Paper: Numerical Methods
Lesson: Solutions of Transcendental and Polynomial Equations
Course Developer : Parvinder Kaur
Department / College: Department of Mathematics,
Motilal Nehru College
University of Delhi

Table of contents

- References

1. Learning Outcomes

After reading this chapter, students will be able to understand,

→ Transcendental functions

→ Importance of initial approximation

→ Principle of iteration

→ Importance of Intermediate value theorem

→ Bisection Method

→ Secant Method

→ Newton-Raphson Method

→ Regula falsi Method

→ Rate of convergence of Iterative Methods and their importance

2. Transcendental and polynomial equations

2.1 Introduction

The practitioners of Applied Mathematics and Engineering are often required to determine the roots of an equation of the form

$$f(x)=0 \qquad (1)$$

If the equation is, quadratic, cubic or biquadratic, we have formulae and methods for getting roots in terms of the coefficients. But when $f(x)$ is either polynomial of higher degree or an expression having transcendental functions like exponential, logarithmic, trigonometric functions, algebraic methods are not available and we have to use numerical methods and be satisfied with approximate values of the roots of the equation.

We describe here some numerical methods for solving equations of the form (1) where $f(x)$ is either an algebraic function of higher degree or involves transcendental functions.

2.2 Iterative Methods

These methods are based on the idea of successive approximation i.e. starting with one or more initial approximations to the root that gives us a sequence of approximations or iterates $\{x_k\}$ which in the limit converges to the root. These methods may give only one root at a time.

Definition : A sequence of iterates $\{x_k\}$ is said to converge to the root ξ, if

$$\lim_{k\to\infty}|x_k - \xi| = 0$$

In practice, except in rare cases, it is not possible to find ξ which satisfy the given equation exactly. We, therefore, attempt to find an approximate root ξ^* such that either $|f(\xi^*)| < \in$ or $|x_{k+1} - x_k| < \in$ where x_k and x_{k+1} are two consecutives iterates and $\in$ is the prescribed *error tolerance*.

I.Q.1

Value addition:

i) An algebraic equation of degree *n* is of the type

$$P(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

Where the coefficients $a_0, a_1, \ldots a_n$ are real numbers and $a_0 \neq 0$. Here $n \geq 1$. Transcendental equations are non-algebraic equations involving transcendental functions like exponential, logarithmic,trigonometric function etc.

ii) Geometrically, a root of the equation (1) is the value of *x* at which the graph of *y* = *f* (*x*) intersects the x-axis.

I.Q.2

2.3 Initial Approximations

Numerical methods are based on the idea of initial approximations to the root. We can obtain initial approximations to the roots from the physical considerations of the given problem; else graphical methods are generally used to obtain initial approximations to the root. Since the value of *x*, at which the graph of the equation $y = f(x)$ intersects the x-axis, gives the root of $f(x) = 0$, any value in the neighbourhood of this point may be taken as an initial approximation to the root. If the equation $f(x) = 0$ can be conveniently written in the form $f_1(x) = f_2(x)$, then the point of intersection of the graphs of the equation $y = f_1(x)$ and $y = f_2(x)$ gives the root of $f(x) = 0$ and therefore any value in the neighbourhood of this point can be taken as an initial approximation to the root.

Example 1: Obtain an initial approximation to a root of the equation

$$f(x) = \cos x - x\,e^{x} = 0$$

Solution : We can write the above equation as

$$\cos x\, e^{-x} = x$$

The point of intersection of the graphs $f_1(x) = \cos x\, e^{-x}$ and $f_2(x) = x$ is shown in Figure 1.

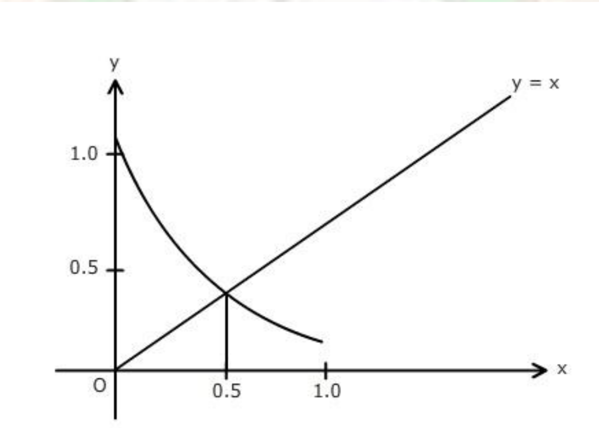

Figure 1.

x	0	0.5	1	1.5	2
f (*x*)	1	0.0532	−2.1780	−6.6518	−15.1972

From the graph we find that the equation $f(x)=0$ has atleast one root in the interval (0.5, 1). The exact root correct to ten decimal places is 0.5177573637.

I.Q.3

Another commonly used method to obtain the initial approximation to the root is based upon the *Intermediate value theorem*, which states:

Theorem: If $f(x)$ is a continuous function on some interval $[a,b]$ and $f(a)f(b)<0$, then the equation $f(x)=0$ has at least one real root or an odd number of roots in the interval (a,b).

