Discipline Courses-I
Semester-I
Paper: Calculus-I
Lesson: Sketching of Graphs using Derivatives

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Sketching of Graphs using Derivatives

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1. Learning outcomes:

After studying this chapter you should be able to

- understand the increasing, decreasing and monotonic functions.
- determine the critical points
- determine the maxima and minima of function
- determine the concavity of the function
- determine the points of inflection

- draw the graph of functions using derivatives
- determine the limits to infinity
- determine the vertical and horizontal asymptotes
- determine the tangents and cusps.

2. Introduction:

One of the aims of calculus is to be able to tell as much as possible about a given function. In particular, Our goal is to understand how the graph of a function relates to its derivative, and vice versa. In this lesson we will find all ingredients that can help in drawing the graph of a function using the derivative.

3. Increasing and Decreasing Function:

3.1. Increasing function:

Let a function f(x) is defined on an interval I, then f(x) is called increasing function on I if

$$f(x_1) \leq f(x_2) \quad \textit{whenever} \ x_1 < x_2$$

for all $x_1, x_2 \in I$

Value Addition: Strictly Increasing Function
A function f(x) is called strictly increasing function on I if $f(x_1) < f(x_2)$ *whenever* $x_1 < x_2$ for all $x_1, x_2 \in I$

3.2. Decreasing function:

Let f(x) be a function defined on interval I, then f(x) is called decreasing function on I if

$$f(x_1) \geq f(x_2) \quad \textit{whenever} \ x_1 < x_2$$

for all $x_1, x_2 \in I$

Value Addition: Decreasing Function
The function f(x) is called strictly decreasing function on I if $f(x_1) > f(x_2)$ *whenever* $x_1 < x_2$ for all $x_1, x_2 \in I$

Value Addition: Note
For a increasing function the graph of the function moves in the upward direction in the right while for a decreasing function it moves downward in the right.

3.3. Monotonic function:

A function f(x) defined on an interval I, is said to be monotonic on I if it is either strictly increasing on I or strictly decreasing on I.

Vale Addition: Note
The monotonic behavior of a function on an interval I is related to the sign of first derivative of the function on I.

3.4. Monotone function Theorem:

Theorems 1: Let f(x) be a function which is differentiable on an open interval (a, b)

(i) If $f'(x) > 0$ for all x$\in$ (a, b), then f(x) is strictly increasing on (a, b).

(ii) If $f'(x) < 0$ for all x$\in$ (a, b), then f(x) is strictly decreasing on (a, b).

Proof : (i) Let $f'(x) > 0$ for all x$\in$ (a, b). and let x_1, x_2 be any two numbers chosen arbitrarily from this interval, with $x_1 < x_2$.

Using the mean value theorem, we know that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

or $$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

where c is number lies in (a, b) such that $x_1 < c < x_2$

Because $f'(x) > 0$ for all $x \in (a, b)$, therefore $f'(c) > 0$

and $x_2 - x_1 > 0$ $\left[\text{since } x_1 < x_2 \textit{ by choise}\right]$

then we have

$$f(x_2) - f(x_1) = f'(c).(x_2 - x_1) > 0$$

$$\Rightarrow \quad f(x_2) > f(x_1) \quad \text{whenever } x_1 < x_2$$

That is x_1 and x_2 are any two numbers in (a, b) such that $x_1 < x_2$ then $f(x_1) < f(x_2)$.

Therefore f(x) is strictly increasing function on (a, b).

(ii) The proof of II part is similar to the Ist part and left as an exercise

4. Critical Points:

Let f(x) be a function defined on an interval I, the points in the interval I at which the derivative of f(x) is either zero or does not exist are called critical points.

Example 1 : Find the critical points of the function.

$$f(x) = x^3 - 3x^2 - 9x + 1$$

Solution : Given function is $f(x) = x^3 - 3x^2 - 9x + 1$

On differentiating, we have

$$f'(x) = 3x^2 - 6x - 9$$

for the critical points, we have

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 - 6x - 9 = 0$$

$$\Rightarrow \quad x^2 - 2x - 3 = 0$$

$$\Rightarrow \quad (x+1)(x-3) = 0$$

$$x = -1 \qquad x = 3$$

Since f'(x) exists everywhere therefore $x = -1$ and 3 are the only critical points.

Value Addition: Interval of Increase or Decrease
In order to determine the interval of increase or decrease of a function, we first find all the critical points of the function and then mark these points on a number line. If the coefficient of highest power of x is positive in the derivative of f(x), mark each interval as + and – alternatively from right to left. Then the function will increase in all intervals which have the + sign and will decrease in all intervals which have – sign. If the coefficient of highest power of x is negative in the derivative of f(x), mark each interval as - and + alternatively from right to left. then the function will increase in all intervals which have the + sign and will decrease in all intervals which have – sign.

Example 2 : For the function $f(x) = \frac{1}{3}x^3 - 9x + 2$. Final all the critical points and also find where the function is increasing or decreasing.

