Discipline Course-I

Semester -I

Paper: Mathematical PhysicsI IA

Lesson: Differential Calculus

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Learning Objectives

After reading this chapter the student will be able to learn

- **how to find the Limiting value of a function?**
- **what we mean by Continuity of a function?**
- **when is a function Differentiable?**
- **limits, continuity and differentiation are used in many fields of physics. Students will find direct application in Newtonian Mechanics, thermodynamics and many other fields.**
- **how to calculate the Average rate of change and the Instantaneous rate of change of a function? We need while reading physics the average and instantaneous values. So knowledge developed on these topics is important here.**
- **the Theorem of Mean Value & Taylor's Theorem of the Mean; here the student will be able to write a function with respect to any reference point and express it in the form of power series.**

Differential Calculus

You may have studied differential calculus in your school. However, we here go through the basics of differential calculus in order to be able to understand the differential equations.

3.1 Limits

Let a be a point on the real line and $\delta > 0$ be a real number. The δ-neighbourhood of the point a, denoted by $N(a)$ is defined as the interval

$$x - \delta < x < x + \delta \;\; or \;\; |x - a| < \delta$$

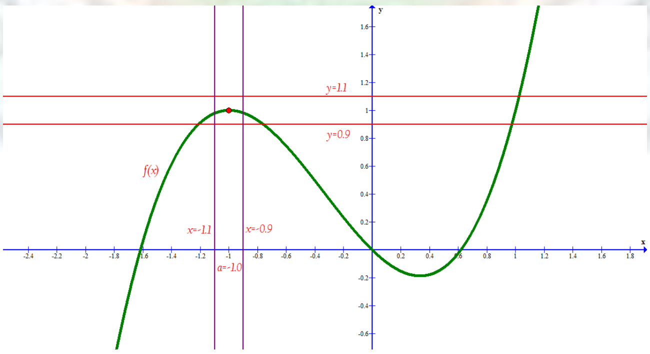
A real valued function $f(x)$ of a real variable x is said to have a limit l if for any pre-assigned arbitrary positive number ϵ, no matter however small, there corresponds a positive number δ such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$ but $x \neq a$. In a compact form it is written as

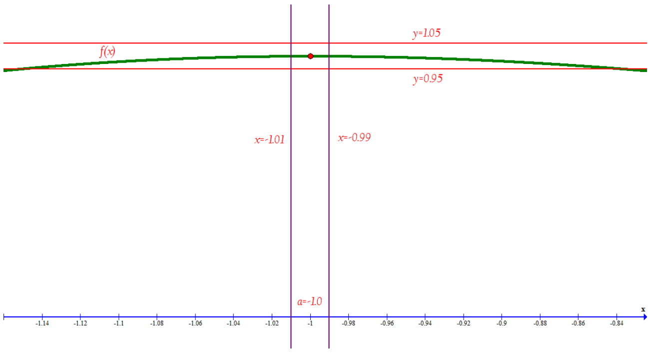
$$\lim_{x \to a} f(x) = l$$

The meaning is that for every neighbourhood $(l - \epsilon, l + \epsilon)$ of l, there exists a neighbourhood $(a - \delta, a + \delta)$ excluding the point $\boldsymbol{a}$ itself such that $f(x)$ is in $(l - \epsilon, l + \epsilon)$ for every x in $(a - \delta, a + \delta)$ excluding $x = a$ itself.

This definition does not require the behaviour of $f(x)$ at $x = a$.

Geometrically, this means that for every x in the two open intervals $a - \delta < x < a$ and $a < x < a + \delta$, the graph of a function $f(x)$ can always be confined to lie between the horizontal lines $y = l - \epsilon$ and $y = l + \epsilon$. The following figures presents two cases wherein limits are checked for a) $\delta = 0.1$ and b) $\delta = 0.01$





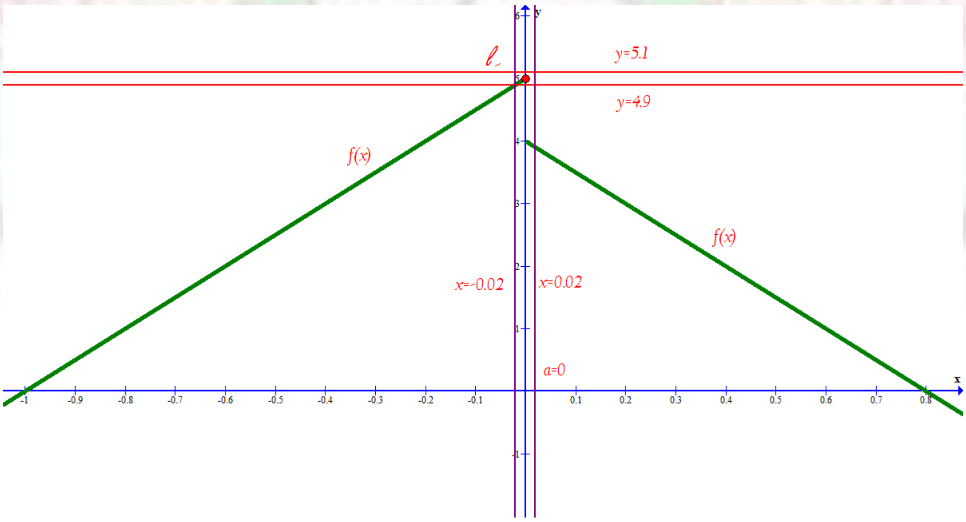
3.1.1 Left Limit

A real valued function $f(x)$ of a real variable x is said to have a limit l_{-} if for any pre-assigned arbitrary positive number ϵ, no matter however small, there corresponds a positive number δ such that $|f(x) - l_{-}| < \epsilon$ whenever $a - \delta < x$ but $x \neq a$. In a compact form it is written as

$$\lim_{x \to a_{-}} f(x) = l_{-}$$

where $x \to a_{-}$ represents the fact that x is approaching the value a from the left of the number line.