3. Numerical Methods

3.1 Bisection Method (or Bolzano Method)

This method is based on the repeated application of the intermediate value theorem. If we know that a root of $f(x)=0$ lies in the interval $I_0=(a_0,b_0)$, we bisect I_0 at the point $m_1=\frac{1}{2}(a_0+b_0)$. Denote by I_1 the interval (a_0, m_1) if $f(a_0)f(m_1)<0$ or the interval (m_1,b_0) if $f(m_1)f(b_0)<0$. Therefore the interval I_1 also contains the root. We bisect the interval I_1 and get a subinterval I_2 at whose end points $f(x)$ takes the values of opposite signs and therefore by intermediate value theorem it contains the root. Continuing this procedure, we obtain a sequence of nested intervals $I_0 \supset I_1 \supset I_2$ … such that each subinterval contains the root. After repeating the Bisection process *q* times, we either find the root or find the interval I_q of length $\frac{(b_0-a_0)}{2^q}$ (because the interval is reduced by one half at each step) which contains the root. We take the mid-point of the last subinterval as the desired approximation to the root. Which is given by, we have,

$$b_k - a_k = \frac{1}{2^k}(b_0 - a_0), \quad k = 0,1,2...$$

So, $m_{k+1} = \frac{1}{2}(a_k + b_k), \quad k = 0,1,2,...$

where

$$(a_{k+1}, b_{k+1}) = \begin{cases} (a_k, m_{k+1}) & \text{if } f(a_k) f(m_{k+1}) < 0 \\ (m_{k+1}, b_k) & \text{if } f(m_{k+1}) f(b_k) < 0 \end{cases}$$

Note 1: This method uses only the end points of the interval $[a_k, b_k]$ for which $f(a_k) f(b_k) < 0$ and not the values of $f(x)$ at these end points, to obtain the next approximation to the root.

Note 2: In this method the sequence of approximations always converge to the root for any $f(x)$ which is continuous in the interval that contains the root. If the permissible error is $\in$, then the approximate number of iterations required can be determined as we have,

$$\frac{(b_0 - a_0)}{2^n} \le \in \qquad \text{or} \qquad 2^n \ge \frac{(b_0 - a_0)}{\in}$$

$$\Rightarrow n \ge \frac{\log(b_0 - a_0 / \in)}{\log 2}$$

or $$n \ge \frac{\log(b_0 - a_0) - \log \in}{\log 2}$$

Which gives us the number of iterations required to achieve the desired accuracy for given $\in$.

Example 2: Find the minimum number of iterations required for converging to a root in the interval (0, 1) for a given $\in = 10^{-2}$.

Solution : we have $n \ge \frac{\log(b_0 - a_0) - \log \in}{\log 2}$

$$\Rightarrow n \geq \frac{\log(1-0) - \log 10^{-2}}{\log 2}$$

$$\Rightarrow n \geq \frac{\log 10^{2}}{\log 2}$$

$$\Rightarrow n \geq \log_2 10^2$$

$$\Rightarrow n \geq \frac{2}{\log_{10} 2} = 6.64$$

$$\Rightarrow n = 7.$$

- The minimum number of iterations required for converging to a root in the interval (0, 1) for a given $\in$ are listed in table 1.

Table 1 Number of Iterations

$\in$	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	10	14	17	20	24

Value Addition:

i) **Nested Intervals:** In mathematics, a sequence of nested intervals is understood as a collection of real numbers such that each set I_n in an intervals of the real line, for n = 1, 2, 3, ... and that further for all n I_{n+1} is a subset of I_n.

ii) The bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the root. It requires one function evaluation for each iteration.

Example 3: Perform five iterations of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

Solution: Since $f(0) > 0$ and $f(1) < 0$, the smallest positive root lies in the interval (0, 1). Taking $a_0 = 0$ and $b_0 = 1$, we get

$$m_1 = \frac{1}{2}(a_0 + b_0)$$

$$= \frac{1}{2}(0+1) = 0.5$$

$f(m_1) = f(0.5) = -1.375$ and $f(a_0)f(m_1) < 0$.

Thus the root lies in the interal (0, 0.5). The sequence of intervals are given in table 2.

Table 2 Sequence of Intervals for the Bisection method

k	a_{k-1}	b_{k-1}	m_k	$f(m_k)f(a_{k-1})$
1	0	1	0.5	< 0
2	0	0.5	0.25	< 0
3	0	0.25	0.125	> 0
4	0.125	0.25	0.1875	> 0
5	0.1875	0.25	0.21875	< 0

Hence the root lies in (0.1875, 0.21875). The approximate root is taken as 0.203125.

Example 4: Find the root of the equation $x^3 - 4x + 1 = 0$, lying in (0, 1). Using Bisection method. Performing 10 iterations.

Solution: Here $f(x) = x^3 - 4x + 1$

then $f(0) > 0$ and $f(1) < 0$

hence at least one root lies in (0, 1). We can take $m_1 = \frac{0+1}{2} = 0.5$

Next $f(0.5) < 0$.