Solution : Given function is

$$f(x) = \frac{1}{3}x^3 - 9x + 2$$

The derivative of f(x) is $f'(x) = x^2 - 9$

For the critical points, we have

$$f'(x) = 0$$

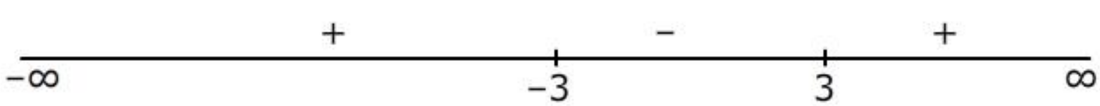
$$x^2 - 9 = 0$$

$$\Rightarrow \quad x^2 = 9$$

$$\Rightarrow \quad x = \pm 3$$

Since f'(x) exists everywhere therefore, $x = \pm 3$ are the only critical points.

For the region of increasing or decreasing. We have



Thus, the function $f(x) = \frac{1}{3}x^3 - 9x + 2$ is increasing in the region $(-\infty, -3) \cup (3, \infty)$ and decreasing in the region $(-3, 3)$.

5. Maxima and Minima of a function (Extrema):

Let f(x) be a function defined on an interval I. A point in I is called a maxima or minima if the function f(x) attains the maximum value or minima value at that point respectively.

5.1. First Derivative Test for Relative Extrema:

In order to find relative extremum, we use the following steps.

Step 1 : Find all the critical points.

Step 2 : Now classify each critical point (c, f(c)) as follows

(i) The point (c, f(c)) is a relative minimum if $f'(x) < 0$ (graph falling) for all $x \in (a, c)$ to the left of c and $f'(x) > 0$ (graph rising) for all x in an open interval (c, b) to the right of c.

(ii) The point (c, f(c)) is a relative maximum if $f'(x) > 0$ (graph rising) for all $x \in (a, c)$ to the left of c and $f'(x) < 0$ (graph falling) for all $x \in (c, b)$ to the right of c

(iii) The point (c, f(c)) is not an extremum if the derivative $f'(x)$ has the same sign for all x is open interval (a, c) and (c, b) on each side of c.

Value Addition: Note
Every relative extrmum is a critical point but every critical point need not be a relative extremum.

Example 3 : Find all critical points of $f(x) = \frac{x-1}{x^2+3}$. Determine whether each corresponds to a relative maximum a relative minimum or neither.

Solution : Given function is

$$f(x) = \frac{x-1}{x^2+3}$$

The derivative of f(x) is

$$f'(x) = \frac{(x^2+3).1-(x-1).2x}{(x^2+3)^2} = \frac{x^2+3-2x^2+2x}{(x^2+3)^2}$$

$$\Rightarrow \quad f'(x) = \frac{-x^2+2x+3}{(x^2+3)^2} = \frac{-(x+1)(x-3)}{(x^2+3)^2}$$

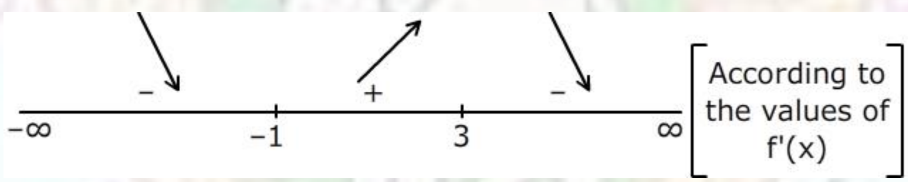
for the critical points, we have

$$\Rightarrow \quad \frac{-(x+1)(x-3)}{(x^2+3)^2} = 0 \quad \Rightarrow \quad (x+1)(x-3) = 0$$

$$\Rightarrow \quad x = -1, 3$$

Since f'(x) does not attain infinity at any point therefore, $x = -1$ and 3 are the only critical points

Now, we have



Now, for the point $x = -1$, $f'(x) < 0$ for all $x \in (-\infty, -1)$ to the left of -1 and $f'(x) > 0$ for all $x \in (-1, 3)$ to the right of –1.

Thus, the point $x = -1$ a relative minimum.

For the point $x = 3$, $f'(x) > 0$ for all $x \in (-1, 3)$ to the left of 3 and $f'(x) < 0$ for all $x \in (3, \infty)$ to the right of 3.

Thus, the point $x = 3$ is a relative maximum.

The graph of $f(x) = \dfrac{x-1}{x^2+3}$ is

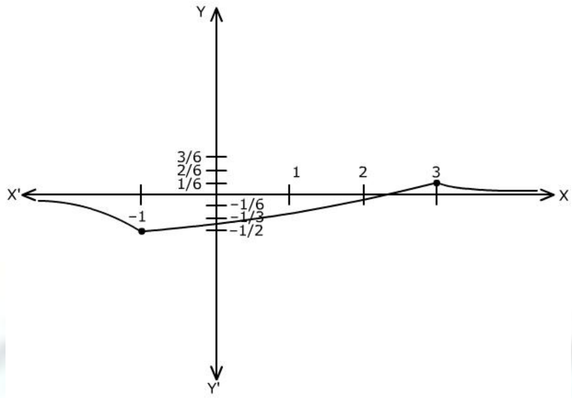


Figure. Graph of $f(x) = \dfrac{x-1}{x^2+3}$

5.2. Second–Derivative Test for Relative Extrema:

Let f(x) be a function defined on an interval I. Then to find the Relative Extrema of f(x), we use the following steps.

