Subject: Maths, Algebra-I
Discipline Courses-1
Semester-1
Lesson : Sets and Functions
Lesson Developer: Gurudatt Rao Ambedkar
College/Department : A.N.D. College, Delhi University

Table of Contents

Sets and Functions

1. Learning outcomes:

After studying this chapter you should be able to

- Understand the meaning of the term 'Set'
- How to represent the sets
- Distinguish between the different types of sets
- Find the union, intersection, difference and complement of sets
- Meaning of the term 'function'
- Understand different type of function
- Plot different type of plots

2. Introduction:

The theory of sets was developed at the end of 19^{th} century. George Cantor (1845-1918), a German mathematician introduced the theory of sets which is now being used in many concepts of mathematics like sequences, probability etc. In this chapter we are presenting a brief idea about the theory of sets.

Quantities – There are two kinds of Quantities:

a) Constants **b)** Variables

a) Constants –If any quantity does not change in mathematical operation then it is called constant. There are two types of Constants-

i) Arbitrary constants
ii) Absolute constants

The constants remain unchanged in particular problems is called arbitrary constants; these are represented by $a, b, c, \ldots, k.$ The value of *absolute constant* remains fixed in all conditions; for example $3, 6, -5, \sqrt{2}, \pi$ etc. are absolute constants.

b) Variables – Variable are those quantities which are capable of assuming unlike values in a particular argument. These variables are represented by $x, y, z, u, v, w\ etc$. Variables are of two types-

i) Dependent variables.
ii) Independent variables.

Independent variables are those variables whose value can be changed independently and the dependent variables are those variables which depend on independent variables.

For Example: Diameter of circle is d = 2r. Here diameter 'd' depends on the radius 'r' so r is the independent variable and d is dependent variable.

3. Sets:

"A well defined collection of distinct objects/things is called a ***Set***."

We regularly speak the words which describes a particular category of objects like class, team, rivers etc. The adjective 'well defined' is most important which tells us that the object must have some definition. It is necessary to decide whether object belongs to a group or not. Students, players, numbers, alphabets, cities etc. are few examples of sets.

Examples of Sets:

- ✓ The cities of Uttar Pradesh.
- ✓ States of India.
- ✓ Solutions of the equations x^2 – 4=0 i.e. 2 and -2.
- ✓ Natural numbers N.
- ✓ Letters in the word ALLAHABAD.

Examples which are not set:

- The collection of all intelligent boys.
- The collection of all rich persons.

3.1. Representation of Sets:

Generally, we represent a set with capital letter (X, Y, Z etc) and the elements of sets i.e. object are denoted by small letters (a, b, c, etc).

If X = {a, b, c, 1, 2, 3} then we say that a, b, c, 1, 2, 3 are the elements of set X or element a belongs to the set X. In mathematics we use Greek letter called epsilon, ϵ , means 'belongs to' to tell an element of a set.

For Example: b $\in$ X i.e. b is an element of the set X.

b $\notin$ X i.e. b is not an element of the set X.

3.2. Description of Sets:

There are two ways to describe or specify the elements of a set:

a) **Roster method/ Tabular method:** We list all the members of a set separated by commas. The list of members should be enclosed in curly bracket.

e.g., X = {1, 2, 3, 4, 5}

Y= {a, l, h, b, d}

b) **Set builder method or rule method:** We use a rule or definition to describe all the members of a set.

e.g., X is a set whose elements are the first five natural numbers or X= $\{x: x \in$ N and $x \leq 5\}$. In this notation, the colon (":") means "such that",

Y is a set whose elements are the letters used in ALLAHABAD

Value addition: Do you Know?
➢ **Note 1:** Two element of a set may not be identical, Every element of a set must be unique; {a, b} = {b, a} = {b, a, a, b, a} ➢ **Note 2**: The enumeration of elements can be abbreviated for sets with many elements. For example the set of all positive integer may be specified by tabular method as: N = {1, 2, 3, _ _ _}.

4. Types of Sets:

4.1. Null Set:

A set with no element is call null set or void set or empty set. It is denoted by standard notation ∅ i.e. ∅ = { }

For Example: ∅ = The set of countries in India

$\emptyset = \{ x : x \in N, x \leq 4 \text{ and } x \geq 5\}$

4.2. Equal Sets:

If two sets have exactly same element, then they are said to be equal & we write A=B.

For Example: A= {p, q, r}, B={r, q, p}

=> A=B

4.3. Singleton Sets:

A set with exactly one element is called a *singleton set*.

For Example: A= {a}, B= {Sun}

4.4. Subsets:

If every element of set A is also an element of set B. Then A is called a subsets of B and B is called superset of A. We denote this by $A \subseteq B$.

