Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: Differential Equations

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Learning Objectives

From this chapter the student learns

- ***What is a Differential Equation (DE)?***
- ***Different types of Differential Equations***
  - ***Difference between Linear and Non Linear DE***
  - ***To check when the DE is Homogenous and Non Homogenous***
- ***The understand Order and Degree of the DE***
- ***To determine the solutions of the DE***
  - ***What are Explicit and Implicit solutions?***
  - ***How to find a general solution and for a given constraint a particular solution?***
  - ***Any solution other than the above ; singular solution***
- ***The role of Number of Arbitrary Constants in the general solution.***

Differential Equations

Differential equations are important in science, engineering and social sciences. They are a link between mathematics and science. Their abundance helps in understanding the sciences.

2.1 Introduction to Differential Equations

We are all familiar with Newton's Second Law of motion. Let us consider the motion of a body of mass m along a straight line (say x-*axis*). Let this mass be subjected to a force $x(t)$ along the axis at a time t. Then,

$$m\frac{d^2x(t)}{dt^2} = F(t)$$

where $x(t)$ is the mass's displacement measured from the origin.

(i) Now if $x(t)$ the displacement is prescribed we can determine $F(t)$ by merely differentiating $x(t)$ twice by multiplying by m.

(ii) However, if applied force $F(t)$ is known and $x(t)$ is to be determined then we need to 'undo' the differentiations i.e. We need to integrate the above equation twice.

If say $F(t) = F_0$ was a constant force then integrating (1) with respect to t we get

$$m\frac{dx(t)}{dt} = F_0t + A$$

where A is an arbitrary constant of integration. Integrating again we get

$$mx(t) = \frac{1}{2}F_0t^2 + At + B$$

or

$$x(t) = \frac{1}{m}\left(\frac{1}{2}F_0t^2 + At + B\right)$$

If displacement $x(t)$ and velocity $v(t)$ are prescribed at the initial time $(t = 0)$, then constants A and B can be found. If say initially

$x(0) = 0$ then $B = 0$,

and

$\dot{x}(0) = 0$ then $A = 0$

then

$$x(t) = \frac{1}{2m}F_0t^2$$

is the solution.

We need to however understand that not all differential equations can be solved by merely undoing the derivatives. Suppose that a mass is restrained by a coil (spring) that supplies a restoring force proportional to the displacement $x(t)$ with constant of proportionality k. The differential equation for this motion is

$$m\frac{d^2x(t)}{dt^2} = -kx + F(t)$$

After one integration

$$m\frac{dx(t)}{dt} = -k\int x(t)dt + \int F(t)dt + A$$

since $F(t)$ is prescribed function its integral is known. Since $x(t)$ is unknown, the integral of $x(t)$ cannot be evaluated. Thus we see that solving the differential equation *is not merely a matter of undoing the derivatives by direct integration.*

2.2 Types of Differential Equations

Differential Equation (*DE*): An equation containing one or more derivatives of the unknown function under consideration is a DE.

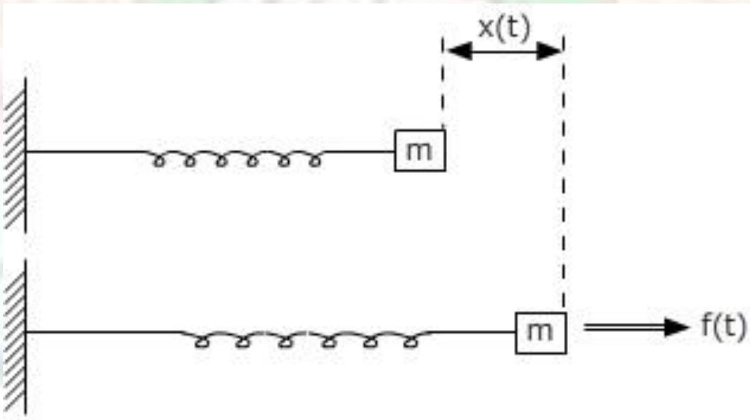
2.2.1 Ordinary Differential Equation (*ODE*)

A DE is classified as an ODE if it contains ordinary derivatives with respect to a single independent variable.

A few examples are:

(i) The ODE governs the linear displacement $x(t)$ of a body of mass m, when subjected to an applied force $f(t)$ and a restraining spring of stiffness k.