Step 1 : First find the critical points of f(x).

Step 2 : Find the second derivative of $f(x)$ i.e. $f''(x)$

Step 3 : Now check the sign of $f''(x)$ at the critical points. Let c be a critical point, then

(i) If $f''(c) > 0$, then $f(x)$ is relative minimum at x = c

(ii) If $f''(c) < 0$, then $f(x)$ is relative maximum at x = c

(iii) If $f''(c) = 0$, then $f(x)$ is neither maximum nor minimum at x =c and further investigation is needed.

6. Concavity of a function:

The bending of a curve is measured in terms of concavity. If the slope of a graph increases on an interval then the graph is called concave up in the interval. If the slope is decreasing on the interval then the graph is called concave down in the interval.

6.1. Derivative Test for the Concavity:

Let f(x) be a function defined on an interval I.

(i) If the slope of graph of f(x) is increasing on I. i.e.,

$f'(x)$ is increasing on I

$\Rightarrow$ $f''(x) > 0$ for all $x \in I$

Then f(x) is called concave up in the interval I.

(ii) If the slope of graph of $f(x)$ is decreasing on I i.e.

$f'(x)$ is decreasing on I

$\Rightarrow$ $f''(x) < 0$ for all $x \in I$

Then f(x) is called concave down in the interval I.

Value Addition: Note
A function f(x) defined on interval I is called concave up if $f''(x) > 0$ for all $x \in I$ and called concave down if $f''(x) < 0$ for all $x \in I$.

7. Point of Inflection:

Let f(x) be a function defined on an interval I, a point $c \in I$ is called point of inflection if the graph of f(x) is concave up on one side of c and concave down on other side of c.

In other words we can say that if $f''(x) > 0$ on one side of c and $f''(x) < 0$ on the other side of c then the point c is called the point of inflection or inflection point.

Value Addition: Note
1. The Points at which either $f'(x) = 0$ or does not exists are called the critical points of first order or first order critical points. 2. The points at which either $f''(x) = 0$ or does not exists are called the critical points of second order or second order critical points.

Example 4 : For the function $f(t) = (t+1)^2(t-5)$

(i) Find all the critical points

(ii) Find where the function is increasing or decreasing

(iii) Find the relative maximum, relative minimum or neither

(iv) Find the second order critical points and tell where the graph is concave up and where it is concave down

(v) Find the point if inflection if any

Solution: Given function is $f(t) = (t+1)^2(t-5)$

First order derivative is

$$f'(t) = 2(t+1)(t-5) + (t+1)^2$$

$$= (t+1)[2t - 10 + t + 1] = 3(t+1)(t-3)$$

Second order derivative is

$$f''(t) = 3(t+1) + 3(t-3) = 3[2t-2]$$

$$= 6(t-1)$$

(i) For the critical points, we have

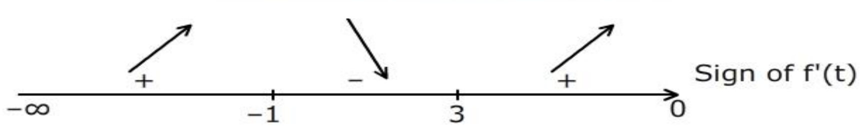
$$f'(t) = 0$$

$$\Rightarrow \quad 3(t+1)(t-3) = 0$$

$$\Rightarrow \quad t = -1, 3$$

since $f'(t)$ exists everywhere, thus $t = -1, 3$ are the only critical points.

(ii) For the region of increasing and decreasing, we have



since $f'(t) > 0$ for $t \in (-\infty, -1) \cup (3, \infty)$

Thus, $f(t)$ is increasing in $(-\infty, -1) \cup (3, \infty)$.

$$f'(t) < 0 \text{ for } t \in (-1, 3)$$

Thus, $f(t)$ is decreasing in $(-1, 3)$

(iii) For the point $t = -1$

$f'(t) > 0$ for all $t \in (-\infty, -1)$ left of $t = -1$.

and $f'(t) < 0$ for all $t \in (-1, 3)$ right of $t = -1$.

Thus, $t = -1$ is the relative maximum.

For the point $t = 3$.

$f'(t) < 0$ for all $t \in (-1, 3)$ left of $t = 3$.

and $f'(t) > 0$ for all $t \in (3, \infty)$ right of $t = 3$.

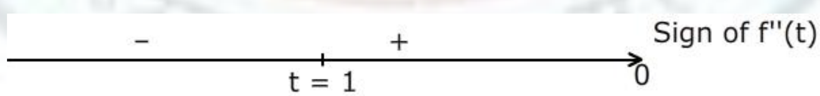
Thus, $t = 3$ is the point of relative minimum.

(iv) For the second order critical points, we have

$$f''(t) = 0$$

$$\Rightarrow \quad 6(t-1) = 0 \quad \Rightarrow \quad t = 1$$

since $f''(t)$ exists everywhere, therefore t = 1, is only second order critical point



since $f''(t) > 0$ for all $t > 1$, therefore $f(t)$ in concave up in the region $t > 1$.