For Example: A= {5, 2, a}, B= {a, 2, 5} Then $A \subseteq B$

4.5. Proper Subsets:

A is called a proper subset of a set B if B has at least one more element than A. It is denoted by $A \subset B$.

For Example: A= {Delhi, U.P., Haryana}, B= {U.P., Delhi, Haryana, H.P.}. Then $A \subset B$

For Example: P={x,y,5,6,0}, Q= {x,5,0,6,z,y,7}, Then $P \subset Q$

4.6. Power Set:

A set of all the subsets of a set A is called power set of that set. It is denoted by P(A)

A= {a, b}, then

P (A) = {{a}, {b}, {a, b}, ∅}

Value addition: Remarks
➢ ∅ is the subset of every set ➢ If A has n element, then P (A) has 2^n element.

4.7. Universal Set:

A superset of all the sets under consideration is called Universal Set and is denoted by capital letter U.

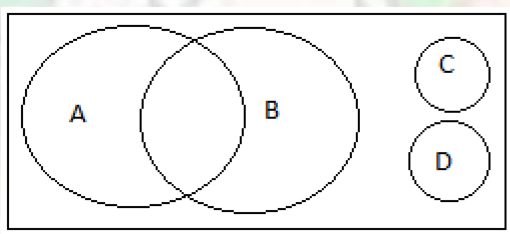
For Example: A= {1,2}, B={a, b}, C={p, q, r, 1, 5}, then

U={1, 2, a, b, p, q, r, 5}

5. Venn-Euler Diagram:

To understand the relationship between sets, some simple diagram is suggested by John Venn (1834-1923), called Venn diagram.

In Venn diagram, the inner portion of a rectangle represents the universal set U and all the subsets are represented by the closed curve (circle/ellipse or any other shape) lying completely within the rectangle:



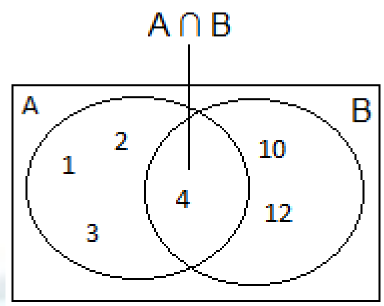
6. Operation on Sets:

6.1. Intersection of sets:

The intersection of two sets A & B is the set of all common elements from A and B. It is denoted by A ∩ B.

i.e $A \cap B = \{ x : x \in A \text{ and } x \in B \}$

For Example: A = {1, 2, 3, 4}, B= {4,10,12}, Then A ∩ B= {4}

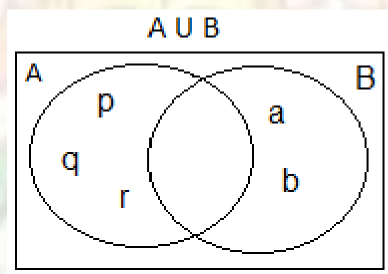


6.2. Union of Sets:

Let A and B are two sets. Then union of A & B is the set of all those elements which belong to either the set A or the set B. It is denoted by $A \cup B$.

i.e. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

For Example: A= {p, q, r}, B= {a, b}, Then $A \cup B$= {a, b, p, q, r}



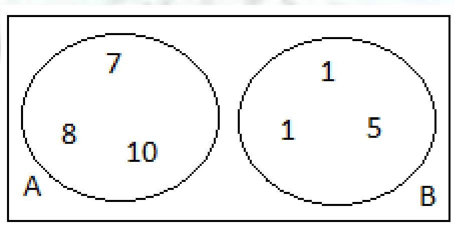
6.3. Disjoint Sets:

Two sets A & B with no common element are called disjoint sets.

i.e. $A \cap B = \emptyset$

For Example: A= {7, 8, 9}, B= {1, 2, 5}, Then $A \cap B = \emptyset$

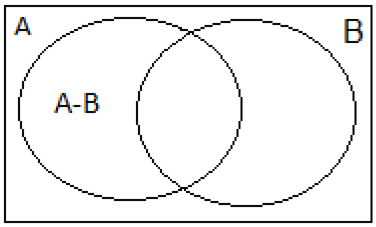
Thus, A and B are disjoint sets.