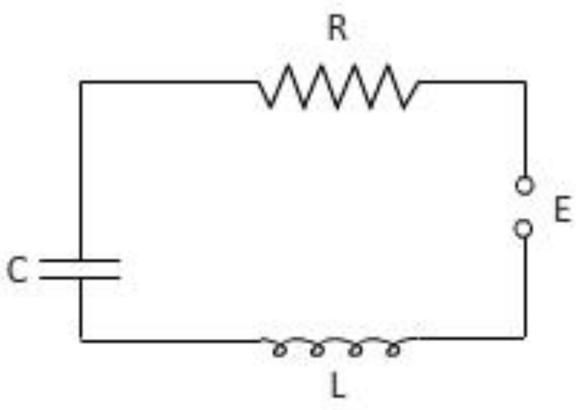
$$m\frac{d^2x(t)}{dt^2} + kx = f(t)$$



Mass/spring system

(ii) This is the ODE governing the current $i(t)$ in an electrical circuit containing an inductor L, a capacitance C and a resistor R with an applied voltage source strength of $E(t)$ in the circuit at time t.

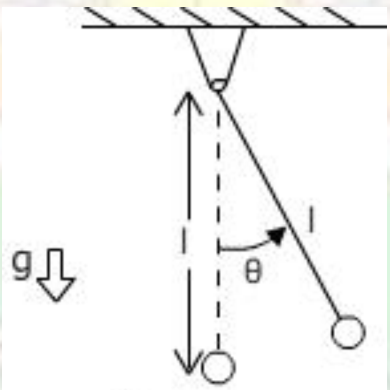
$$L\frac{d^2i(t)}{dt^2} + Ri(t) + \frac{i(t)}{C} = \frac{dE}{dt}$$



Electrical circuit

(iii) In this ODE, the angular motion $\theta(t)$ of a pendulum of length l under the action of gravity g (acceleration due to gravity) with time t is governed.

$$l\frac{d^2\theta(t)}{dt^2} + gsin\theta = 0$$



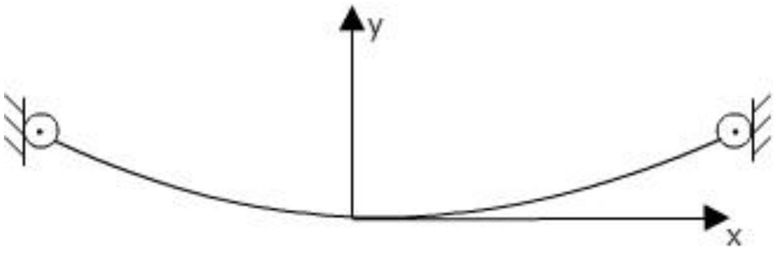
Pendulum

(iv) The ODE governs the population (of humans, animals, bacteria, etc) where t is the time and the net birth/death rate constant is "a".

$$\frac{dy}{dt} = ay$$

(v) This ODE governs the shape of the flexible cable or string which hangs under the action of gravity with $y(x)$ as the deflection and A the constant that depends on the mass density of the string or cable, and the tension at the midpoint $x = 0$.

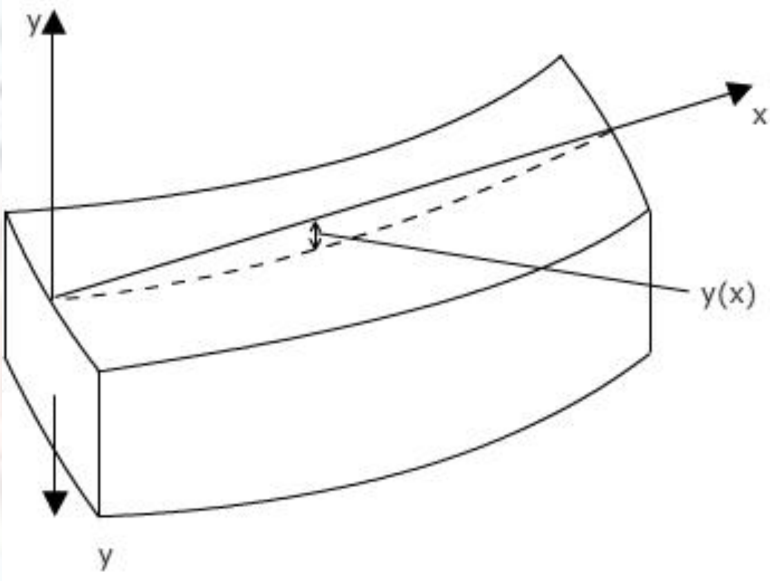
$$\frac{d^2y}{dx^2} = A\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$



Hanging Cable

(vi) This ODE governs the deflection $y(x)$ of a beam which is subjected to a load $f(x)$ where E and I are physical constants of the beam material and cross section.