Hence root lies in (0, 0.5). Here we take $m_2 = \frac{0+0.5}{2} = 0.25$,

Next $f(0.25) > 0$, thus root lies in (0.25, 0.5).

Continuing like this, we get a sequence of intervals as listed:

k	a_{k-1}	b_k	m_k	new interval	$f(m_k)$
1	0	1	0.5	(0, 0.5)	< 0
2	0	0.5	0.25	(0.25, 0.5)	> 0
3	0.25	0.5	0.35	(0.25, 0.35)	< 0
4	0.25	0.35	0.30	(0.25, 0.30)	< 0
5	0.25	0.30	0.275	(0.25, 0.275)	< 0
6	0.25	0.275	0.2625	(0.25, 0.2625)	< 0
7	0.25	0.2625	0.25625	(0.25, 0.25625)	< 0
8	0.25	0.25625	0.253125	(0.253125, 0.25625)	> 0
9	0.253125	0.25625	0.2546875	(0.253125, 0.2546875)	< 0
10	0.253125	0.2546875	0.25390625	(0.25390625, 0.2546875)	> 0

The approximate root is $\xi = \dfrac{0.25390625 + 0.2546875}{2}$

= 0.254296875 ≈ 0.2540.

3.2 Regula falsi Method (or Method of false position)

It is an improvement over the bisection method. In this method we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs i.e., $f(x_0)\, f(x_1) < 0$. Since the graph of $y = f(x)$ crosses the x-axis between these two points, at least one real root must lie between these points.

The equation of the Chord Joining the points $A[x_0,\, f(x_0)]$ and $B[x_1,\, f(x_1)]$ is

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad (2)$$

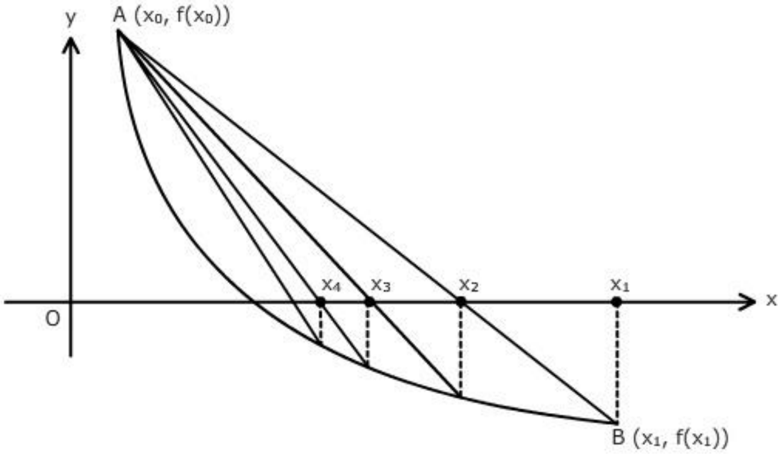

Figure 2

Suppose $f\left(x_0\right)>0$ and $f\left(x_1\right)<0$ (see Fig. 2 above)

In this method we replace the curve AB by the chord AB joining the points A and B and take the point of intersection of the chord with the x-axis as an approximation to the root. In the present case the point of intersection is obtained by setting *y* = 0 in equation (2) which gives

$$x = x_0 - \frac{\left(x_1 - x_0\right) f\left(x_0\right)}{f\left(x_1\right) - f\left(x_0\right)}$$

Let us call it x_2 and we write

$$x_2 = x_0 - \frac{\left(x_1 - x_0\right) f\left(x_0\right)}{f\left(x_1\right) - f\left(x_0\right)} = \frac{x_0\, f\left(x_1\right) - x_1\, f\left(x_0\right)}{f\left(x_1\right) - f\left(x_0\right)}$$

Now, if $f\left(x_2\right)$ and $f\left(x_0\right)$ are of opposite signs then the root lies between x_0 and x_2. Continuing in the same manner we get the next approximation and call it x_3. The procedure is repeated till the root is obtained to the desired accuracy.

In general, we get,

$$x_{k+1} = \frac{x_{k-1}\, f\left(x_k\right) - x_k\, f\left(x_{k-1}\right)}{f\left(x_k\right) - f\left(x_{k-1}\right)}$$

Which may also be written as

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1}) f(x_k)}{(f(x_k) - f(x_{k-1}))}.$$

Value Addition:
In the Regula falsi method above if we violate the condition $f(x_0) f(x_1)$ < 0 then the above method will be called as **Secant Method** (or **Chord Method**)

I.Q.4

Example 5: Using Regula falsi method compute the real root of the equation $xe^x = 2$. Correct to four decimal places.