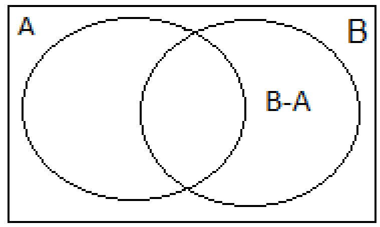


6.4. Difference of two Sets:

Let A and B be two sets. The difference of A and B in that order is the set of all elements of A which do not belong to B. It is denoted by A-B or A\B.

i.e. A-B= $\{x: x \in A \text{ and } x \notin B\}$

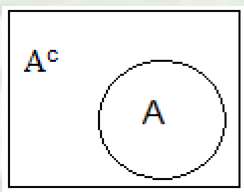




6.5. Complement of a Set:

The difference between universal set U and set A is called the complement of the set A with respect to the universal set U. It is denoted by A' or $\bar{A}$ or A^c or –A.

i.e. $A^c = \{x: x \in U \text{ and } x \notin A\}$



6.6. Cartesian Product of Sets:

Let A and B be two sets, then $A \times B$ is called Cartesian product of set A and set B if

$$A \times B = \{(m,n) : m \in A and\ n \in B\}$$

If A has p elements and B has q elements, then $A \times B$ has pq elements.

For Example: Let $A = \{a,b\}, B = \{1,2,3\}$, then $A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$

Here $A \times B$ has $2*3$ i.e., 6 elements.

7. Some useful Results:

Let A, B, C are finite Sets and U is finite universal set then

(i) $A \cup \varphi = A$

(ii) $A \cup A = A$

(iii) $A \cup B = B \cup A$ $\quad i.e, \cup$ *is commutative,*

(iv) $(A \cup B) \cup C = A \cup (B \cup C)$ $\quad i.e, \cup$ *is associative,*

(v) $A \cup U = U$ *and* $\quad A \cup A' = U$

(vi) $A \cap A = A$ *and* $\quad A \cap A' = \varphi$

(vii) $A \cap B = B \cap A$ $\quad i.e, \cap$ *is commutative,*

$(viii)$ $(A \cap B) \cap C = A \cap (B \cap C)$ $\quad i.e, \cap$ *is associative,*

(ix) $A \cap U = A$

(x) $A \cap \varphi = \varphi$

(xi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\quad i.e, \cap$ *is distributive over union.*

(xii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad$ i.e., $\cap$ *is distributive over* int*er* sec*tion.*

$(xiii)$ $A - B = A \cap B' = B' - A'$

(xiv) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(xv) $A \Delta B = (A - B) \cup (B - A)$

$= (A \cup B) - (A \cap B),$ $A \Delta B$ *is called symmetric difference of* A *and* $B.$

(xvi) $(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B',$ *De Morgan's Law*

$(xvii)$ (a) $A \Delta \varphi = A$

(b) $A \Delta A = \varphi$

(c) $A \Delta B = B \Delta A$

(d) $A \Delta B = \varphi \Rightarrow A = B$

(e) $A \Delta B = A \cup B \Leftrightarrow A \cap B = \varphi$

$(xviii)$ (a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(b) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A)$
$+ n(A \cap B \cap C)$

(c) $n(A') = n(U) - n(A)$

(d) $n(A \cap B') = n(A) - n(A \cap B)$

Value addition: Remarks
Cardinal number of a set: The number of elements in a set is called the cardinal number of the set. **For Example:** A = { $x : x \in$ N, $x \leq$ 5}= {1, 2, 3, 4, 5} and B= { p, q, r, s, t}, Then Cardinal number of set A and set B = 5.

8. Functions:

A function or mapping is a law which assigns every element x of a set P to some unique element, called f(x) of set Q.

Function is written as f : P → Q. Here set P is called the domain of f and set Q is called the co-domain of f. Mappings are also called function and correspondence. Range of 'f' is given by $\{f(x): x \in P\}$.

Value addition: Note
Mapping f : P → Q is called well defined if- (i) All the elements of P are mapped on some member of Q (ii) A member of P should be mapped on unique element of Q If any of above two conditions fails, we say f : P → Q is not well defined.

8.1. Domain of a function: The domain of a function y=$f(x)$ is the set of real number for which the function is defined.

For Example: $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow Domain = (-1, \infty)$

8.2. Range of a function: The set of all real possible value of y in the function y=$f(x)$ is called the range of the function.

For Example: $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow Range = (0, \infty)$

9. Types of Function:

9.1. One-One Mapping: If different elements of domain set P have different f - images in co-domain set B, then the function f : P → Q is said to be *one-one* i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
or equivalently $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

One - one are also called as *injection.*

9.2. Many–One Mapping: A mapping f : P $\rightarrow$ Q is said to be *many-one* if and only if two or more different elements in P have the same f - image in Q.

9.3. Into Mapping: A mapping f is said to be *into* if and only if there is at least one element in Q which is not the f - image of any element in P.
In this case f (P) $\subset$ Q
i.e., range of P is proper subset of co-domain B.

9.4. Onto Mapping: A mapping f is said to be *onto* if and only if every elements in Q is the f - image of at least one element in P.