Geometrically, this means that for every x in the two open intervals $a - \delta < x < a$ the graph of a function $f(x)$ can always be confined to lie between the horizontal lines $y = l - \epsilon$ and $y = l + \epsilon$.



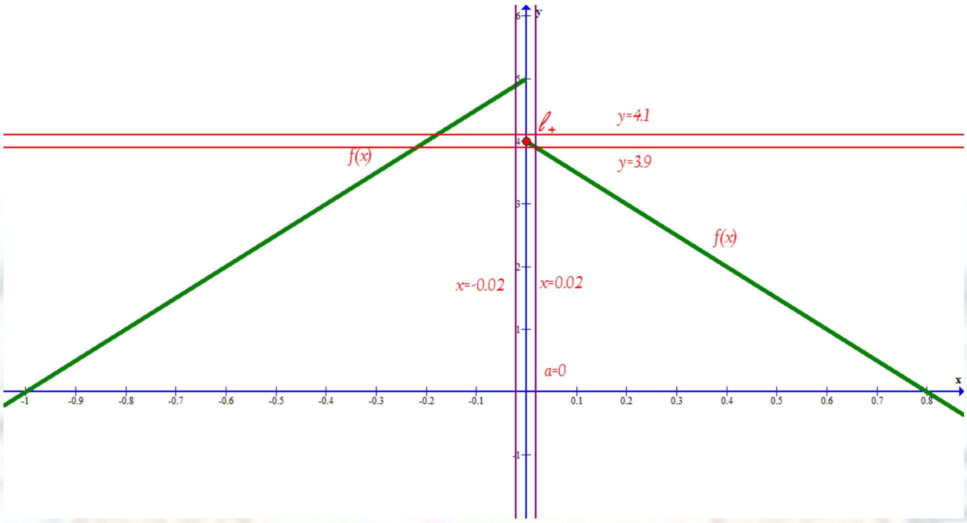
3.1.2 Right Limit

A real valued function $f(x)$ of a real variable x is said to have a limit l_{+} if for any pre-assigned arbitrary positive number ϵ, no matter however small, there corresponds a positive number δ such that $|f(x) - l_{+}| < \epsilon$ whenever $x < a + \delta$ but $x \neq a$. In a compact form it is written as

$$\lim_{x \to a_+} f(x) = l_+$$

where $x \to a_+$ represents the fact that x is approaching the value a from the right of the number line.

Geometrically, this means that for every x in the two open intervals $a < x < a + \delta$ the graph of a function $f(x)$ can always be confined to lie between the horizontal lines $y = l - \epsilon$ and $y = l + \epsilon$.



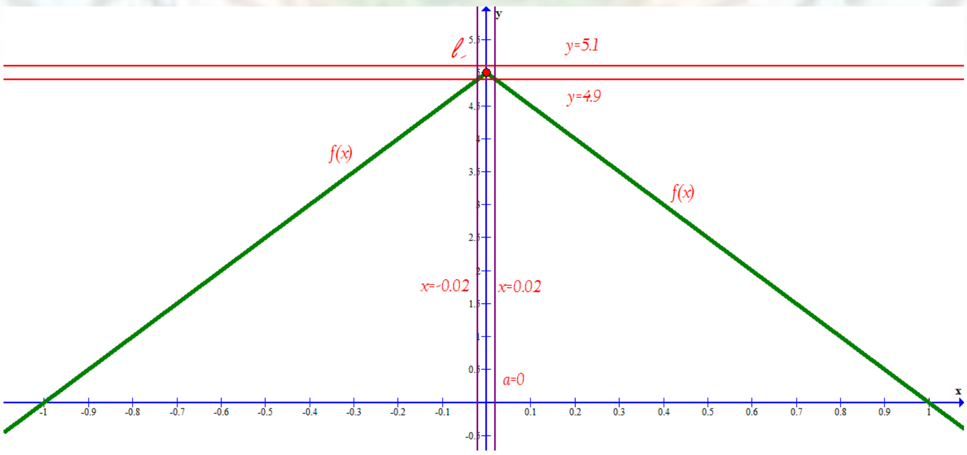
The limit of a real valued function $f(x)$ of a real variable x

$$\lim_{x \to a} f(x) = l$$

exits if and only if

a) the left limit $\lim_{x \to a_-} f(x) = l_-$ and $\lim_{x \to a_+} f(x) = l_+$ right limit exists
b) and $l_- = l_+$

where we denote $l = l_- = l_+$.



