Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Dynamics of system of particles
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Ch.2. DYNAMICS OF A SYSTEM OF PARTICLES

Objective

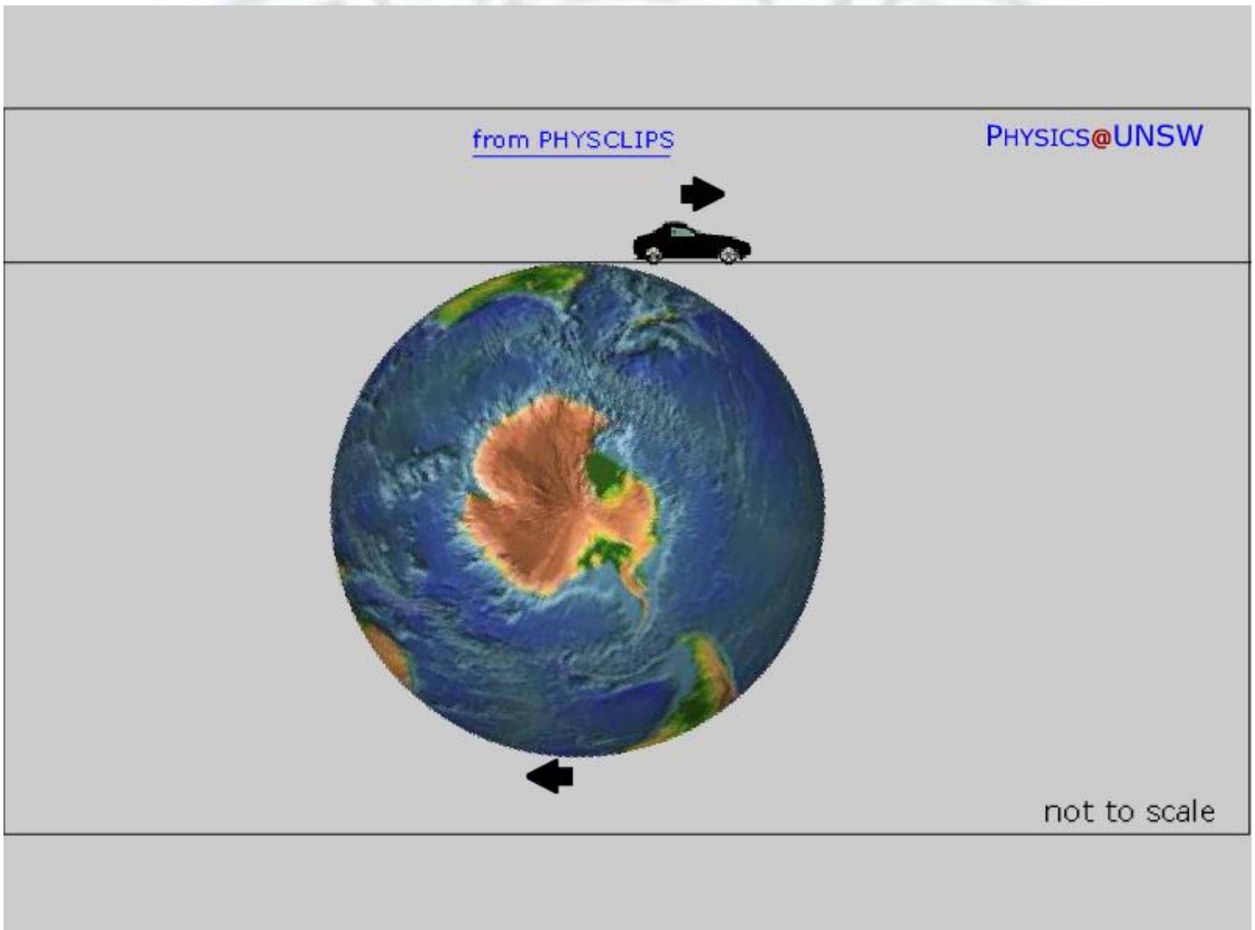
After studying this chapter you will be able to understand:

- The concept of frame of reference
- The type of frame of reference-(a) Inertial and (b) non-inertial
- The transformation rules for inertial frames called the Galilean transformation
- The invariance of Galilean transformation with respect to laws of motion- called the Galilean Invariance
- Descriptions of motion of particles in terms of displacement, velocity acceleration vectors
- The concept of instantaneous and average displacement, vector and acceleration
- The concept of system of mass particle and the description of the motion of the system of particles and its dynamics

1. Reference frames and Inertial Frame of reference

When we study motion of a body or collections of bodies we have to consider some stationary reference points or axes relative to which the body is moving. That stationary reference is known as ***the frame of reference.***

Consider for example motion of a car as shown in the figure below, now the observer sitting in the car observes the motion of car by looking at the motion of outside objects like trees, buildings, other peoples etc., opposite to the direction of his motion. So here we can attach our Frame of reference to the outside tree or buildings etc.