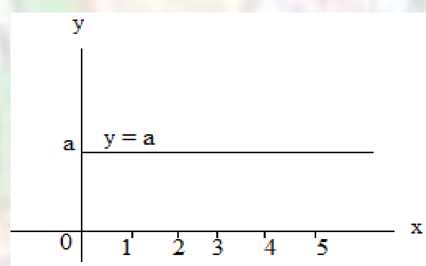
In this case f (P) = Q.
i.e., co-domain= the range of mapping f.
The other name of onto mapping is surjection.

9.5. Bijection Mapping: A mapping is said to be bijection if mapping is one-one and onto both.

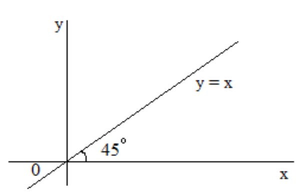
9.6. Inverse function: Inverse funtion f^{-1} exists only if f is one-one and onto from P $\rightarrow$ Q, and it carries elements of Q back to 'P' i.e. f^{-1} is one-one and onto mapping from Q $\rightarrow$ P.

9.7. Constant Function:

If $f(x)$= a $\forall$ $x \in$ R, a is a constant. Then $f(x)$ = a, is called the constant function.



9.8. Identity Function: A function $f(x)$ = x $\forall$ $x \in$ R, is called the identity function. A straight line making 45° angle with positive direction of X-axis and passing through origin is the graph of Identity function.



9.9. Even Function: If $f(-x) = f(x)$ $\forall$ $x \in$ R, then f is called an even function.

For Example: $y = \cos x, \quad y = x^2$

9.10. Odd Function: If $f(-x) = -f(x)$ $\forall$ $x \in$ R, then f is called an odd function.

For Example: $y = \sin x, \quad y = x^3$

9.11. Increasing Function: A function y= $f(x)$ is called monotonically increasing function if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$ and strictly increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

For Example: y = 5x + 9, is a strictly increasing function.

9.12. Decreasing Function: A function y= $f(x)$ is called monotonically decreasing function if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ and strictly decreasing if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

For Example: y= -5x + 9, is a strictly decreasing function.

9.13. Polynomial Function: A function with finite number of term involving constant and an independent variable is called a polynomial function.

For Example: y= $x^2 + 2x + 1$

9.14. Rational Function: The quotient of two polynomial functions is called a rational function.

For Example: $y = \dfrac{x^3 + 2x^2 + 5x + 1}{x^3 + 2x + 1}$

9.15. Periodic Function: If $f(x + T) = f(x)$ $\forall$ $x \in$ R and T is smallest positive real number then f is said to be Periodic function and T is called the period of f.

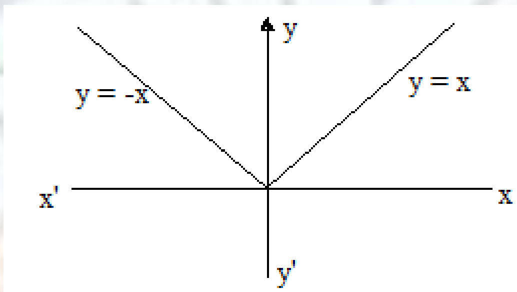
For Example: y= sin x is a periodic function with period 2π

$$\Rightarrow \sin(x + 2\pi) = \sin x$$

9.16. Modulus Function: The function

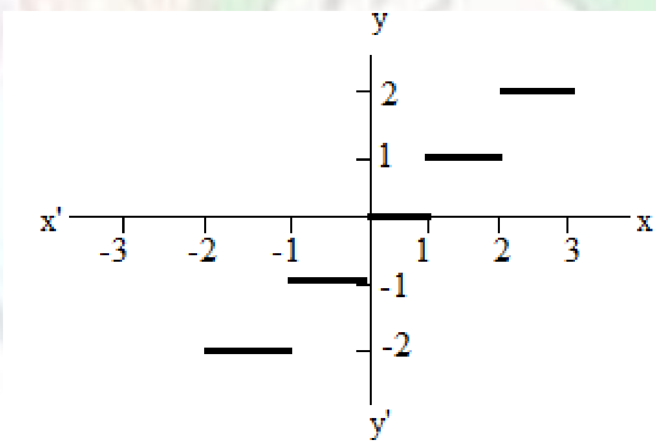
$$f(x)=\begin{cases}-x, & if \;\; x<0\\ 0, & if \;\; x=0\\ x, & if \;\; x>0\end{cases}$$

is called modulus function or absolute value function. It is denoted by | x |.