Solution: Here $f(x) = xe^x - 2$ and $f(0) = -2,\ f(0.5) = -1.175$

$f(0.8) = -0.2196,\ f(0.9) = 0.2136$ and $f(1.0) = 0.715$

therefore root lies between 0.8 and 0.9

we take $x_0 = 0.8$ and $x_1 = 0.9$

therefore $x_2 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$

$$x_2 = 0.8 + \frac{0.2196(0.9 - 0.8)}{0.2136 + 6.2196} = 0.851$$

$$\therefore \quad f(x_2) = -0.00697$$

thus $x_3 = 0.851 + \frac{0.00697}{0.2136 + 0.00697}(0.9 - 0.851)$

$$x_3 = 0.851 + 0.0015484 = 0.85256$$

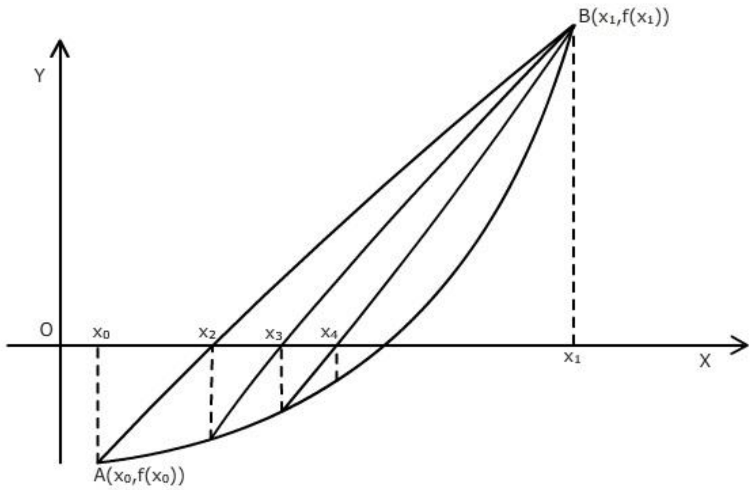

Figure 3

Now $f\left(x_3\right)=-0.0001977$

Thus $x_4=0.85256+\dfrac{0.0001977}{0.2136+0.0001977}\left(0.9-0.85256\right)$

$x_4=0.85256+0.000043868=0.8526$

Again, $f\left(x_4\right)=-0.0000239$

$x_5=0.8526+\dfrac{0.0000239}{0.2136+0.0000239}\left(0.9-0.8526\right)$

$x_5=0.8526+0.0000053$

$=0.8526$

$\therefore$ Approximate root is 0.8526

3.3 Newton-Raphson Method (or Method of Tangents)

Newton's method gives a better approximation of a root as compared to the approximations obtained by bisection method or Regula falsi method. This method consists of replacing the part of the curve between point $\left[x_0,\ f\left(x_0\right)\right]$ and the x-axis by means of the tangent to the curve at the point and is described graphically in the adjoining Figure 4. The intercept

OT on the x-axis, of the tangent to the curve at the point P is taken as the first approximation.

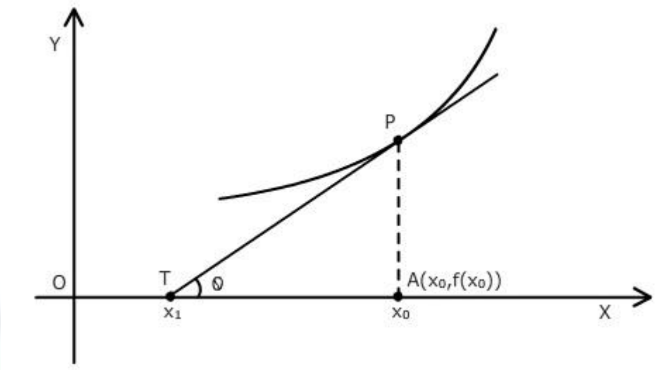

Figure 4

From the given figure 4 we have,

$$\tan\theta = \frac{f(x_0)}{x_0 - x_1}, \qquad \text{but} \qquad \tan\theta = \frac{dy}{dx} = f'(x_0)$$

This gives, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

Repeating the process replacing x_0 by x_1, we get the second approximation as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{and so on.}$$

In general, after $(n+1)$ iterations, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Obviously this method fails if the slope of the tangent to the curve becomes zero.

As an alternative approach

Draw a tangent to the curve at B_0 which meets x-axis at x_1. Then draw a tangent at B_1 which meets x-axis at x_2. Continuing this process, the root ξ is obtained as shown in Fig. 5.

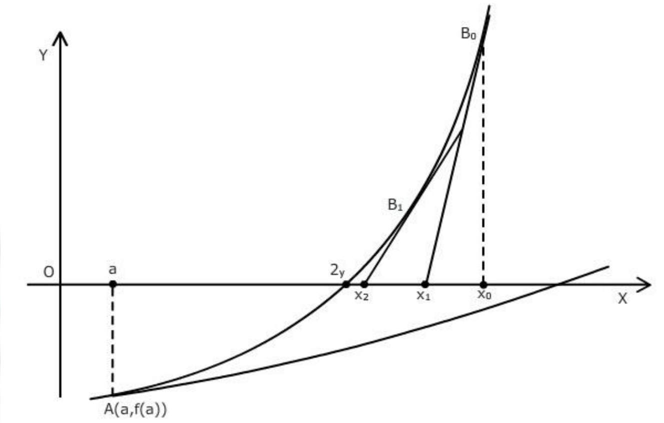

Figure 5

Suppose $\xi = x + h$ where h is a small quantity. Then applying Taylor's formula, we have

$$0 = f(x+h) \approx f(x) + h\, f'(x)$$

or $$h = \frac{-f(x)}{f'(x)}$$

Thus $\xi = x + h = x - \dfrac{f(x)}{f'(x)}$.