Since $f''(t) < 0$ for all $t < 1$, thus, $f(t)$ is concave down in the region $t < 1$

(v) Point of infection

Since $f(t)$ is concave up in right of $t=1$ and concave down in the left of $t=1$, therefore $t=1$, is the point of inflection.

8. Curve Sketching Using the First and Second Derivatives:

In order to sketch the graph of a function, we use the following steps.

Step 1 : First find the critical points of f(x).

Step 2 : Determine the region in which the function is increasing or decreasing

Step 3 : Determine the concavity of the graph.

Step 4 : Determine the points of inflexion.

Step 5 : Find the relative extrema.

Step 6 : Find the value of f(x) at the critical points and points of inflection.

Step 7 : Find the points where the f(x) is either zero or does not exist.

Step 8 : Now Draw the graph.

Example 5 : Sketch the graph of the function

$$f(x) = \frac{x}{x^2+1}$$

Solution : Given function is $f(x) = \frac{x}{x^2+1}$

First derivate of f(x) is

$$f'(x) = \frac{(x^2+1).1 - x.\,2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

Second derivate of f(x) is

$$f''(x) = \frac{(x^2+1)^2 \,.\, (-2x) - (1-x^2) \,.\, 2(x^2+1) \,.\, (2x)}{(x^2+1)^4}$$

$$= \frac{2x\left(x^2+1\right)\left[-x^2-1-\left(1-x^2\right).2\right]}{\left(x^2+1\right)^4}$$

$$= \frac{2x\left(-x^2-1-2+2x^2\right)}{\left(x^2+1\right)^3} = \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}$$

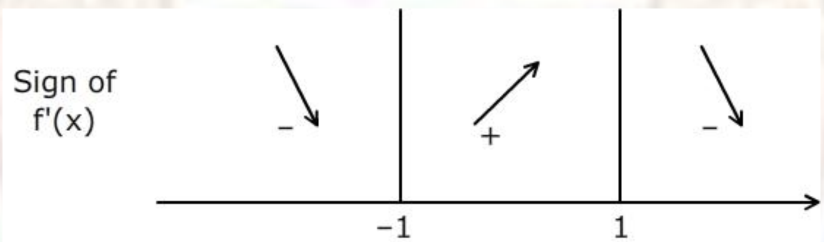
Step 1 : Critical points

For critical points $f'(x) = 0 \Rightarrow \frac{1-x^2}{\left(x^2+1\right)^2} = 0$

$\Rightarrow \quad x = \pm 1$

since $f'(x)$ exists everywhere thus $x = \pm 1$ are the only critical points.

Step 2 : Region for increase and decrease



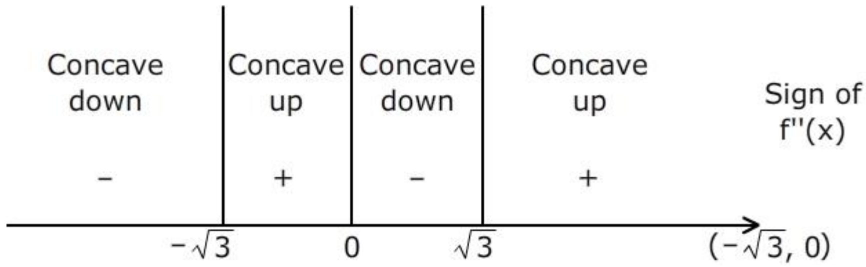
Thus $f(x)$ decreases in the region $(-\infty, -1) \cup (1, \infty)$ and increases in the region (-1, 1).

Step 3 : Concavity

for the concavity, we have

$$f''(x) = 0 \Rightarrow \frac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3} = 0$$

$\Rightarrow \quad x = 0, \pm\sqrt{3}$



Thus f(x) is concave up in the region $\left(-\sqrt{3}, 0\right) \cup \left(\sqrt{3}, \infty\right)$ and concave down in the region $\left(-\infty, -\sqrt{3}\right) \cup (0, 3)$.

Step 4 : The point x = 0 and $\pm\sqrt{3}$ are the points of inflection since $f''(x)$ changes the sign on both sides of these points.

Step 5 : Relative Extrema

For the critical points $x = \pm 1$, we have

$$f''(-1) = \frac{-2(1-3)}{(1+1)^3} = \frac{4}{8} = \frac{1}{2} > 0$$

$$f''(1) = \frac{2(1-3)}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2} < 0$$

Thus f(x) is relative minimum at the point $x = -1$. and $f(x)$ is relative maximum at the point $x = 1$.