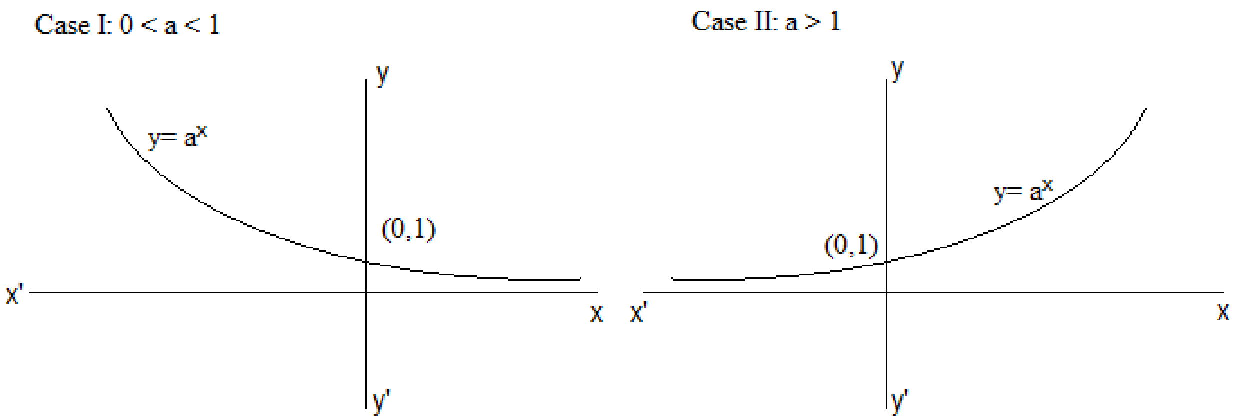
9.17. Greatest Integer Function: The function y= $[x]$ is called greatest Integer Function, where $[x]$ is the greatest integer ≤ x.

f(x)= $[x]$ = {n, n ≤ x ≤ n +1}, where n is any integer.

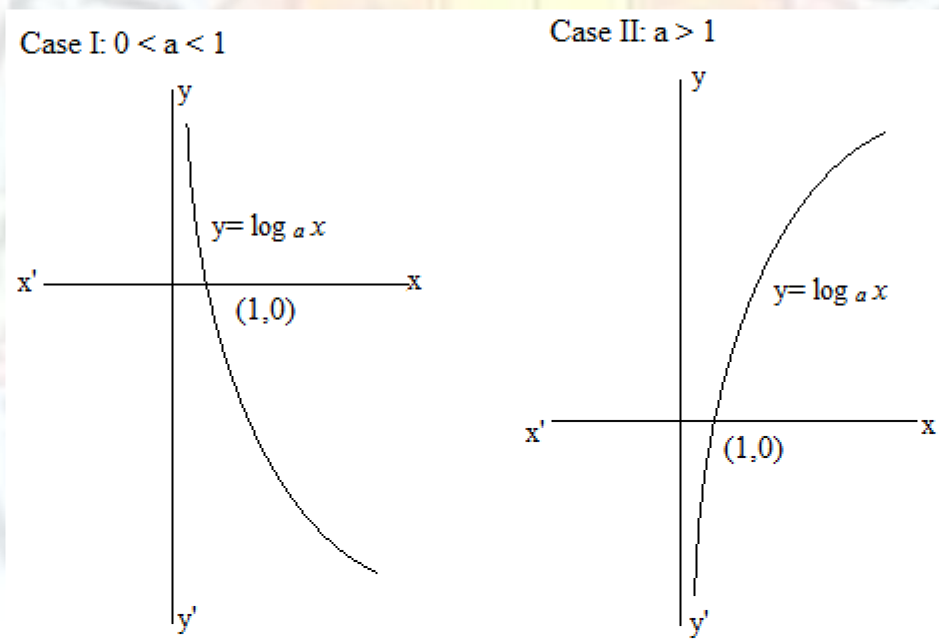


9.18. Transcendental Function: The transcendental function is the functions which is not an algebraic function.

9.19. Exponential Function: The function y **=** $f(x)$ = a^x, a > 0 and x ∈ R is called an exponential function.



9.20. Logarithmic Function: The function y= $\log_a x$ where a ≠ 1 is called logarithmic function



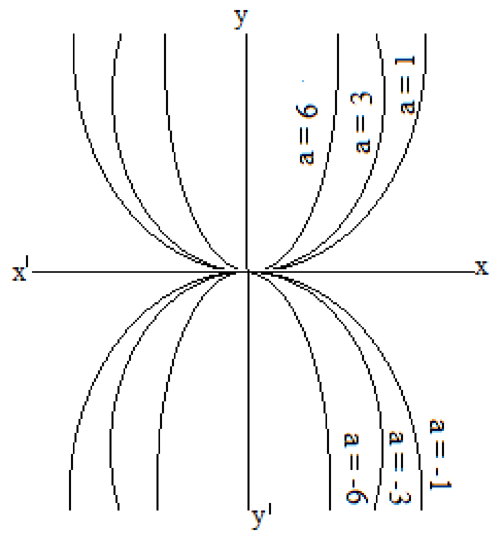
9.21. Power Function: A function of the type y=ax^n is called a power function, where a is any constant i.e. y $\propto$ x^n

9.22. Quadratic function: A power function with power/degree 2 is called quadratic function or second degree function.

i.e. y = ax^2 is general form of quadratic function.