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Example 6: Find the root of the equation

$\cos x - x e^x = 0$ using

(i) Regula falsi Method.

(ii) Newton Raphson Method.

Solution: $f(x) = \cos x - xe^x$

Here $f(0) = 1$ and $f(1) = \cos 1 - e = -2.17798$

(i) We take $x_0 = 0$ and $x_1 = 1$. By Regula falsi method

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} f(x_1)$$

$$= 1 - \frac{(1-0)}{(-2.17798 - 1)}(-2.17798)$$

= 0.314665

$f(x_2) = 0.51987$

$$\therefore \quad x_3 = x_2 - \frac{(x_2 - x_1)}{(f(x_2) - f(x_1))} f(x_2)$$

$$= 0.314665 - \frac{(0.314665 - 1)}{0.51987}(0.51987)$$

= 0.446728

Continuing the process, we get $x_4 = 0.491015$, $x_5 = 0.5099461$, $x_6 = 0.5152$.

(ii) Let the initial value of the root be $x_0 = 1$

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

or $$x_{n+1} = x_n - \frac{\cos x_n - x_n e^{x_n}}{-\sin x_n - e^{x_n} - x_n e^{x_n}}$$

$$= x_n + \frac{\cos x_n - x_n e^{x_n}}{\sin x_n + (1 + x_n) e^{x_n}}$$

$$\therefore \qquad x_1 = x_0 + \frac{\cos x_0 - x_0 e^{x_0}}{\sin x_0 + (1 + x_0) e^{x_0}}$$

$$= 1 + \frac{\cos 1 - e}{\sin 1 + 2e} = 0.65308$$

$$f(x_1) = -0.4606$$

$$x_2 = x_1 + \frac{\cos x_1 - x_1 e^{x_1}}{\sin x_1 + (1 + x_1) e^{x_1}} = 0.531343$$

Continuing the process, we get

$$x_3 = 0.51791, \; x_4 = 0.51776$$

Example 7: Using Newton-Raphson Method compute $\sqrt{5}$.

Solution: $\sqrt{5}$ will be calculated as the root of the equation $x^2 - 5 = 0$.

So that $f(x) = x^2 - 5$ and $f'(x) = 2x$.

The starting value of the root is obviously 2 hence we take $x_0 = 2$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n}$$

$$= x_n - \frac{1}{2} x_n + \frac{5}{2x_n}$$

$$= \frac{x_n}{2} + \frac{5}{2x_n} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$$

$\therefore$ for $n = 0, \; x_1 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$

$$n=1, x_2=\frac{1}{2}\left(2.25+\frac{5}{2.25}\right)=2.236111$$

$$n=2,\ x_3=\frac{1}{2}\left(2.236111+\frac{5}{2.236111}\right)=2.2360679$$

$$n=3,\ x_4=\frac{1}{2}\left(2.2360679+\frac{5}{2.2360679}\right)=2.23606797$$

Thus, the value of the root is 2.23606797 correct to nine significant digits.

I.Q.5

4. Rate of Convergence

In numerical analysis, the speed at which a convergent sequence approaches its limit is called the **rate of convergence**. We now study the rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root.

4.1 Order of a Convergence

Order of a root

- A root of order *m* = 1 is called a simple root.
- A root of order *m* > 1 is called a multiple root.
- A root of order *m* = 2 is sometimes called as double root and so on.

Definition order of convergence : Let the sequence $\{x_n\}$ converges to ξ. Denote the difference between x_n and ξ by ϵ_n,

i.e. $\epsilon_n = x_n - \xi$. If there exists a positive number $p \geq 1$ and a constant $c \neq 0$ such that

$$\lim_{n\to\infty}\frac{|x_{n+1}-\xi|}{|x_n-\xi|^p}=\lim_{n\to\infty}\frac{|\epsilon_{n+1}|}{|\epsilon_n|^p}=c$$

then p is called the order of convergence of the sequence. The constant c is called the *Asymptotic error constant.*

- If p is large, then the sequence $\{x_n\}$ converges rapidly to ξ.
- If $p = 1$ and $c < 1$, then the convergence is said to be linear and c is called the rate of convergence.
- If $p = 2$, then it is quadratic.

Example of Convergence

$$\left\{\frac{11}{2}, \frac{21}{4}, \frac{41}{8}, \dots, 5+\frac{1}{2^n}, \dots\right\} \quad \text{and} \quad \left\{\frac{11}{2}, \frac{41}{8}, \frac{641}{128}, \dots, 5+\frac{1}{2^{2^n-1}}, \dots\right\}$$

Both sequences converges to 5. However it seems that the second sequence converges faster to 5 than the first one.

For the first sequence, we have

$$\epsilon_n = x_n - \xi = \frac{1}{2^n} \quad \text{and} \quad \lim_{n\to\infty}\frac{|\epsilon_{n+1}|}{|\epsilon_n|} = \lim_{n\to\infty}\frac{2^n}{2^{n+1}} = \frac{1}{2},$$

First sequence converges linearly $(p = 1)$ to 5.