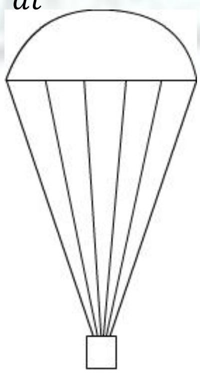
$$EI\frac{d^4y}{dx^4} = f(x)$$



Deformation of Bean

(vii) This ODE is of a parachutist of mass m falling with velocity $v(t)$ at a time t under a force of attraction mg by the earth and an air resistance bv^2.

$$m\frac{dv}{dt} = mg - bv^2$$



Parachutist

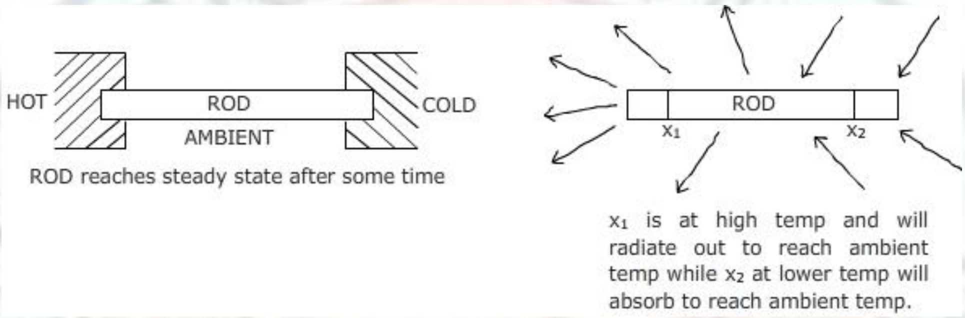
In all the equations mentioned above are ODE, the independent variable is t in equations (1-4) and (7) and x in (5) and (6).

2.2.2 Partial Differential Equation (*PDE*)

A *PDE* contain an unknown function of two or more independent variables. Some important PDE's are

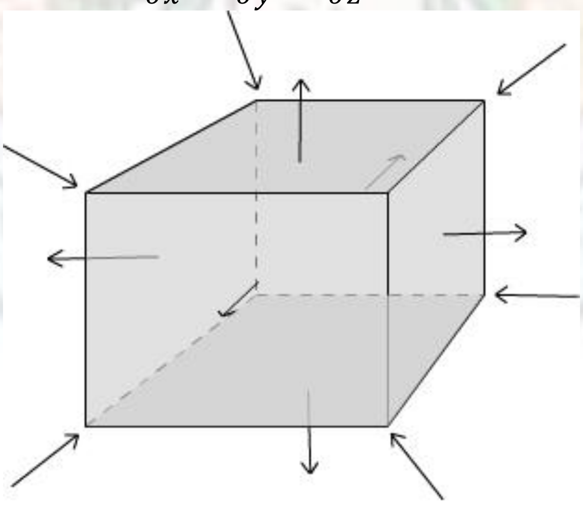
(i) This is the heat equation which governs the time varying temperature distribution $u(x,t)$ in a *1D* rod or slab. x is the point under consideration within the material, t is the time and h^2 is the material property called diffusivity.

$$h^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$



(ii) This is the Laplace equation which governs the steady state temperature distribution $u(x,y,z)$ within a *3D* body; and x,y,z are the coordinates of the point within the material.

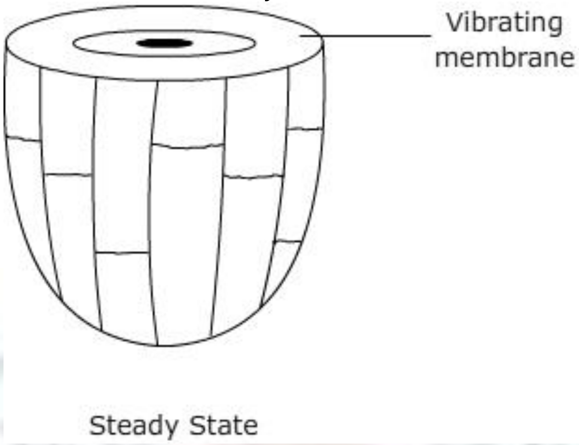
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$



Steady State

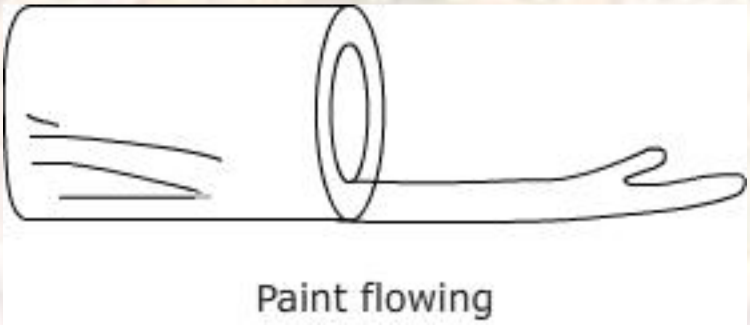
(iii) This is the wave equation that governs the deflection $u(x,y,t)$ of the vibrating membrane such as a drum, tabla etc.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$



*(iv)*This is a biharmonic equation that governs the stream function $u(x,y,z)$ in the case of a viscous fluid such as a wet paint (or honey).