Step 6 : Value of the function at the critical points and points of inflection

$$f(-1) = \frac{-1}{1+1} = -\frac{1}{2}$$

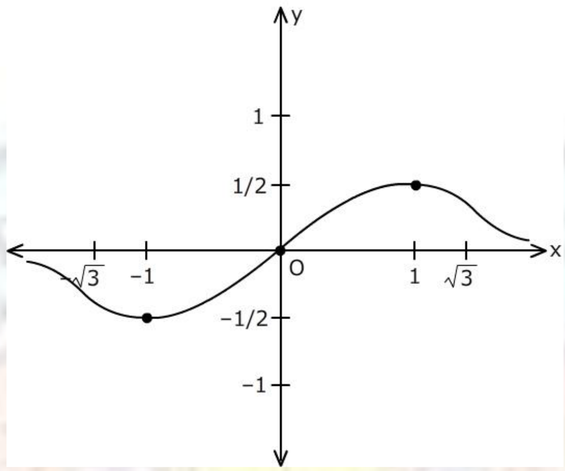
$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f(0) = 0$$

Step 7 : Intersection of x–axis

$$f(x) = 0 \qquad \Rightarrow \qquad \frac{x}{x^2+1} = 0 \qquad \Rightarrow \quad x = 0$$

Step 8 : Now drawing of sketch



Graph of the function $f(x) = \dfrac{x}{x^2+1}$

Example 6: Sketch the graph of the function

$$f(\theta) = \sin\theta - \cos\theta \quad \text{for} \quad 0 \le \theta \le 2\pi$$

Solution : We have

$$f'(\theta) = \cos\theta + \sin\theta$$

$$f''(\theta) = -\sin\theta + \cos\theta$$

Step 1 : Critical points

for critical points, we have $f'(\theta) = 0$

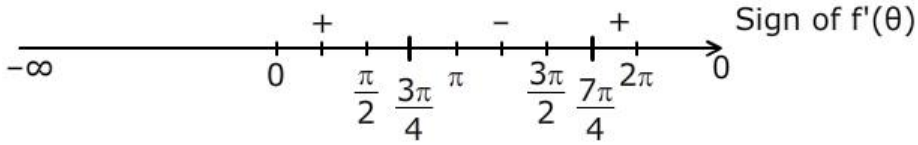
$$\Rightarrow \qquad \cos\theta + \sin\theta = 0$$

$$\Rightarrow \qquad \tan\theta = -1 = \tan\frac{3\pi}{4} \ \ or \ \ \tan\frac{7\pi}{4}$$

thus, $\theta = \dfrac{3\pi}{4} \ \ and \ \ \dfrac{7\pi}{4}$

Since $f'(\theta)$ exists everywhere for $0 \le \theta \le 2\pi$ therefore $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ are the only critical points

Step 2 : Increasing and decreasing



Thus f(0) is increasing in the region $\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$ and decreasing in the region $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$

Step 3 : Concavity

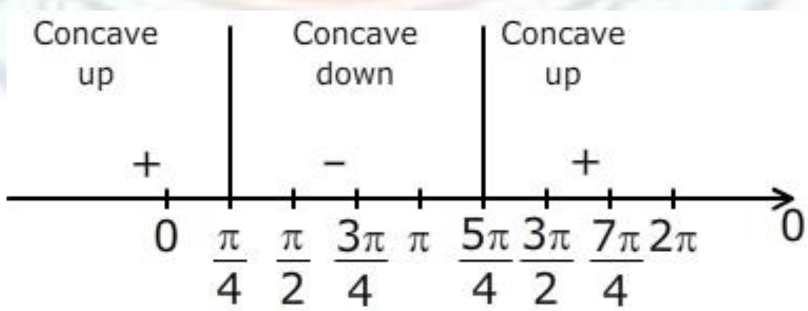
we have $f''(\theta) = 0$

$$\Rightarrow \quad -\sin\theta + \cos\theta = 0$$

$$\Rightarrow \quad -\tan\theta = 1 = \tan\frac{\pi}{4} \ or \ \tan\frac{5\pi}{4}$$

Thus, $\theta = \frac{\pi}{4} \ or \ \frac{5\pi}{4}$

Since $f''(\theta)$ exists everywhere, therefore $\theta = \frac{\pi}{4}$ *and* $\frac{5\pi}{4}$ are the only second order critical points.



Step 4 : Thus, $\theta = \frac{\pi}{4}$ *and* $\frac{5\pi}{4}$ are the points of inflection

Step 5 : Relative extrema

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} > 0$$

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = +\sqrt{2} < 0$$

Thus $f(\theta)$ is maximum at $\frac{3\pi}{4}$ and attains minimum at $\frac{7\pi}{4}$

Step 6 : Value of the $f(\theta)$ at critical points and inflection points.

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = +\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = +\sqrt{2}$$

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} - \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$$

$f(0) = \sin\theta - \cos\theta = -1$ and $f(2\pi) = \sin 2\pi - \cos 2\pi = -1$

Step 7 : Points where $f(\theta)$ is either zero or does not exists, we have

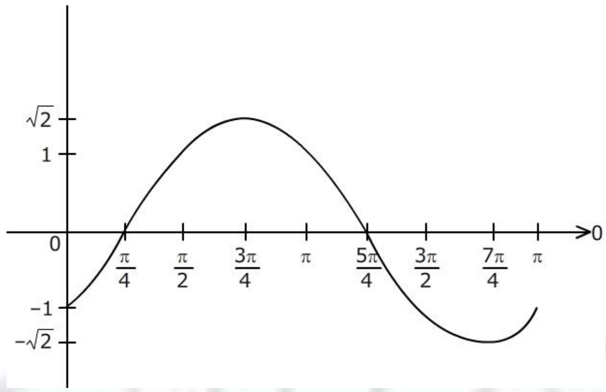
$$f(\theta) = 0 \quad \Rightarrow \sin\theta - \cos\theta = 0$$

$$\Rightarrow \tan\theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4} \ \& \ \frac{5\pi}{4}$$