For the second sequence, we have

$$\epsilon_n = x_n - \xi = \frac{1}{2^{2^n-1}} \quad \text{and} \quad \lim_{n\to\infty}\frac{|\epsilon_{n+1}|}{|\epsilon_n|} = \lim_{n\to\infty}\frac{1/2^{2^{n+1}-1}}{1/\left(2^{2^n-1}\right)^2} = \frac{1}{2}$$

Second sequence converges quadratically $(p = 2)$ to 5.

Value Addition:

The concept of rate of convergence is of practical importance if we deal with a sequence of successive approximations for an iterative method, as than typically fewer iterations are needed to yield a useful approximation if the rate of convergence is higher. This may even make the difference between needing ten or a million iterations.

I.Q.6

4.2 Rate of Convergence of Secant Method

We assume that ξ is a simple root of $f(x)=0$.

Let us define the error ϵ_k as

$$\epsilon_k = x_k - \xi$$

The secant method reads,

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})\, f(x_k)}{f(x_k) - f(x_{k-1})} \qquad (3)$$

To figure out the convergence order, we have to find a relation between ϵ_{k+1} and ϵ_k.

Using Taylor's theorem, we have

$$f(x_k) = f\left(\xi + (x_k - \xi)\right) = f(\xi + \epsilon_k)$$

$$= f(\xi) + f'(\xi)\epsilon_k + \frac{1}{2} f''(\xi)\epsilon_k^2 + 0\left(\epsilon_k^3\right).$$

Similarly we can write $f(x_{k-1})$ as

$$f(x_{k-1}) = f\left(\xi + (x_{k-1} - \xi)\right) = f(\xi + \epsilon_{k-1})$$

$$= f(\xi) + f'(\xi)\varepsilon_{k-1} + \frac{1}{2} f''(\xi)\varepsilon_{k-1}^2 + 0\left(\varepsilon_{k-1}^3\right).$$

Furthermore, we have

$$x_k - x_{k-1} = \left(x_k - \xi\right) - \left(x_{k-1} - \xi\right) = \varepsilon_k - \varepsilon_{k-1}.$$

Subtracting ξ from both sides of equation (3) and keeping in mind that by definition we have $f(\xi) = 0$, gives then

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\left(f'(\xi)\varepsilon_k + \frac{1}{2} f''(\xi)\varepsilon_k^2\right)\left(\varepsilon_k - \varepsilon_{k-1}\right)}{f'(\xi)\left(\varepsilon_k - \varepsilon_{k-1}\right) + \frac{1}{2} f''(\xi)\left(\varepsilon_k^2 - \varepsilon_{k-1}^2\right)},$$

Which can be rewritten using,

$$\left(\varepsilon_k^2 - \varepsilon_{k-1}^2\right) = \left(\varepsilon_k - \varepsilon_{k-1}\right)\left(\varepsilon_k + \varepsilon_{k-1}\right) \quad \text{as}$$

$$\varepsilon_{k+1} = \varepsilon_k - \frac{f'(\xi)\varepsilon_k + \frac{1}{2} f''(\xi)\varepsilon_k^2}{f'(\xi) + \frac{1}{2} f''(\xi)\left(\varepsilon_k + \varepsilon_{k-1}\right)},$$

or $$\varepsilon_{k+1} = \frac{\frac{1}{2} f'(\xi)'\varepsilon_k\,\varepsilon_{k-1}}{f'(\xi) + \frac{1}{2} f''(\xi)\left(\varepsilon_k + \varepsilon_{k-1}\right)}$$

or $$\varepsilon_{k+1} = \frac{f''(\xi)}{2f'(\xi)}\varepsilon_k\,\varepsilon_{k-1} + 0\left(\varepsilon_k^{\,3}\right).$$

The relation $\varepsilon_{k+1} = \frac{f''(\xi)}{2f'(\xi)}\varepsilon_k\,\varepsilon_{k-1} + 0\left(\varepsilon_k^{\,3}\right)$ is of the form $\varepsilon_{k+1} = c\,\varepsilon_k\,\varepsilon_{k-1}$ (4)

where $c = \frac{f''(\xi)}{2f'(\xi)}$, and higher powers of ε_k are neglected.

The relation (4) is called as *Error equation*. Keeping in view the definition of the rate of convergence, we seek a relation of the form

$$\epsilon_{k+1} = A\,\epsilon_k^p \qquad (5)$$

Where A and p are to be determined.

From (5), we have

$$\epsilon_k = A\,\epsilon_{k-1}^p \quad \text{or} \quad \epsilon_{k-1} = A^{-1/p}\,\epsilon_k^{1/p}$$

Substituting the values of ϵ_{k+1} and ϵ_{k-1} in eq. (4), we get

$$\epsilon_k^p = c\,A^{-(1+1/p)}\epsilon_k^{(1+1/p)} \qquad (6)$$

Comparing the powers of ϵ_k on both sides, we get

$$p = 1 + \frac{1}{p}$$

Which gives $p = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$

Neglecting the minus sign, we find that the rate of convergence for the secant method is p = 1.618. From (6), we also obtain $A = c^{p/(p+1)}$.