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$



2.2.3 Order

The order of a *DE* is the order of the highest derivative therein. From the above examples, the equation of order one is eq (4), equations (1, 2, 3, 5) are of second order and equation (6) is of order 4.
In general if

$$F\{x, y(x), y'(x), y''(x), .., y^{(n)}(x)\} = 0$$

is said to be an nth order DE of the unknown function $y(x)$ where n is the order of the highest derivative present. $y'(x)$ represents $\frac{dy}{dx}$, $y''(x)$ represents $\frac{d^2y}{dx^2}$ and so on ....
The independent variable is x and y is the dependent variable.

The general form of nth order partial differential equation is

$$F\left\{x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \dots, \frac{\partial^n z}{\partial x^n}, \frac{\partial^n z}{\partial y^n}\right\} = 0$$

2.2.4 Degree

If a *DE* can be rationalised and cleared of fraction with regard to all derivatives present, the exponent of the highest order derivative is called the degree of the DE. e.g., we have

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} + 2y = 0$$

$$\left(\frac{d^3y}{dx^3}\right)^{2/3} = a + b\frac{dy}{dx}$$

The first equation has degree one as highest order term $\frac{d^2y}{dx^2}$ has power one and the second equation has degree two as highest order term $\left(\frac{dy}{dx}\right)^2$ has power two. The third equation is of degree 2 because on cubing both sides it takes the form

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(a + b\frac{dy}{dx}\right)^3$$

in which exponent of highest order derivative is 2.

It may be noted that the degree of a DE may not be defined,

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = log\left(\frac{d^2y}{dx^2}\right)$$

is a *DE* for which degree is not defined.

2.2.5 Explicit and Implicit solution

Consider an equation

$$F(x, y) = 0$$

The functional relation between x and y may be explicit or implicit. If we can express

$$y = h(x)$$

then y is said to be an explicit function. It has been defined as a rule that is to be applied on the independent variable x. Otherwise, y is an implicit function.

Again, if we have a relation of the form

$$F(x, y, z) = 0$$

It is possible to have any variable as a function of other two. If x and y can be assigned values, they are the independent variables and z which takes on these values is the dependent variable. If z can be defined as a function of x and y as

$$z = g(x, y)$$

then z is an explicit function of x and y. Thus, parallelly if we have a first order *DE*

$$F(x, y, y') = 0$$

then sometimes a solution of differential equation will appear as an implicit function if

$$H(x, y) = 0$$

which is called the implicit solution or it may be solution of the type

$$y = h(x) \text{ or } y' = f(x, y)$$

which is an explicit solution.

Singular Solution

An additional solution that cannot be obtained from the general solution is called a Singular solution.

Consider nth order general ODE

$$F\{x, y(x), y'(x), y''(x), .., y^{(n)}(x)\} = 0$$

Its general solution is a solution containing n essential arbitrary constants. Any solution obtained from the general solution by giving particular values to one or more of n essential arbitrary constants is a **particular solution.**

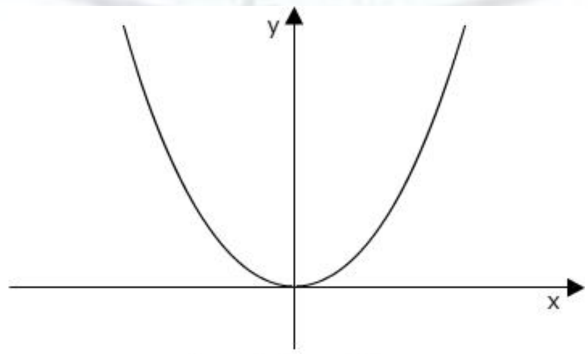
A solution that cannot be obtained from the general solution by any choice of the essential arbitrary constants is called **singular solution.**

Q A DE $xy' = 2y$ for all values of x has a solution $x^2 = y$. What type of solution is that?

Soln : $x^2 = y$ is the solution then we see from differentiation that
$y' = 2x$ and substituting in the DE we have the equation as

$$xy' = 2y$$
$$x(2x) = 2y$$
$$x^2 = y$$

It is an **explicit** solution.