Thus, $f(\theta)$ cut the θ– axis at $\theta = \frac{\pi}{4} \ \& \ \frac{5\pi}{4}$

Step 8 : Sketch of the graph



Graph of the function $f(\theta) = \sin\theta - \cos\theta$

9. Limits to infinity:

Let f(x) be a function defied, then the limit to infinity is denoted by

$$\lim_{x \to +\infty} f(x) \quad or \quad \lim_{x \to -\infty} f(x)$$

and defined as

$$\lim_{x \to +\infty} f(x) = L_1$$

means that for any number $\varepsilon > 0$ there exists a number n, such that

$$\left| f(x) - L_1 \right| < \in \quad whenever \quad x > n,$$

for all x in the domain of f.

Similarly

$$\lim_{x \to -\infty} f(x) = L_2$$

means that for an $\varepsilon > 0$, there exists a number n_2

such that

$$\left| f(x) - L_2 \right| < \in \quad whenever \quad x > n_2 .$$

for x in the domain of f.

9.1. Special Limits to infinity:

Theorem 2: If A is any real number and r is a positive rational number, then

$$\lim_{x \to +\infty} \frac{A}{x^r} = 0$$

Furthermore, if r is such that x^r is defined for x < 0,

Then,

$$\lim_{x \to -\infty} \frac{A}{x^r} = 0$$

Proof: We will start the proof of the theorem by proving that

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

For $\varepsilon > 0$, let $n = \frac{1}{\varepsilon}$, then for $x > n$, we have

$$x > n = \frac{1}{\varepsilon} \quad \text{so that} \quad \frac{1}{x} < \varepsilon$$

Thus, $\left| \frac{1}{x} - \varepsilon \right| < 0$

Then, by the definition of limit, we have

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

Now let r be a rational number, say $r = {}^{p}\!/_{q}$. Then

$$\lim_{x \to +\infty} \frac{A}{x^r} = \lim_{x \to +\infty} \frac{A}{x^{p/q}} = A. \lim_{x \to +\infty} \left[\frac{1}{\sqrt[q]{x}} \right]^p$$

$$= A\left[\sqrt[q]{\lim_{x \to +\infty} \frac{1}{x}} \right]^p$$

$$= A\left[\sqrt[q]{0} \right]^p = A.0 = 0$$

Hence $\lim\limits_{x \to +\infty} \dfrac{A}{x^r} = 0$

The proof of the $\lim\limits_{x \to -\infty} \dfrac{A}{x^r} = 0$ for $x < 0$ is similar to above proof. Thus, we left it as an exercise.

9.2. Evaluation of the Limits of the form $\lim\limits_{x \to +\infty} \dfrac{p(x)}{q(x)}$ or $\lim\limits_{x \to -\infty} \dfrac{p(x)}{q(x)}$, where p(x) and q(x) are polynomials:

In order to evaluate the limits of the form

$$\lim_{x \to +\infty} \frac{p(x)}{q(x)} \quad or \quad \lim_{x \to -\infty} \frac{p(x)}{q(x)}$$

where p(x) and q(x) are polynomials.

We divide both p(x) and q(x) by the highest power of x that occurs either in p(x) or q(x) and then will evaluate the limit.

Example 7 : Evaluate $\lim\limits_{x \to -\infty} \dfrac{(2x+5)(x-3)}{(7x-2)(4x+1)}$.

Solution : We have

$$\lim_{x \to -\infty} \frac{(2x+5)(x-3)}{(7x-2)(4x+1)} = \lim_{x \to -\infty} \frac{x^2\left(2+\frac{5}{x}\right)\left(1-\frac{3}{x}\right)}{x^2\left(7-\frac{2}{x}\right)\left(4+\frac{1}{x}\right)}$$

$$= \lim_{x \to -\infty} \frac{\left(2+\frac{5}{x}\right)\left(1-\frac{3}{x}\right)}{\left(7-\frac{2}{x}\right)\left(4+\frac{1}{x}\right)}$$

$$= \frac{\lim\limits_{x \to -\infty}\left(2+\frac{5}{x}\right)\left(1-\frac{3}{x}\right)}{\lim\limits_{x \to -\infty}\left(7-\frac{2}{x}\right)\left(4+\frac{1}{x}\right)}$$

$$= \frac{\lim_{x\to-\infty}\left(2+\frac{5}{x}\right) \cdot \lim_{x\to-\infty}\left(1-\frac{3}{x}\right)}{\lim_{x\to-\infty}\left(7-\frac{2}{x}\right)\lim_{x\to-\infty}\left(4+\frac{1}{x}\right)}$$

$$= \frac{2.1}{7.4} = \frac{2}{28} = \frac{1}{14}$$

Then $\lim_{x\to-\infty}\frac{(2x+5)(x-3)}{(7x-2)(4x+1)} = \frac{1}{14}$

Example 8 : Evaluate $\lim_{x\to-\infty} e^x \sin x$.