3.1.3 Some Theorems

If $\lim_{x \to a} f(x) = l_f$ and $\lim_{x \to a} g(x) = l_g$

Theorem 1.1.3.1

$$\lim_{x\to a}\{f(x)+g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l_f + l_g$$

Theorem 1.1.3.2

$$\lim_{x\to a}\{f(x)-g(x)\} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l_f - l_g$$

Theorem 1.1.3.3

$$\lim_{x\to a}\{f(x)g(x)\} = \left\{\lim_{x\to a} f(x)\right\}\left\{\lim_{x\to a} g(x)\right\} = l_f l_g$$

Theorem 1.1.3.4

$$\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l_f}{l_g} \quad provided\ l_g \neq 0$$

Example 1.1.1 Show that the limit

$$\lim_{x\to 2}\left\{\frac{x^2-4}{x-2}\right\} = 4$$

Solution: For the limit to be correct, given the confinement value $\epsilon > 0$ for the function $f(x) = \frac{x^2-4}{x-2}$ we should be able to find out some region $\delta > 0$ about the limit point $x = 2$

$$\left|\frac{x^2-4}{x-2} - 4\right| < \epsilon$$

for

$$0 < |x-2| < \delta$$

Now

$$\left|\frac{x^2-4}{x-2} - 4\right| = \left|\frac{(x+2)(x-2)}{x-2} - 4\right| = |(x+2)-4| = |x-2|$$

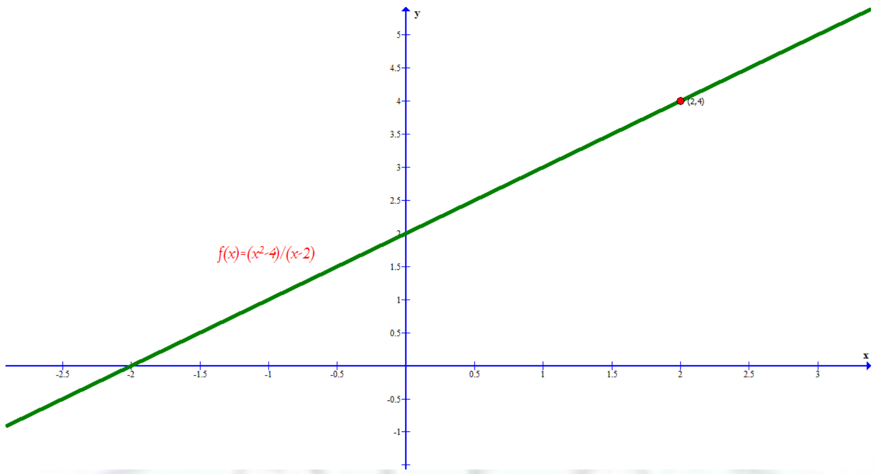
which means given $\epsilon > 0$

$$|x-2| < \epsilon$$

Obviously if we chose $\delta = \epsilon$ then can satisfy

$$0 < |x-2| < \delta$$

Therefore, we find $\left|\frac{x^2-4}{x-2} - 4\right| < \epsilon$ for $0 < |x-2| < \epsilon$ and so the limit $\lim_{x\to 2}\left\{\frac{x^2-4}{x-2}\right\} = 4$.



Example 1.1.2 Evaluate the limit

$$\lim_{x\to 2}\left\{\frac{x^2-4}{x-2}\right\}$$

Solution: Know that

$$\frac{x^2-4}{x-2}=\frac{(x+2)(x-2)}{x-2}=(x+2)$$

we get

$$\lim_{x\to 2}\left\{\frac{x^2-4}{x-2}\right\}=\lim_{x\to 2}(x+2)$$

Intuitively the limit value must be 4, so we test for it. For the limit to be correct, given the confinement value $\epsilon>0$ for the function $f(x)=x+2$ we should be able to find out some region $\delta>0$ about the limit point $x=2$

$$|(x+2)-4|<\epsilon$$

for

$$0<|x-2|<\delta$$

which is obvious as the two inequalities looks the same provided $\delta=\epsilon$.

Example 1.1.3 Show that the limit

$$\lim_{x\to 0}\left\{\frac{|x|}{x}\right\}$$

does not exist.

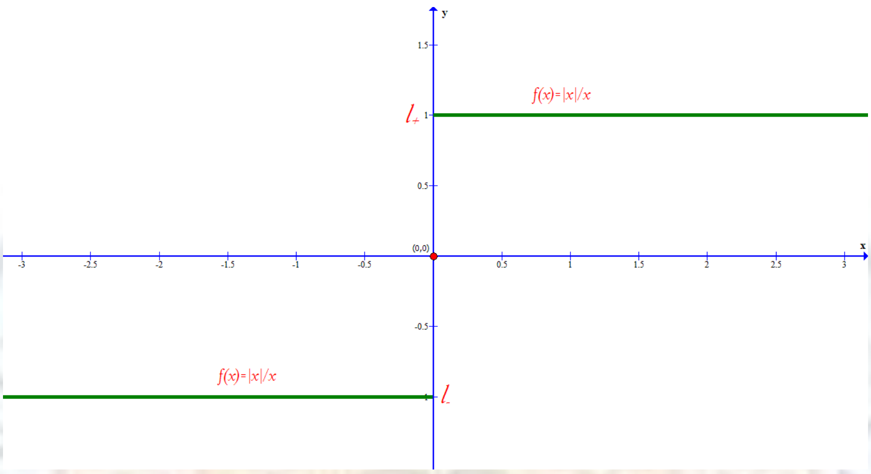
Solution: Let's look at the limit of the function left of $x=0$ i.e. $x<0$ and for which $|x|=-x$

$$\lim_{x\to 0_-}\left\{\frac{|x|}{x}\right\}=\lim_{x\to 0_-}\left\{\frac{-x}{x}\right\}=\lim_{x\to 0_-}\{-1\}=-1$$