The motion of car relative to earth. To play the movie click Mechanics with animations and film clips: Physclips.

Credits: Authored and Presented by Joe Wolfe

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The actual motion of an object is determined by observing the positional change of the object for a given time period. For this measurement we need a ***frame of reference***. To understand the ***frame of reference***, let us take the example of a moving bus. The change in the bus's position in a given time period has one value if measured by an observer standing on the ground, has another value if the observer is on a moving bike and has zero value if the bike is moving with same speed and direction as that of the bus. And each of these values is equally correct from the point of view of each observer. So in general, the measured value of any physical quantity depends upon **the reference frame** of observer in which the observer is taking measurement. ***To specify a physical quantity, each observer may fix a zero of the time scale, an origin in space and an appropriate coordinate system. These collectively are known as a frame of reference.***

Now studying about frames of reference, we are going to study some special types of frames, which we call INERTIAL FRAMES. Those frames of references in which Newton's laws of motion remains valid are known *as* ***THE INERTIAL FRAMES***.

These frames are either stationary or move with a constant velocity. So here acceleration of these frames is zero. A frame of reference, which is either stationary or moving with constant velocity with respect to an inertial frame, is itself an inertial frame of reference.

When we have accelerated frames, we'll see Newton's laws of motion have to be modified; we call these frames as ***NON-INERTIAL FRAMES***.

Our normal motions just require inertial frames of reference.

2. Galilean Transformations and Galilean Invariance

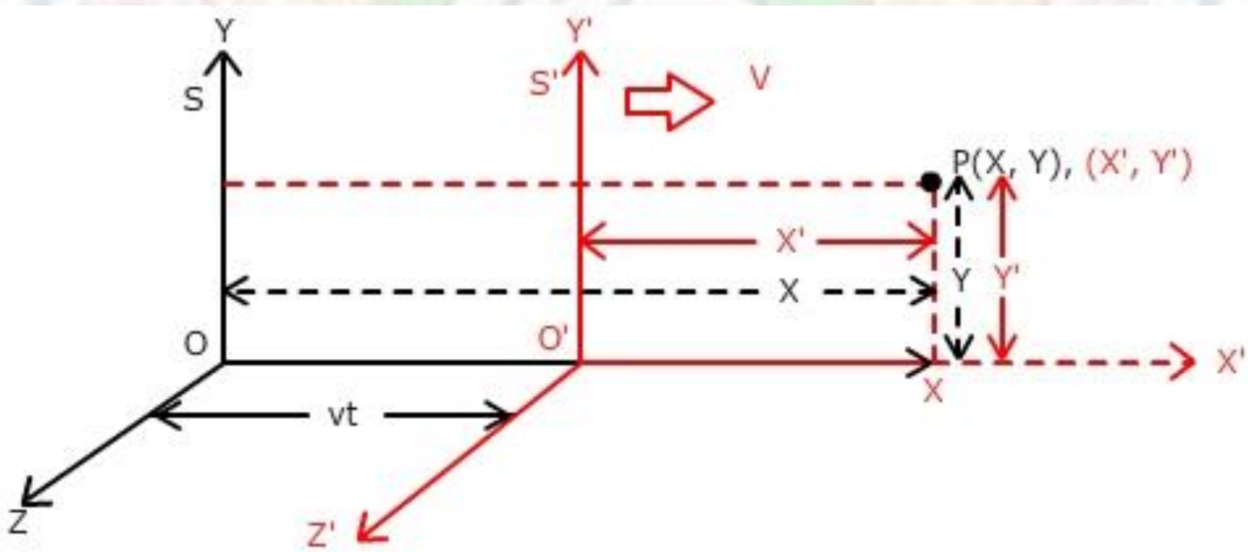


Fig 2.1 Frames of reference S and S' in Uniform relative motion along the x-axis

A physical phenomenon, which is observed simultaneously in two different frames of reference, has two different sets of coordinates corresponding to the two frames of references. So if we wish to establish some relations between the two sets of equations of motion of a particle or the system of particles we have to frame some rules or laws. These set of rules are known as the transformation equations, since they enable the observations made in one frame to be transformed into those made in the other.