Solution : We have $\lim_{x\to-\infty} e^x \sin x$

The product rule of limits cannot be applied here because the $\lim_{x\to-\infty} \sin x$ does not exists [it diverges by oscillation]

Let x = –y, then $\lim_{x\to-\infty} e^x \sin x = \lim_{y\to\infty} e^{-y}\sin(-y)$

$$= \lim_{x\to\infty}\left(-e^{-y}\sin y\right)$$

However, the magnitude of y → ∞, $\frac{\sin y}{e^y}$, must become smaller and smaller as y→∞, sine siny is bounded between -1 and 1, while e^y increase relentlessly larger with y.

Thus by squeeze rule

$$\lim_{x\to\infty} -e^{-y}\sin y = 0$$

Hence, $\lim_{x\to-\infty} e^x \sin x = \lim_{y\to\infty} -e^{-y}\sin y = 0$.

Value Addition: Limit Rule
1. If $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist, then for constant a& b
(a) Power Rule : $\lim_{x\to+\infty}\left[f(x)\right]^n = \left[\lim_{x\to+\infty} f(x)\right]^n$
(b) Linearity Rule : $\lim_{x\to+\infty}\left[a\,f(x)+b\,g(x)\right] = a\lim_{x\to+\infty} f(x) + b\lim_{x\to+\infty} g(x)$

	(c)	Product Rule : $\lim_{x\to+\infty}\left[f(x)\,.\,g(x)\right] = \left[\lim_{x\to+\infty} f(x)\right] + \left[\lim_{x\to+\infty} g(x)\right]$
	(d)	Quotient Rule : $\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} f(x)}$ $\quad provided \lim_{x\to+\infty} g(x) \neq 0$
2.	Similar results also hold for $\underset{x\to-\infty}{Lim} f(x)$, if it exists.	

9.3. Infinite Limits:

Let f(x) be a function, then the limit

$$\lim_{x\to x_0} f(x) = +\infty$$

means that the function f(x) increases without bound as x approaches x_0 from either side.

Similarly

$$\lim_{x\to x_0} g(x) = -\infty$$

means that function g(x) decreases without bound as x approaches x_0.

Value Addition: Note	
1.	In mathematics, the symbol ∞ not a number, but it is used to describe either the process of unrestricted growth or the result of such growth.
2.	The limit $\lim_{x\to x_0} f(x) = +\infty$ mean that for any number $n > 0$ (no matter how large) there exists a number $\delta > 0$ such that $f(x) > n \quad whenever \quad 0 < \lvert x - x_0 \rvert < \delta$.
3.	$\lim_{x\to x_0} g(x) = -\infty$ means for any $n > 0$, there exists a number $\delta > 0$ such that $g(x) < -n \quad whenever \quad 0 < \lvert x - x_0 \rvert < \delta$.

Example 9 : Find $\lim_{x\to 3^+}\frac{2x-4}{x-3}$ *and* $\lim_{x\to 3^+}\frac{2x-4}{x-3}$

Solution : We notice that $\frac{1}{x-3}$ increases without bound as x approaches 3 from the right and $\frac{1}{x-3}$ decreases without bound as x approaches 3 from the left.

Thus,

$$\lim_{x \to 3^+} \frac{1}{x-3} = +\infty \quad and \quad \lim_{x \to 3^-} \frac{1}{x-3} = -\infty$$

we have

$$\lim_{x \to 3} (2x-4) = 2$$

Thus,

$$\lim_{x \to 3^+} \frac{2x-4}{x-3} = +\infty \quad and \quad \lim_{x \to 3^-} \frac{2x-4}{x-3} = -\infty$$

10. Vertical and Horizontal Asymptote:

Let f(x) a function defined on an interval I.

A line x = a is called a vertical asymptote to the graph of f(x) if either of the one sided limits

$$\lim_{x \to a^-} f(x) \quad or \quad \lim_{x \to a^+} f(x)$$

is infinite.

A line y = b is called a horizontal asymptote of the graph f(x) if

$$\lim_{x \to \infty} f(x) = b \quad or \quad \lim_{x \to -\infty} f(x) = b$$

11. Vertical Tangents and Cusps:

Let f(x) be a function defined on an interval I, Let P(a, f(a)) be any point.

1. Then the graph of f(x) has a vertical tangent at P if

$\lim_{x \to a^+} f'(x) \quad and \quad \lim_{x \to a^-} f'(x)$ are either both +∞ or both -∞.

2. The graph of f(x) has a cusp at P if $\lim_{x \to a^-} f'(x)$ and $\lim_{x \to a^+} f'(x)$ are both infinite with opposite signs i.e. one is +∞ and the other is -∞.