Now let's look at the limit of the function right of $x = 0$ i.e. $x > 0$ and for which $|x| = x$

$$\lim_{x \to 0_-} \left\{ \frac{|x|}{x} \right\} = \lim_{x \to 0_-} \left\{ \frac{x}{x} \right\} = \lim_{x \to 0_-} \{1\} = 1$$

Although the left and right limit exists *but they are not equal* therefore the limit $\lim_{x \to 0} \left\{ \frac{|x|}{x} \right\}$ do not exist .



3.2 Continuity

A real valued function $f(x)$ of a real variable x is said to be continuous at $x = a$ if the following conditions are fulfilled

a) the limit $\lim_{x \to a} f(x) = l$ exists
b) the function is defined and single valued in the neighbourhood of $x = a$ & at $x = a$ and $f(a)$ exits
c) and finally $f(a) = l$

If the function $f(x)$ fails to be continuous at some point $x = a$ then that point is known as the *point of discontinuity* and the function is said to be *discontinuous* at that point.

3.2.1 Left Continuity

A real valued function $f(x)$ of a real variable x defined *only* for $x \leq a$ is said to *continuous on the left* at $x = a$ if

$$\lim_{x \to a_-} f(x) = l_- = f(a)$$

3.2.2 Right Continuity

A real valued function $f(x)$ of a real variable x defined *only* for $x \geq a$ is said to *continuous on the right* at $x = a$ if

$$\lim_{x \to a_+} f(x) = l_+ = f(a)$$

It seems obvious that if a function $f(x)$ is continuous both from the left and from the right at a point $x = a$ then it is continuous at $x = a$.

Example 1.2.1 Show that the function

$$\boldsymbol{f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}}$$

is continuous at $x = 0$.

Solution: First we consider the difference

$$|f(x) - f(0)| = \left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| = |x^2| \left|\sin\left(\frac{1}{x}\right)\right| < |x|^2$$

since the $\sin$ function is always less than or equal to 1. If we impose the condition that for some arbitrarily small $\epsilon > 0$ we have $|f(x) - f(0)| < \epsilon$ then $|x|^2 < \epsilon$ and we find

$$|x - 0|^2 < \epsilon$$
$$|x - 0| < \sqrt{\epsilon}$$

This means that there always exist a positive $\delta = \sqrt{\epsilon}$ for the given ϵ such that $|f(x) - f(0)| < \epsilon$ and $|x - 0| < \delta$. Thus the given function is continuous $x = 0$.

Example 1.2.2 Show that the function

$$\boldsymbol{f(x) = \begin{cases} x^3 \ \forall\ 0 < x < 1 \\ x \ \ \forall\ 1 \leq x < 2 \\ x^2 \ \forall\ 2 \leq x < 3 \end{cases}}$$

is continuous at $x = 1$ and discontinuous at $x = 2$.

Solution: First we consider the left limit at the point $x = 1$

$$\lim_{x \to 1_-} f(x) = \lim_{x \to 1_-} x^3 = 1$$

and the right limit at the point $x = 1$

$$\lim_{x \to 1_+} f(x) = \lim_{x \to 1_+} x = 1$$

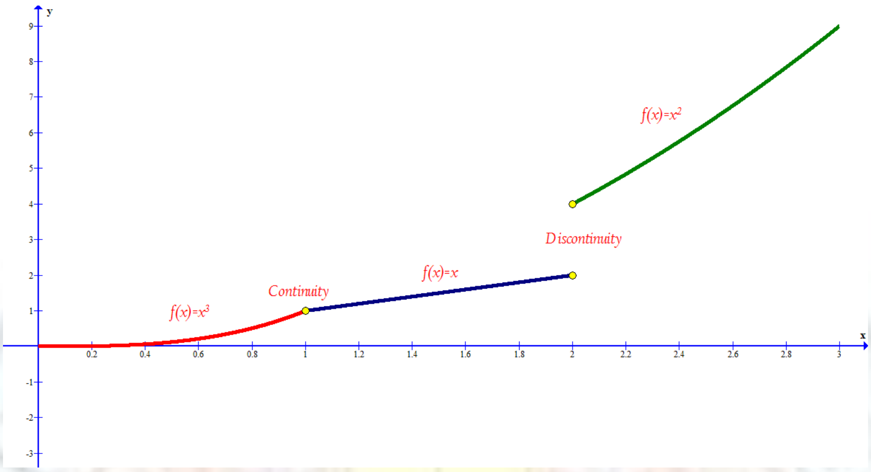
Since the two limit exist and have the same value the function $f(x)$ is continuous at $x = 1$.
Now we consider the left limit at the point $x = 2$

$$\lim_{x \to 2_-} f(x) = \lim_{x \to 2_-} x = 2$$

and the right limit at the point $x = 2$

$$\lim_{x \to 2_+} f(x) = \lim_{x \to 2_+} x^2 = 4$$

Though the two limit exist but they do not have the same value, the function $f(x)$ is discontinuous at $x = 2$.