A parabolic solution

2.3 Linear and Non Linear, Homogeneous and Non Homogeneous

Linear *DE*s are those which contain dependent variable and its derivatives in their first degree. An nth order differential equation is said to be **linear** if it can be expressed in the form

$$F\{x, y(x), y'(x), y''(x), .., y^{(n)}(x)\} = 0$$

or

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y(x) = f(x)$$

where, $a_0(x) \neq 0$ and $a_0(x), \ldots, a_n(x), f(x)$ are functions of the independent variable x alone and **non linear otherwise**. These functions are continuous on an interval I and $a_0(x) \neq 0$ for x element of I. The interval I may be open or closed. The solution of this nth order DE will have the property that all derivatives are continuous on I.

Equations that we have seen so far that are linear are

$$m\frac{d^2x(t)}{dt^2} + kx = f(t)$$

$$L\frac{d^2i(t)}{dt^2} + Ri(t) + \frac{i(t)}{C} = \frac{dE}{dt}$$

$$\frac{dy}{dt} = ay$$

$$EI\frac{d^4y}{dx^4} = f(x)$$

while the non-linear equations are

$$l\frac{d^2\theta(t)}{dt^2} + gsin\theta = 0$$

$$\frac{d^2y}{dx^2} = A\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$m\frac{dv}{dt} = mg - bv^2$$

For linear equations solutions can generally be found either in closed form or as infinite series, however for non-linear equations one would instead obtain qualitative information about the solution, rather than the solution itself or getting numerical solutions by computer simulation. In equation (A) if $f(x) = 0$ we say that equation (A) is homogeneous if not, it is non-homogeneous.

Number of Arbitrary Constants

If a relation involving a certain set of constants is a solution of a differential equation regardless of the specific values assigned to the constants and if the set of the constants cannot be replaced by a smaller set then the constants are called essential arbitrary constants.

Consider solutions of the type

$$y = Asinx + Bcosx$$
$$y = Ae^{x} + Be^{-x}$$
$$y = (A + B)e^{2x}$$
$$y = Ae^{x+B}$$

The constants A and B are essential arbitrary constants as they cannot be replaced by a single arbitrary constant in equation (1) and (2).

Summary

Introduction to Differential Equation (DE)

- A DE expresses a relation between rates of an unknown function and the unknown function itself.

$$F\{x, y(x), y'(x), y''(x), .., y^{(n)}(x)\} = 0 \qquad (1)$$

$$F\left\{x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \ldots, \frac{\partial^n z}{\partial x^n}, \frac{\partial^n z}{\partial y^n}\right\} = 0 \qquad (2)$$

- Not all DE’s can be solved by merely undoing the derivatives.

Types of Differential Equations

- A DE is classified as an ***O****rdinary* ***D****ifferential* ***E****quation (ODE)* if it contains ordinary derivatives with respect to a single independent variable (equation 1).
- A **P**artial **D**ifferential **E**quation (PDE) contains an unknown function of two or more independent variables (equation 2).

Order

- The order of a *DE* is the order of the highest derivative therein.

Degree

- The exponent of the highest order derivative is called the degree of the DE.
- It may be noted that the degree of a DE may not be defined.

Explicit and Implicit solution

- If the solution y of the DE can be expressed as a function of the independent variable x i.e., $y = h(x)$ then y is said to be an explicit function.
- Otherwise, y is an implicit function.

Solution

- The general solution of an nth order DE is a solution containing n essential arbitrary constants.
- Any solution obtained from the general solution by giving particular values to one or more of n essential arbitrary constants is a ***particular*** solution.
- An additional solution that cannot be obtained from the general solution by any choice of the essential arbitrary constants is called ***singular*** solution.

Linear and Non Linear

- Linear *DE*s are those which contain dependent variable and its derivatives in their first degree
- Otherwise, the DE is non-linear.
- For linear equations solutions can generally be found either in closed form or as

infinite series, however for non-linear equations one would instead obtain qualitative information about the solution, rather than the solution itself or getting numerical solutions by computer simulation.

Homogeneous and Non Homogeneous

With the DE rewritten as

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y'(x) + a_n(x)y(x) = f(x)$$

- if $f(x) = 0$ we say that the DE is homogeneous
- Otherwise, it is non-homogeneous.

Number of Arbitrary Constants

- If a relation involving a certain set of constants is a solution of a differential equation regardless of the specific values assigned to the constants and if the set of the constants cannot be replaced by a smaller set then the constants are called essential arbitrary constants.

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