Example 10 : Determine whether the graph of the given function has a vertical tangent or cusp :

$$f(x) = x^{1/3}(x-4)$$

Solution : Given function

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$$

Derivative of $f(x)$ is

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}(x-1)$$

Thus, $f'(x)$ become infinity only when x = 0

Now we find

$$\lim_{x \to a^-} f'(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{4\left[(0-h)-1\right]}{4\left[(0-h)^{2/3}\right]}$$

$$= \lim_{h \to 0} \frac{4\left[(-h-1)\right]}{3.h^{2/3}} = -\infty$$

and $\lim_{x \to a^+} f'(x) = \lim_{h \to 0} f'(0+h)$

$$= \lim_{h \to 0} \frac{4\left[(0+h)-1\right]}{3.[0+h]^{2/3}}$$

$$= \lim_{h\to 0} \frac{4\left[h-1\right]}{3h^{2/3}} = -\infty$$

since the limit approaches -∞ as x→0 from both sides, thus a vertical tangent occurs at the origin (0.0).

Value Addition: Note
The graph of a function f(x) has a vertical tangent at the point if the graph turns smoothly through the point and a cusp at that point if the graph changes direction abruptly there i.e. f(x) is a continuous function at a point x = a and that $f'(x)$ become infinity as x→a.

Exercise:

1. For the following functions,

   a. Find all critical numbers.

   b. Find where the function is increasing and decreasing.

   c. Find the critical points and identify each as a relative maximum, relative minimum, or neither.

   d. Find second–order critical numbers and tell where the graph is concave up and where it is concave down.

(i) $f(x) = x + \frac{1}{x}$ (ii) $f(x) = x\ln x$

2. For the following functions, determine the intervals of increase and decrease and concavity, and then use those intervals to sketch there graph.

(i) $f(t) = t^2 e^{-3t}$ (ii) $f(x) = \sqrt{x^2+1}$ (iii) $f(t) = (\ln x)^2$

(iv) $f(x) = \frac{x}{x^2+1}$ (v) $g(t) = (t^3+t)^2$ (vi) $f(x) = e^x + e^{-x}$

(vii) $f(x) = (x-12)^4 - 2(x-12)^3$ (viii) $f(x) = 2x - \sin^{-1} x \quad \text{for} \quad -1 \le x \le 1$

3. For the following functions, use the first–derivative test to find a relative minimum, a relative maximum, or neither.

(i) $f(x) = \left(x^3 - 3x + 1\right) \; at \;\; x = 1, \;\; x = -1$

(ii) $f(x) = \left(x^2 - 4\right)^4 \left(x^2 - 1\right)^3 \; at \;\; x = 1, \;\; x = 2$

(iii) $f(x) = \sqrt[3]{x^3 - 48x} \; at \;\; x = 4$

(iv) $f(x) = \dfrac{e^{-x^2}}{3 - 2x} \; at \;\; x = 1, \;\; x = \dfrac{1}{2}$

4. For the following functions, use the second–derivative test to find a relative minimum, a relative maximum, or neither.

(i) $f(x) = \dfrac{x^2 - x + 5}{x + 4}$

(ii) $f(x) = \sin x + \dfrac{1}{2}\cos 2x \;\; at \;\; x = \dfrac{x}{6}, \;\; x = \dfrac{\pi}{2}$

(iii) $f(x) = \left(x^3 - 3x + 1\right)e^{-x} \; at \;\; x = 1, \;\; x = 4$

5. Evaluate the following limits

(i) $\lim\limits_{x \to 0^+} \dfrac{x^2 - x + 1}{x - \sin x}$ (ii) $\lim\limits_{x \to 0^+} \dfrac{x^2}{1 - \cos x}$ (iii) $\lim\limits_{x \to (x/4)^+} \dfrac{sex\, x}{\tan x - 1}$

(iv) $\lim\limits_{x \to 0^+} \dfrac{\ln \sqrt[3]{x}}{\sin x}$ (v) $\lim\limits_{x \to +\infty} \left(x \sin \dfrac{1}{x}\right)$ (vi) $\lim\limits_{x \to +\infty} \dfrac{\tan^{-1} x}{e^{0.1x}}$

6. For the following functions, find all vertical and horizontal asymptotes of the graph of each function. Find where the function is increasing and where it is decreasing, determine concavity, and determine all critical points and point of inflection. Finally, sketch the graph.

(i) $f(x) = \left(x^2 - 9\right)^2$ (ii) $f(x) = \dfrac{3x + 5}{7 - x}$ (iii) $f(x) = \dfrac{x^3 + 1}{x^3 - 8}$

(iv) $f(x) = 4 + \dfrac{2x}{x - 3}$ (v) $f(x) = \ln\left(4 - x^2\right)$ (vi) $f(\theta) = \tan^{-1}\theta - \tan^{-1}\dfrac{\theta}{3}$

7. Sketch the graph of each of the following functions:

(i) $f(x) = \dfrac{x^2 - 4x - 5}{2x^2 + x - 10}$ (ii) $f(x) = \dfrac{3x^2 - x - 7}{-12x^2 + 4x + 8}$

Summary:

In this chapter, we have emphasized on the following topics

- increasing, decreasing and monotonic functions.
- critical points
- maxima and minima of function
- concavity of the function
- points of inflection
- limits to infinity
- vertical and horizontal asymptotes
- tangents and cusps
- drawing of the graph of functions using derivatives.

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