3.3 Differentiability

A real valued function $f(x)$ of a real variable x is said to be differentiable at $x = a$ if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exits and this limiting value is denoted as $f'(a)$. Thus, the derivative of the function $f(x)$ of a real variable x at $x = a$ can be represented as (by writing $x - a = h$)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We find that to be differentiable $x = a$ the function $f(x)$ must be continuous at $x = a$.

If the function $f(x)$ fails to be differentiable at some point $x = a$ then the function is said to be *non-differentiable* at that point.

3.3.1 Left Differentiability

A real valued function $f(x)$ of a real variable x is said to be left differentiable at $x = a$ if

$$f'_{-}(a) = \lim_{h \to 0_-} \frac{f(a+h) - f(a)}{h}$$

3.3.2 Right Differentiability

A real valued function $f(x)$ of a real variable x is said to be right differentiable at $x = a$ if

$$f'_{+}(a) = \lim_{h \to 0_+} \frac{f(a+h) - f(a)}{h}$$

It seems obvious that if a function $f(x)$ is differentiable both from the left and from the right at a point $x = a$ then it is differentiable at $x = a$.

Example 1.3.1 Show that the function

$$\boldsymbol{f(x) = e^x}$$

is differentiable everywhere.

Solution: First we look for left differentiability at an arbitrary point $x = a$

$$f'_{-}(a) = \lim_{h \to 0_-} \frac{f(a+h) - f(a)}{h}$$
$$f'_{-}(a) = \lim_{h \to 0_-} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h \to 0_-} \frac{e^h - 1}{h}$$
$$f'_{-}(a) = e^a \lim_{h \to 0_-} \frac{\left(1 + h + \frac{h^2}{2!} + \cdots\right) - 1}{h}$$
$$f'_{-}(a) = e^a \lim_{h \to 0_-} \frac{h + \frac{h^2}{2!} + \cdots}{h} = e^a \lim_{h \to 0_-} \frac{h\left(1 + \frac{h}{2!} + \cdots\right)}{h}$$
$$f'_{-}(a) = e^a \lim_{h \to 0_-} \left(1 + \frac{h}{2!} + \cdots\right) = e^a.1$$

The $\lim_{h \to 0_-} \left(1 + \frac{h}{2!} + \cdots\right)$ is independent of the sign of h and the limiting value is 1. Thus,

$$f'_{-}(a) = e^a$$

Then we look for right differentiability at $x = a$

$$f'_{+}(a) = \lim_{h \to 0_+} \frac{f(a+h) - f(a)}{h}$$
$$f'_{+}(a) = \lim_{h \to 0_+} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h \to 0_+} \frac{e^h - 1}{h}$$
$$f'_{+}(a) = e^a \lim_{h \to 0_+} \frac{\left(1 + h + \frac{h^2}{2!} + \cdots\right) - 1}{h}$$
$$f'_{+}(a) = e^a \lim_{h \to 0_+} \frac{h + \frac{h^2}{2!} + \cdots}{h} = e^a \lim_{h \to 0_+} \frac{h\left(1 + \frac{h}{2!} + \cdots\right)}{h}$$
$$f'_{+}(a) = e^a \lim_{h \to 0_+} \left(1 + \frac{h}{2!} + \cdots\right) = e^a.1$$

Again $\lim_{h \to 0_+} \left(1 + \frac{h}{2!} + \cdots\right)$ is independent of the sign of h and the limiting value is 1. Thus,

$$f'_{+}(a) = e^a$$

If the function were to be differentiable then

$$f'_{-}(a) = f'_{+}(a)$$

which is true.

Example 1.3.2 Show that the function

$$\boldsymbol{f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right) & \forall\, x \neq 0 \\ 0 & \forall\, x = 0 \end{cases}}$$

is differentiable if $m > 1$.

Solution: First we look for left differentiability at $x = 0$

$$f'_{-}(0) = \lim_{h \to 0_-} \frac{f(0+h) - f(0)}{h}$$

$$f'_{-}(0) = \lim_{h \to 0_-} \frac{h^m \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0_-} \left\{h^{m-1} \sin\left(\frac{1}{h}\right)\right\}$$

where $h \to 0_-$ means $h < 0$. To make h a positive number, we can rewrite it as

$$f'_{-}(0) = \lim_{h \to 0_+} \left\{(-h)^{m-1} \sin\left(\frac{1}{(-h)}\right)\right\}$$

$$f'_{-}(0) = \lim_{h \to 0_+} \left\{(-1)^{m-1} h^{m-1} (-1) \sin\left(\frac{1}{h}\right)\right\}$$

$$f'_{-}(0) = (-1)^m \lim_{h \to 0_+} \left\{h^{m-1} \sin\left(\frac{1}{h}\right)\right\}$$

where now $h > 0$.
Then we look for right differentiability at $x = 0$

$$f'_{+}(0) = \lim_{h \to 0_+} \frac{f(0+h) - f(0)}{h}$$

$$f'_{+}(0) = \lim_{h \to 0_+} \frac{h^m \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0_+} \left\{h^{m-1} \sin\left(\frac{1}{h}\right)\right\}$$

where $h \to 0_+$ means $h > 0$.