The graph of $y = ax^2$ for different values of a are shown in figure:

a = 1,	$y = x^2$
a = 3,	$y = 3x^2$
a = 6,	$y = 6x^2$
a = -1	$y = -x^2$
a = -3	$y = -3x^2$
a = -6	$y = -6x^2$



The graph of the quadratic function y = ax^2 is a quadratic parabola with vertex at (0,0).

9.23. The graph of a general quadratic function:

The general quadratic function is given by

$$y = ax^2 + bx + c,\ a \neq 0$$

To plot the graph of quadratic function given above , following steps are needed:

- Take coefficient of x^2 as common
- Square the half of the coefficient of x and add and subtract it to the given equation
- Try to make the general form of quadratic function

$$y = ax^2 + bx + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a} +\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

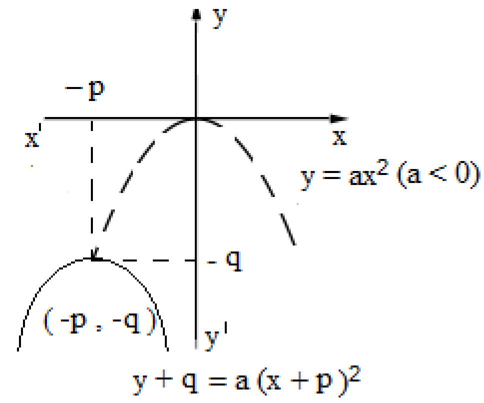
Or $$\left[y + \left(\frac{b^2}{4a} - c\right)\right] = a\left(x + \frac{b}{2a}\right)^2$$

Let $\frac{b^2}{4a} - c = q; and \frac{b}{2a} = p$

Then the above equation can be written as

$(y + q) = a\,(x + p)^2$

If we take Y= y + q and X = x + p, it transforms the above equation to $Y = aX^2$ which is a general form of quadratic parabola with X=0 and Y = 0 i.e , at x = -p, y = -q



Thus graph of the function is obtained by shifting the graph of y $=ax^2$, p unit left to the y axis and q unit downwards to x axis to bring the vertex at the point (- p, - q).

Example 1: Let U be the set of four categories of women: viz

P: Fat and Dark, Q: Fat and Fair

R: Slim and Dark, S: Slim and Fair

Form the subset consisting of fair women. Also find its complement.

Solution: U = {Fat, Dark, Slim, Fair}

Fair = {Q, S}

Complement of Fair = {P, R}

Example 2: In a group of 1000 students, there are 800 students who can speak Hindi and 350 students who can speak English. How many students can speak Hindi only.

Solution: Let H be the set of those students who speak Hindi and E be the set of those who speak English.

Given, $n(H \cup E) = 1000$

$n(H) = 800, \quad n(E) = 350$

$n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$n(H \cap E) = n(H) + n(E) - n(H \cup E)$

$= 800 + 350 - 1000 = 150$

The students who can speak Hindi only.

i.e. $n(H \cap E') = n(H) - n(H \cap E)$

$= 800 - 150 = 650$

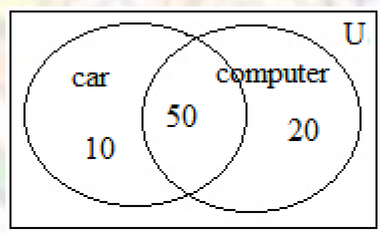
Example 3: In a survey, 60% people having car and 70% having computers. If the people who have both car and computers is 50%. What is the percentage of those people who owned a car or a computer but not both.

Solution: Let $n(U) = 100$,

$n(\text{car}) = 60$

$n(\text{computer}) = 70$

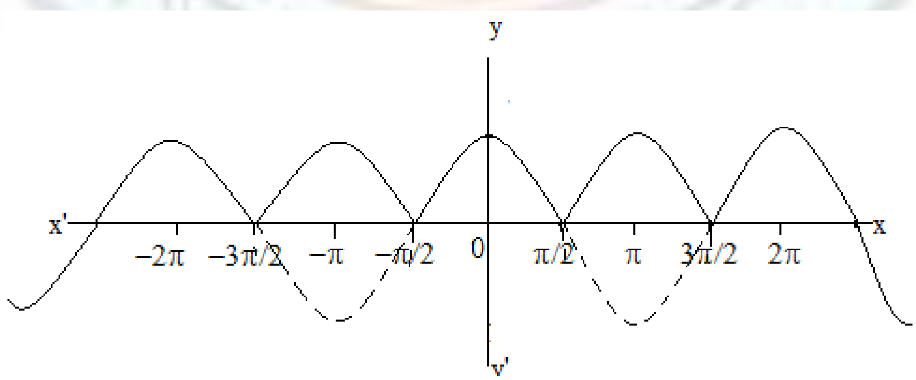
and $n(\text{car} \cap \text{computer}) = 50$