4.3 Rate of convergence of Newton-Raphson Method

The Newton-Raphson formula is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad (7)$$

Let ξ be a root of $f(x) = 0$ also,

Let us define the error at the k^{th} step to be

$\epsilon_k = x_k - \xi$. We assume f'' is continuous near ξ and use a Taylor approximation about x_k, we have

$$0 = f(\xi) = f(x_k - \epsilon_k) = f(x_k) - \epsilon_k f'(x_k) + \epsilon_k^2 f''(x_k)/2 + 0(\epsilon_k^3)$$

If $f'(x_k) \neq 0$, we may write

$$\frac{-f(x_k)}{f'(x_k)} = -\epsilon_k + \frac{\epsilon_k^2 f''(x_k)}{2f'(x_k)} + 0(\epsilon_k^3) \qquad (8)$$

Then $\epsilon_{k+1} = x_{k+1} - \xi = \left(x_k - \dfrac{f(x_k)}{f'(x_k)} \right) - \xi$ using equation (7)

$$\epsilon_{k+1} = x_k - \epsilon_k + \epsilon_k^2 \frac{f''(x_k)}{2f'(x_k)} + 0(\epsilon_k^3) + \epsilon_k - x_k \qquad \text{using equation (8)}$$

$$\epsilon_{k+1} = \epsilon_k^2 \frac{f''(x_k)}{2f'(x_k)} + 0(\epsilon_k^3) \qquad (9)$$

We can write (9) as

$$\epsilon_{k+1} = C\,\epsilon_k^2, \text{ where } C = \frac{f''(\xi)}{2f'(\xi)} \quad \text{as} \quad \begin{matrix} k \to \infty \\ x_k \to \xi \end{matrix}$$

and neglecting higher powers of ϵ_k.

Thus the Newton-Raphson method has second order convergence or quadratic convergence.

Example 8: Show that the initial approximation x_0 for estimating the value of $\frac{1}{N}$ where N is a positive integer, by Newton-Raphson method, must lie in $\left(0, \frac{2}{N}\right)$ for convergence.

Solution: Let us write $f(x)=\frac{1}{x}-N=0$

So that $f'(x)=\frac{-1}{x^2}$

By Newton-Raphson formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\left(\frac{1}{x_n}-N\right)}{\frac{-1}{x_n^2}}=2x_n-Nx_n^2.$$

Now we draw the graph of two curves *y* = *x* and $y=2x-Nx^2$. The second curve is the parabola.

$$\left(x-\frac{1}{N}\right)^2=\frac{-1}{N}\left(y-\frac{1}{N}\right)$$

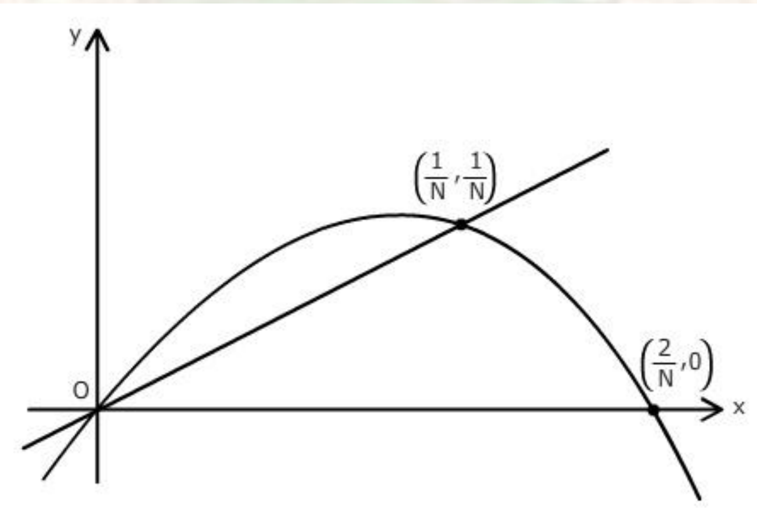

Figure 6

The point of intersection of these two curves is the required value $\frac{1}{N}$. From the Fig. 6, we find that any initial approximation outside $\left(0,\frac{2}{N}\right)$ will diverge. If $x_0=0$, the iteration does not converge to $\frac{1}{N}$ but remains zero always. This clearly tells the importance of choosing suitable initial approximation.

5. Choice of an iterative method and Implementation

- The Bisection method is slow but it never fails. If the evaluation of $f(x)$ is rapid, then the use of the bisection method is strongly advised. It is better to find the bisection point using the formula $x_{k+1} = x_{k-1} + \frac{1}{2}(x_k - x_{k-1})$ rather than using the conventional formula

  $x_{k+1} = \frac{1}{2}(x_k + x_{k-1})$.

- If the function $f(x)$ is smooth enough to have continuous derivatives of first and second order which can be evaluated easily, then the Newton-Raphson method considerably reduce the number of iterations required for attaining the prescribed accuracy.