If the function were to be differentiable then

$$f'_{-}(0) = f'_{+}(0)$$

$$(-1)^m \lim_{h \to 0_+} \left\{h^{m-1} \sin\left(\frac{1}{h}\right)\right\} = \lim_{h \to 0_+} \left\{h^{m-1} \sin\left(\frac{1}{h}\right)\right\}$$

The ***existence of the limits requires that*** $\boldsymbol{m > 1}$ *(otherwise $f'_{-}(0)$ & $f'_{+}(0)$ would blow up)* and the ***equality requires that*** $\boldsymbol{m}$ ***is even***.

Example 1.3.3 Show that the function

$$f(x) = e^x$$

is differentiable at $x = a$.

Solution: First we look for left differentiability at an arbitrary point $x = a$

$$f'_-(a) = \lim_{h \to 0_-} \frac{f(a+h) - f(a)}{h}$$

$$f'_-(a) = \lim_{h \to 0_-} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h \to 0_-} \frac{e^h - 1}{h}$$

$$f'_-(a) = e^a \lim_{h \to 0_-} \frac{\left(1 + h + \frac{h^2}{2!} + \cdots\right) - 1}{h}$$

$$f'_-(a) = e^a \lim_{h \to 0_-} \frac{h + \frac{h^2}{2!} + \cdots}{h} = e^a \lim_{h \to 0_-} \frac{h\left(1 + \frac{h}{2!} + \cdots\right)}{h}$$

$$f'_-(a) = e^a \lim_{h \to 0_-} \left(1 + \frac{h}{2!} + \cdots\right) = e^a.1$$

The $\lim_{h \to 0_-} \left(1 + \frac{h}{2!} + \cdots\right)$ is independent of the sign of h and the limiting value is 1. Thus,

$$f'_-(a) = e^a$$

Then we look for right differentiability at $x = a$

$$f'_+(a) = \lim_{h \to 0_+} \frac{f(a+h) - f(a)}{h}$$

$$f'_+(a) = \lim_{h \to 0_+} \frac{e^{a+h} - e^a}{h} = e^a \lim_{h \to 0_+} \frac{e^h - 1}{h}$$

$$f'_+(a) = e^a \lim_{h \to 0_+} \frac{\left(1 + h + \frac{h^2}{2!} + \cdots\right) - 1}{h}$$

$$f'_+(a) = e^a \lim_{h \to 0_+} \frac{h + \frac{h^2}{2!} + \cdots}{h} = e^a \lim_{h \to 0_+} \frac{h\left(1 + \frac{h}{2!} + \cdots\right)}{h}$$

$$f'_+(a) = e^a \lim_{h \to 0_+} \left(1 + \frac{h}{2!} + \cdots\right) = e^a.1$$

Again $\lim_{h \to 0_+} \left(1 + \frac{h}{2!} + \cdots\right)$ is independent of the sign of h and the limiting value is 1. Thus,

$$f'_+(a) = e^a$$

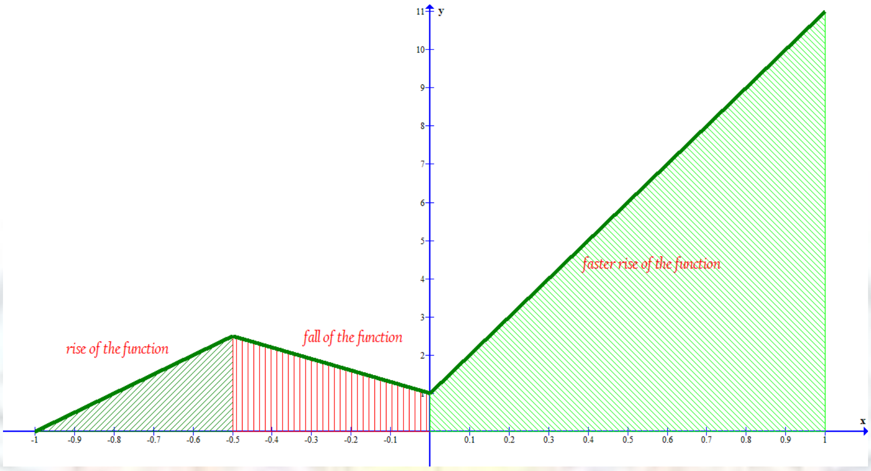
If the function were to be differentiable then

$$f'_-(a) = f'_+(a)$$

which is true.

3.4 Average and Instantaneous Quantities

A real valued function $f(x)$ of a real variable x can be plotted in the $x - y$ coordinate with $f(x)$ on the y –axis as shown. If the x variable represents time then from the graph we can look for region where the function changes faster and also for the region where the change is slower.