It is clear from the Venn diagram that the percentage of those people who owned either a car or a computer but not both = 10 + 20 = 30%

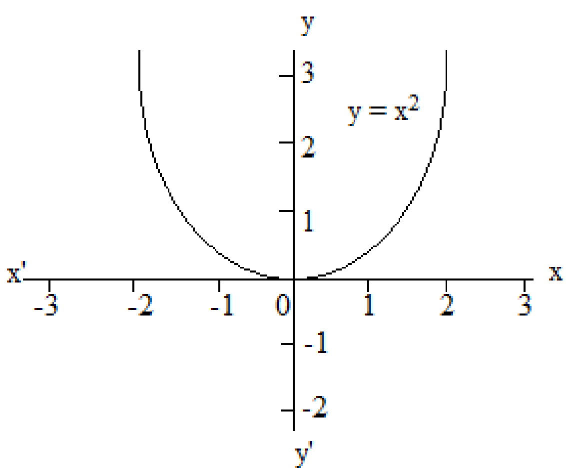
Example 4: Plot the graph of the function $y = |\cos x|$ and find its period.

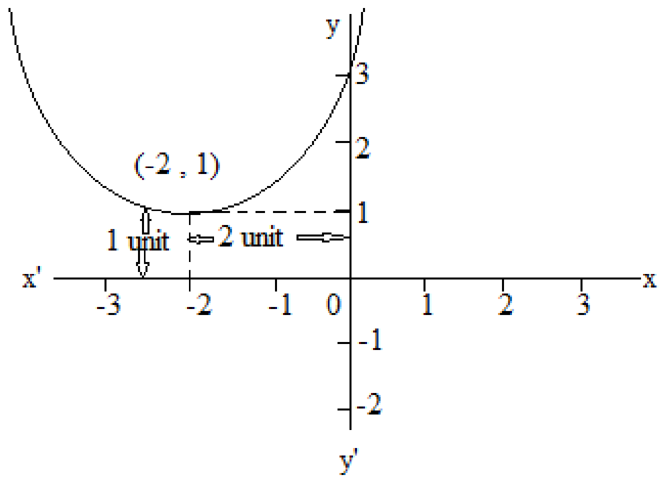
Solution: To plot the graph of $|\cos x|$, first we plot the graph of $\cos x$ and then take reflection of negative portion about x axis towards positive y axis.



Example 5: Shift the graph of y = x^2, two unit to the left and one unit upward.

Solution:





Example 6: Plot the graph of $x^2 = 2x - y$

Solution: $x^2 = 2x - y \Rightarrow y = 2x - x^2$

$y = -(x^2 - 2x)$

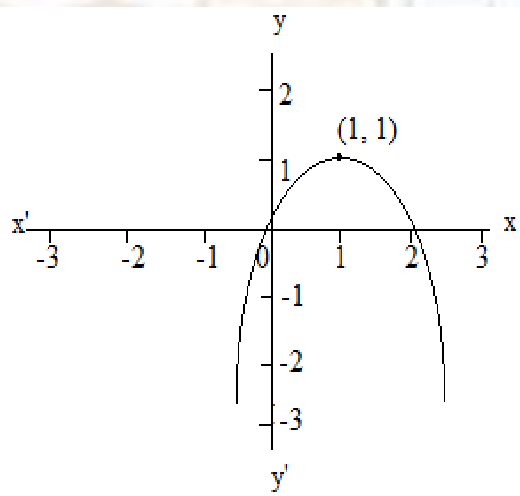
$= -(x^2 - 2x + 1 - 1)$

$= -[(x^2 + 1 - 2x) - 1]$

$= -[(x - 1)^2 - 1]$

$= -(x - 1)^2 + 1$

$y - 1 = -(x - 1)^2$



Example 7: Draw the graph of $y = 2x^2 - 4 + 4$.

