frac{\sin 3x}{x}\frac{x}{\sin 5x}\right)$$

$$\Rightarrow \qquad =\lim_{x\to 0}\left(\frac{3\sin 3x}{3x}\frac{5x}{5\sin 5x}\right)$$

$$\Rightarrow \qquad =\frac{3}{5}\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\frac{5x}{\sin 5x}\right)$$

$$\Rightarrow \qquad =\frac{3}{5}\left[\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\lim_{x\to 0}\left(\frac{5x}{\sin 5x}\right)\right]$$

$$\Rightarrow \qquad =\frac{3}{5}\left[\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\left(\frac{1}{\lim_{x\to 0}\frac{\sin 5x}{5x}}\right)\right]$$

$$\Rightarrow \qquad =\frac{3}{5}(1).(1)=\frac{3}{5}$$

Hence

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\frac{3}{5}.$$

Example 17: Evaluate $\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}$.

Solution: We have

$$\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}=\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}.\frac{\sqrt{x}}{\sqrt{x}}$$

$$=\lim_{x\to 0}\left(\sqrt{x}.\frac{\sin x}{x}\right)$$

$$=\lim_{x\to 0}\left(\sqrt{x}\right)\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$$

$$=0\text{x}1=0$$

Hence

$$\lim_{x\to 0}\frac{\sin x}{\sqrt{x}}=0.$$

Limit Theorems for Functions

Theorem 8: Let X be a non-empty subset of R. If $\lim_{x\to x_0} f(x) > 0 \left(or \lim_{x\to x_0} f(x) < 0\right)$, then there exists a neighborhood $V_\delta(x_0)$ of x_0 such that f(x) > 0 (or f(x) < 0) for all $x \in X \cap V_\delta(x_0),\ x \neq x_0$.

Proof: Let

$$\lim_{x\to x_0} f(x) = \ell$$

and suppose that $\ell > 0$

now take $\varepsilon = \frac{1}{2}\ell > 0$

then there exists a number $\delta > 0$ such that

if $\quad 0 < |x - x_0| < \delta \ and \ x \in X$

then $\ |f(x) - \ell| < \frac{1}{2}\ell$

$\Rightarrow \quad -\frac{1}{2}\ell < f(x) - \ell < \frac{1}{2}\ell$

$\Rightarrow \quad \ell - \frac{1}{2}\ell < f(x) < \ell + \frac{1}{2}\ell$

$\Rightarrow \quad \frac{1}{2}\ell < f(x) < \frac{3}{2}\ell$

$\Rightarrow \quad 0 < \frac{1}{2}\ell < f(x)$

$\Rightarrow \quad f(x) > 0$

Thus, if $x \in X \cap V_\delta(x_0),\ x \neq x_0$, then

$$f(x) > \frac{1}{2}\ell > 0$$

Similarly, it can be proved that if $\ell < 0 \ then \ f(x) < 0$.

Exercise:

1. Apply algebra of limits to determine the following limits

(I) $\quad \lim_{x\to 2}\left[(x^2+3)+\sqrt{x+7}\right]$

(II) $\lim_{x\to 3}[(x-1)(3x-4)]$

(III) $\lim_{x\to 1}\frac{x^2+5}{x^2-5} \quad (x>0)$

(IV) $\lim_{x\to 1}\left[\frac{x}{x^2+1}-\frac{1}{x+2}\right]$

(V) $\lim_{x\to 3}\sqrt{\frac{2x+2}{x+3}}, \quad x>0$

2. Evaluate the following limits and state which theorem is used

(I) $\lim_{x\to -2}\frac{x^2-4}{x+2}$

(II) $\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x+2}$

(III) $\lim_{x\to 1}\frac{(x+1)^2-4x}{x-1}$

(IV) $\lim_{x\to 6}\frac{\sqrt{x-2}-2}{x-6}$

(V) $\lim_{x\to 1}\frac{x^n-1}{x^m-1}, \quad (n>m)$

(VI) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1+2x}}{x-2x^2}$

3. Using the fact that $\lim_{x\to 0}\frac{\sin x}{x}=1$, evaluate the following limits

(I) $\lim_{x\to 0}\sin x$

(II) $\lim_{x\to 0}\frac{x}{\tan x}$

(III) $\lim_{x\to 0}\frac{\tan x}{x}$

(IV) $\lim_{x\to 0}x\sin\sqrt{x}$

4. Determine whether the following limits exists in R.

(I) $\lim_{x\to 0}\left(x\sin\frac{1}{x^2}\right), \quad (x\neq 0)$

(II) $\lim_{x\to 0}\left(x\cos\frac{1}{x^2}\right), \quad (x\neq 0)$

(III) $\lim_{x\to 0}\left(\sqrt{x}\sin\frac{1}{x^2}\right), \quad (x>0)$

(IV) $\lim_{x\to 0}\left(\cos\frac{1}{x^2}\right), \ (x \neq 0)$

5. Let X be a non-empty subset of R and let f(x) and g(x) are two functions defined on X.

(I) Show that if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(f+g)(x)$ exist then $\lim_{x\to a} g(x)$ exist.

(II) If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a}(f.g)(x)$ exist, does it follow that $\lim_{x\to a} g(x)$ exist.

6. Let X be a non-empty subset of R. If $\lim_{x\to a} f(x)$ exists and if $|f(x)|$ denotes the function defined for $x \in X$ by $|f|(x) = |f(x)|$. Prove that $\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right|$.

Summary:

In this lesson we have emphasized on the followings

- Limit at a point.
- Bounded Function
- Sum, Difference, Product and Quotient of Two Functions
- Algebraic Operations on Limits
- Limit Theorems for Functions

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