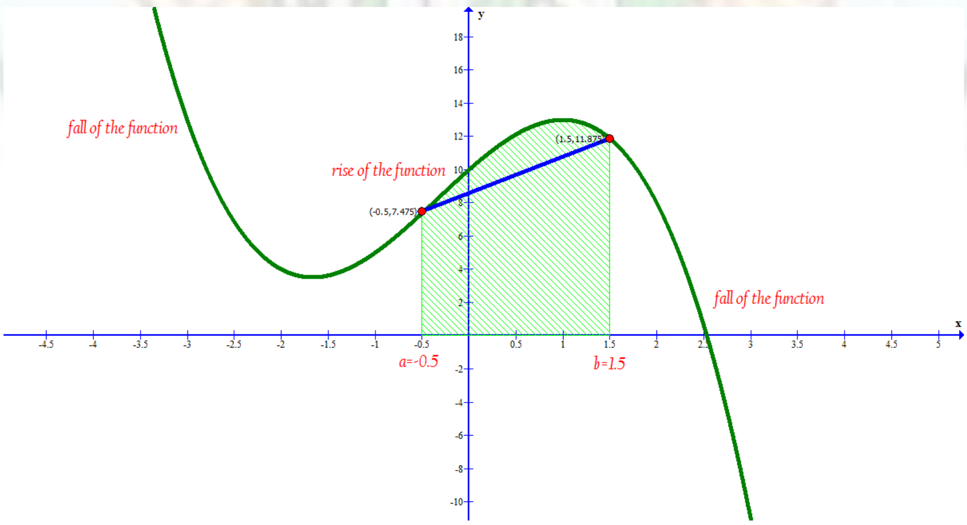


There are two quantities of special interest when we consider such rate of change

3.4.1 Average Quantity

A real valued function $f(x)$ of a real variable x, the average rate of change of the function in the interval $a \leq x \leq b$ is defined as

$$\mathcal{R} = \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

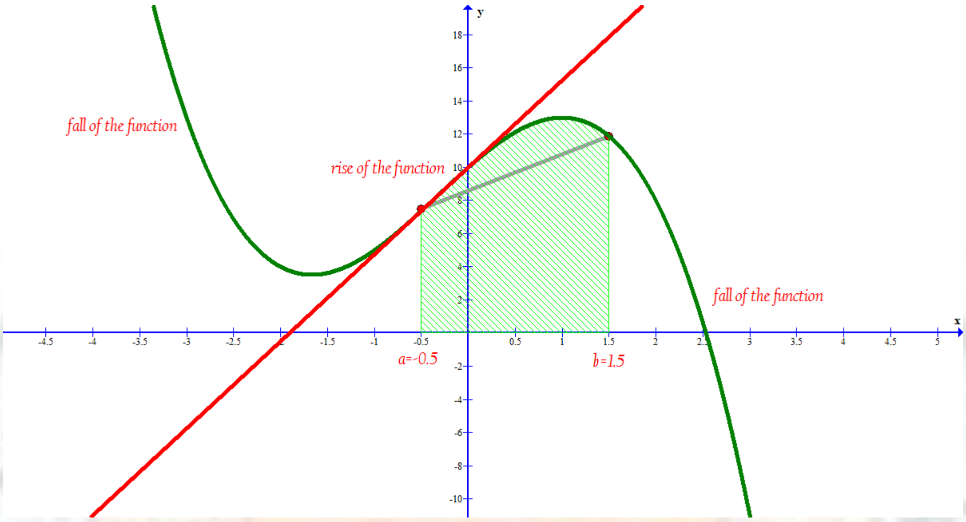


Although the average rate tells the general trend but it can be misleading as in the present case the curve between $a \leq x \leq b$ includes region of rise as well as fall but the rate show only an increase.

3.4.2 Instantaneous Quantity

For a real valued function $f(x)$ of a real variable x, the instantaneous rate of change of the function at a point $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



The red line tells the exact trend at the point $x = a$ but then it doesn't tell the general trend.

Example 1.4.1 Find the mean and the instantaneous value for the function variables

$$\boldsymbol{f(x) = x^2}$$

Solution: The mean value between the interval $a \le x \le b$ will be

$$m = \frac{f(b) - f(a)}{b - a}$$
$$m = \frac{b^2 - a^2}{b - a} = \frac{(b+a)(b-a)}{b - a}$$
$$m = b + a$$

The instantaneous value at $x = a$ would be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{a^2 + h^2 + 2ah - a^2}{h}$$
$$f'(a) = \lim_{h \to 0} \frac{h^2 + 2ah}{h} = h + 2a$$
$$f'(a) = 2a$$

It is to be noted that the instantaneous value at $x = \frac{a+b}{2}$ would be

$$f'\left(\frac{a+b}{2}\right) = 2\left(\frac{a+b}{2}\right) = a + b$$

which is the same as the mean value in the interval $a \leq x \leq b$.

3.4.3 The Theorem of Mean Value

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) then there exists a point θ in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(\theta)$$

The instantaneous rate of change (derivative) of the function $f(x)$ at θ is equal to the mean rate of change of the function $f(x)$ in the interval $[a, b]$.