Solution: $y = 2x^2 - 4x + 4$

$= 2[x^2 - 2x + 4] = 2[x^2 - 2x + 1 + 3]$

$= 2[(x-1)^2 + 3]$

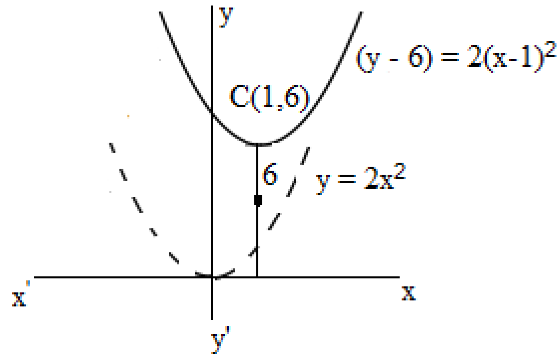
$y - 6 = 2(x-1)^2$

This graph is parabola with vertex at (1, 6).

Its graph can be obtained by shifting the graph of $y = 2x^2$ horizontally and vertically to bring the vertex at (1, 6).

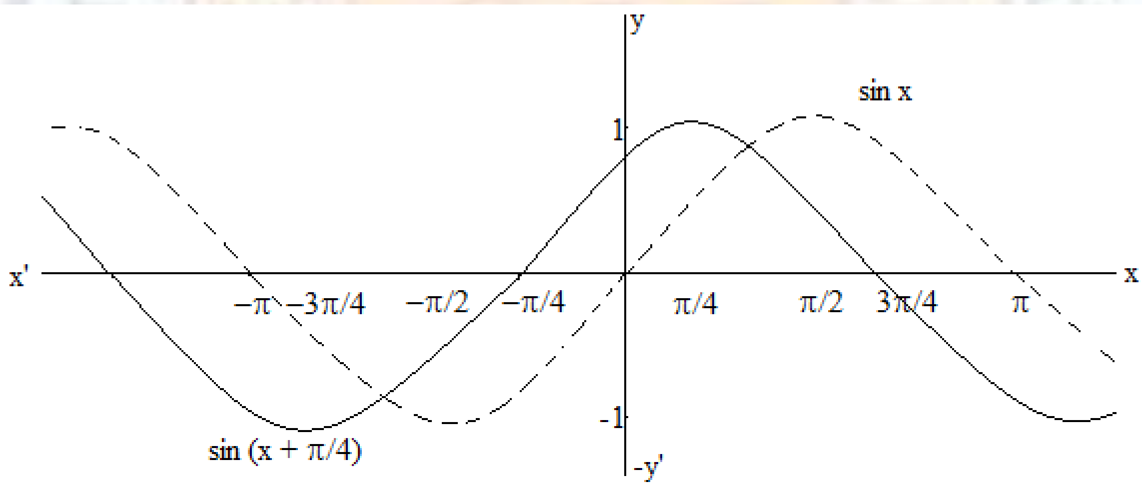
Minimum value of y occurs at the vertex.

There for minimum value of y is 6 at x = 1.



Example 8: Draw the graph of $\sin\left(x+\frac{\pi}{4}\right), in -\pi \le x \le \pi$.

Solution: The graph of y= $\sin\left(x+\frac{\pi}{4}\right)$ first plot the graph of y = $\sin x$ as usual and then shift it $\frac{\pi}{4}$ unit distance to the left of y-axis.



Sets and Functions

Summary:

In this chapter we have discussed the “Sets and Functions”. In particular we discussed:-

- What is set and how we can describe sets in different ways?
- About different sets
- Properties of sets and how we can use sets in day to day problems?
- What is function?
- About different types of functions
- How we can differ about decreasing and increasing behavior of a function?
- How we can plot different types of function, particularly quadratic and trigonometric fuctions?

Exercise:

1. Let A and B have 2 and 7 elements respectively. What will be the maximum and minimum number of elements in $A \cap B$.

2. In a survey of 500 student of a college, it was seen that 130 study life science, 100 study mathematics and 60 study physics, 40 study life-science and mathematics, 30 study mathematics and physics, 50 study life-science and physics and 20 study none of these subjects. How many students study all the subjects?

3. Let L_1 and L_2 be two straight lines in a plane. Then find the all possible values for $L_1 \cap L_2$.

4. If A= { x: x $\geq$ 4 and x ϵN}, B= { x: x $\leq$9 and x ϵN}, then find $A \cap B$, A-B, $A \cup B$.

5. Shift the graph of y = $-(x^2 - 6x)$ one unit right and two unit upwards.

6. Draw the graph of y = $|\sin x|$ in $[-2\pi, 2\pi]$.

References:

- David C. Lay, Linear Algebra and its Applications (3rd edition), Pearson education

- Vijay K Khanna and S K Bhambri, A course in Abstract Algebra (2nd revised edition),Vikas Publishing House Pvt. Ltd.
- Joseph A. Gallian, Contemporary Abstract Algebra (4th edition), Narosa Publishing House
- R. D. Sharma and Umesh Kumar, Basic Applied Mathematics for the Physical Sciences, Pearson education
- http://en.wikipedia.org/wiki/Set_function
- http://www.egyankosh.ac.in