- The secant method also requires lesser iteration than the bisection method. In the secant method we always use x_i and x_{i-1} to generate x_{i+1}. Often secant method given better results than the Regula falsi method. The only disadvantage of the secant method is that it may not converge sometimes. If the evaluation of $f'(x)$ is difficult, then the secant method is a better choice than the Newton-Raphson method. The difficulty with the Newton-Raphson method is in finding a suitable choice of the initial approximation x_0. If this choice is poor, it is possible that the root may never be obtained. Hence, frequently, some convergent method like bisection is used for the first few iterations before using the rapidly convergent Newton-Raphson method. If the evaluation of $f'(x)$ is difficult, then the use of the secant method, preceded by a few iterations by the bisection method is often recommended.

I.Q.7

I.Q.8

I.Q.9

- The bisection method cannot be applied for locating the complex zeros of analytic functions. The secant and Newton-Raphson methods can be used in the complex plane without any change. A good initial approximation to the complex root is necessary for application of these methods. A real initial approximation cannot produce a complex root.

I.Q.10

Value Addition:

Analytic Function: A complex function $f(z)$ is said to be an analytic function at the point z_0 if it is not only differentiable at the point z_0 but also differentiable at every point in some neighbourhood of z_0.

Exercise

Q1. Using four iterations obtain a root of the equation $x^3 - 4x - 9 = 0$ employing Bisection method.

Q2. Apply (a) Newton-Raphson Method

(b) Method of false position to obtain a root of the equation $x^3 - 3x - 5 = 0$.

Q3. Using Newton-Raphson Method obtain the real root of the equation.

(a) $x \sin x + \cos x = 0$

(b) $3x - \cos x - 1 = 0$

Q4. Use Regula falsi Method to find the root of the equation $2x - \log_{10} x = 7$. Lying between 3.5 and 4 correct to three decimal places.

Q5. Find the approximation value of $\sqrt{28}$ correct to four decimal places using Newton-Raphson Method.

Q6. Use Bisection Method to find the real root of the equation $x^3 - 18 = 0$. Correct to three decimal places.

Q7. Find a real root of the equation $x - e^{-x} = 0$ using Newton-Raphson method. Correct to three palces of decimal.

Q8. Using Regula – falsi method, find the roots of the equation $2x - 3\sin x - 5 = 0$. Correct to three decimal places.

Q9. Determine the order of convergence of the iterative method $x_{k+1} = \left(x_0\, f\left(x_k\right) - x_k f\left(x_0\right)\right) \big/ \left(f\left(x_k\right) - f\left(x_0\right)\right)$, for finding a simple root of the equation $f\left(x\right) = 0$.

Q10. Find a positive root of $x^4 - x = 10$ using Newton – Raphson's Method.

Q11. Find all the roots of $x^3 - 4x + 1 = 0$ using Bisection Method.

Q12. Apply Newton-Raphson Method to evaluate $\sqrt{12}$ approximately.

Q13. Solve the equation $5x - \sin x = 2$, correct to four places of decimal.

Q14. Compute $23^{1/3}$ using Newton's Method correct to four decimal places.

Q15. Using Regula – falsi Method find an approximate root of the equation $x^3 - 0.2x^2 - 0.2x - 1.2 = 0$.

Summary:

Numerical analysis involves the study of methods of computing numerical data. In many problems this implies producing a sequence of approximations. Numerical analysis does not strive for exactness. Instead it provides approximate solutions with specified degree of accuracy.

In this chapter, we consider the Bisection, Secant, Regula Falsi and Newton-Raphson's method of obtaining solutions of transcendental equations and their rate of convergence.

We find that Bisection method is slow but it never fails. If the evaluation of f(x) is rapid, then the use of the Bisection method is strongly advised. If the function f(x) is smooth enough to have continuous derivatives of first and second order which can be evaluated easily then the Newton-Raphson method is a better choice.

If the evaluation of $f'(x)$ is difficult, then the use of the secant method, preceded by a few iterations by the bisection method is often recommended. We note that higher is the rate of convergence of a numerical method fewer is the number of iteration needed. The rate of convergence of Newton-Raphson method is 2 whereas the rate of convergence of secant method is 1.618.

References:

1) M.K. Jain, S.R.K. Iyengar, R.K. Jain Numerical Methods for Scientific and Engineering Computation, Wiley Eastern Ltd., New Delhi.

2) S.S. Sastry : Introductory Methods of Numerical Analysis, P.H.I New Delhi.

3) B.V. Ramana : Higher Engineering Mathematics, Tata McGraw-Hill publishing company ltd.

4) V.P. Mishra & Pratibha Mishra : concepts of Engineering Mathematics.

Suggested Readings:

1) Gerald, Wheatley : Applied Numerical Analysis – Pearson Education, Inc. New Delhi.
2) Dr. M.K. Venkatraman: Numerical Methods in Science and Engineering – NPC – Chennai.
3) R.G. Barttle and D.R. Sherbert, Introduction to Real Analysis, John Wiley and Sons Pvt. Ltd., Singapore-2002.
4) K.G. Binmore, Mathematical Analysis, A Straightforward Approach, Cambridge University Press, 1982.