Rolle's Theorem is the special case of the Mean Value Theorem, according to which if $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) and if $f(b) = f(a) = 0$ then there exists a point θ in (a, b) such that

$$f'(\theta) = 0$$

3.5 Taylor and Binomial Series

3.5.1 Taylor's Theorem of the Mean

If the n^{th} derivative of $f(x)$ i.e., $f^{(n)}(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) then at $x \in (a, b)$

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x - a)^n}{n!} + R_n$$

where $R_n = \frac{f^{(n+1)}(\theta)(x-a)^{n+1}}{(n+1)!}$ is known as the remainder for some θ in (a, x). This becomes a Taylor series about the point $x = a$.

Taylor's series is an extension of this theorem when $\lim_{n\to\infty} R_n = 0$, under such circumstance

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!} + \cdots$$

Some examples of Taylor's series are

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

3.5.2 The Binomial Series

If $|x| < 1$ and m is any real number, then

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n$$

where the Binomial Coefficients $\binom{m}{n} = \frac{m!}{(m-n)!n!}$

Example 1.5.1 Expand the function in power series

$$\boldsymbol{f(x) = \frac{1}{\sqrt{1+x}}}$$

Solution: The Binomial expansion would be for the function

$$f(x) = \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$$

which means $m = -1/2$. The coefficients would then be (for $n \geq 1$)

$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)!}{\left(-\frac{1}{2}-n\right)!\,n!}$$
$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}$$
$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{(2n-1)}{2}\right)}{n!}$$
$$\binom{-\frac{1}{2}}{n} = (-1)^n\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\ldots\left(\frac{(2n-1)}{2}\right)}{n!}$$
$$\binom{-\frac{1}{2}}{n} = (-1)^n\frac{1.3.5\ldots(2n-1)}{2^n n!}$$

For $n = 0$ we find that $\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)!}{\left(-\frac{1}{2}-0\right)!0!} = 1$

So the first few coefficients are $1, -\frac{1}{2}, \frac{3}{8}, \ldots$ and hence

$$f(x) = \frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Summary

Limits

- In a compact form it is written as

$$\lim_{x \to a} f(x) = l$$

- It means that for every neighbourhood $(l - \epsilon, l + \epsilon)$ of l, there exists a neighbourhood $(a - \delta, a + \delta)$ excluding the point $\boldsymbol{a}$ itself such that $f(x)$ is in $(l - \epsilon, l + \epsilon)$ for every x in $(a - \delta, a + \delta)$ excluding $x = a$ itself.
- The limit of a real valued function $f(x)$ of a real variable x

$$\lim_{x \to a} f(x) = l$$

exists iff

c) the left limit $\lim_{x \to a_-} f(x) = l_-$ and $\lim_{x \to a_+} f(x) = l_+$ right limit exists
d) and $l_- = l_+$

where we denote $l = l_- = l_+$.

Continuity

- A real valued function $f(x)$ of a real variable x is said to be continuous at $x = a$ if the following conditions are fulfilled
  d) the limit $\lim_{x \to a} f(x) = l$ exists
  e) the function is defined and single valued in the neighbourhood of $x = a$ & at $x = a$ and $f(a)$ exists
  f) and finally $f(a) = l$
- If the function $f(x)$ fails to be continuous at some point $x = a$ then that point is known as the *point of discontinuity* and the function is said to be *discontinuous* at that point.
- *It seems obvious that if a function $f(x)$ is continuous both from the left and from the right at a point $x = a$ then it is continuous at $x = a$.*

Differentiability

- A real valued function $f(x)$ of a real variable x is said to be differentiable at $x = a$ if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and this limiting value is denoted as $f'(a)$.
- We find that to be differentiable at the point $x = a$ the function $f(x)$ must be continuous at $x = a$.
- If the function $f(x)$ fails to be differentiable at some point $x = a$ then the function is said to be *non-differentiable* at that point.
- *It seems obvious that if a function $f(x)$ is differentiable both from the left and from the right at a point $x = a$ then it is differentiable at $x = a$.*

Average Quantity

- A real valued function $f(x)$ of a real variable x, the average rate of change of the function in the interval $a \leq x \leq b$ is defined as

$$\mathcal{R} = \frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Instantaneous Quantity

- For a real valued function $f(x)$ of a real variable x, the instantaneous rate of change of the function at a point $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The Theorem of Mean Value

- If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) then there exists a point θ in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(\theta)$$

- The instantaneous rate of change (derivative) of the function $f(x)$ at θ is equal to the mean rate of change of the function $f(x)$ in the interval $[a, b]$.
- *Rolle's Theorem* is the special case of the Mean Value Theorem, according to which if $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) and if $f(b) = f(a) = 0$ then there exists a point θ in (a, b) such that

$$f'(\theta) = 0$$

Taylor's Theorem of the Mean

- If the n^{th} derivative of $f(x)$ i.e., $f^{(n)}(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) then at $x \in (a, b)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n$$

where $R_n = \frac{f^{(n+1)}(\theta)(x-a)^{n+1}}{(n+1)!}$ is known as the remainder for some θ in (a, x). This becomes a Taylor series about the point $x = a$.
- *Taylor's series* is an extension of this theorem when $\lim_{n \to \infty} R_n = 0$, under such circumstance

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

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