Consider two frames of references S and S′ moving with relative velocity **v**, along x axis.

The observers in the two frames will give two different coordinates to the same particles at point P which is observed by both. The coordinates are related to each other by transformation equations known as the ***Galilean transformations.*** Thus,

$X' = x - vt$

$Y' = y$

$Z' = z$

And $t' = t$

Or in vector form: $\mathbf{R'} = \mathbf{r} - \mathbf{v}t.$

Here, we have assumed that at t=0, the coordinates S and S′ were coincident and the motion of S′ is along x direction with uniform velocity **v**.

The inverse or the conjugate set of transformation equations (transformation from S′-system to S-system) is

$x = X' + vt'$

$y = Y'$

$z = Z'$

And $t = t'.$

Or in vector form: $\mathbf{r} = \mathbf{R'} + \mathbf{v}t'$

Transformation of distance or length:

Let us consider the two frames S and S′ again. The distance between two points in these frames are given by

S frame: distance $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$

And the length L between two points is given by

$L = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$

S′ frame: distance $\Delta X' = X'_2 - X'_1$, $\Delta Y' = Y'_2 - Y'_1$, $\Delta Z' = Z'_2 - Z'_1$

And the length L′ between two points is given by

$L' = [(X'_2 - X'_1)^2 + (Y'_2 - Y'_1)^2 + (Z'_2 - Z'_1)^2]^{1/2}$

Now we know that the two frames are related with each other by Galilean transformation, or

$X' = x - vt$

$Y' = y$

$Z' = z$

And $t' = t$

So

$$\Delta X' = X'_2 - X'_1 = (x_2 - vt) - (x_1 - vt) = x_2 - x_1 = \Delta x$$

$$\Delta Y' = Y'_2 - Y'_1 = y_2 - y_1 = \Delta y$$

$$\Delta Z' = Z'_2 - Z'_1 = z_2 - z_1 = \Delta z$$

Hence the distance between two points remains unchanged or invariant in the two frames. Similarly we can show that the length L is also invariant in the two frames, or

$$L' = [(X'_2 - X'_1)^2 + (Y'_2 - Y'_1)^2 + (Z'_2 - Z'_1)^2]^{1/2}$$

$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$= L$$

Hence

$$L' = L.$$

The transformation equations for the velocity:

Now if we take first derivative of the transformation equations, we have

$V_{X'} = V_x - v$ (since v is constant)

$V_{Y'} = V_y$

$V_{Z'} = V_z$

Or in vector form: $\mathbf{V' = V - v}$

Or we have inverse transformation: $\mathbf{V = V' + v}$

These equations are called as the ***<u>Galilean law of velocity addition</u>***.

These are the transformation equations for the velocity. Here we see that the velocity is not same in the two frames, so velocity is not invariant to Galilean transformations.

Transformation equations for the acceleration:

Now again taking differentiation of velocity transformation equations, we have

$A_{X'} = A_x$

$A_{Y'} = A_y$

$A_{Z'} = A_z$

Since velocity **v** of S' frame is constant hence its differentiation with respect to time will give zero.

Or in vector form: **A'=A.**

HENCE ACCELERATION IS INVARIANT UNDER GALILEAN TRANSFORMATION.

So in the moving S' frame, Newton's second law is written as

$F_{X'} = MA_{X'} = MA_x = F_x$

$F_{Y'} = MA_{Y'} = MA_y = F_y$

$F_{Z'} = MA_{Z'} = MA_z = F_z$

So force components are equal in both frames. So the Newton's laws, which govern the behavior of the system, do not change when we make Galilean transformation.

NEWTON'S LAW ARE THUS SAID TO REMAIN UNCHANGED OR INVARIANT UNDER GALILEAN TRANSFORMATION.

It is obvious that the second law of Newton will have the same form in these two frames of reference since

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$$

Hence, the Newton's second law of motion is said to be invariant with respect to the Galilean Transformations; this is called ***Galilean invariance***.

The behavior of all mechanical systems will thus be identical in all inertial frames in uniform translation (or at rest) with respect to each other. This implies that an observer at rest in a frame will not able to decide by performing any mechanical experiment, if his frame is at rest or in uniform linear motion.

This equivalence of all inertial frames with regards to the laws of motion is known as Newtonian relativity.

3. Motion of a particle

Nothing characterizes our daily lives more than motion itself. A game of cricket or football, the graceful movements of dancer, falling leaves, rising and setting sun are all examples of matter in motion. What is motion? We say that an object is moving if it occupies different positions at different interval of time. The study of motion deals with the questions: where? And when?

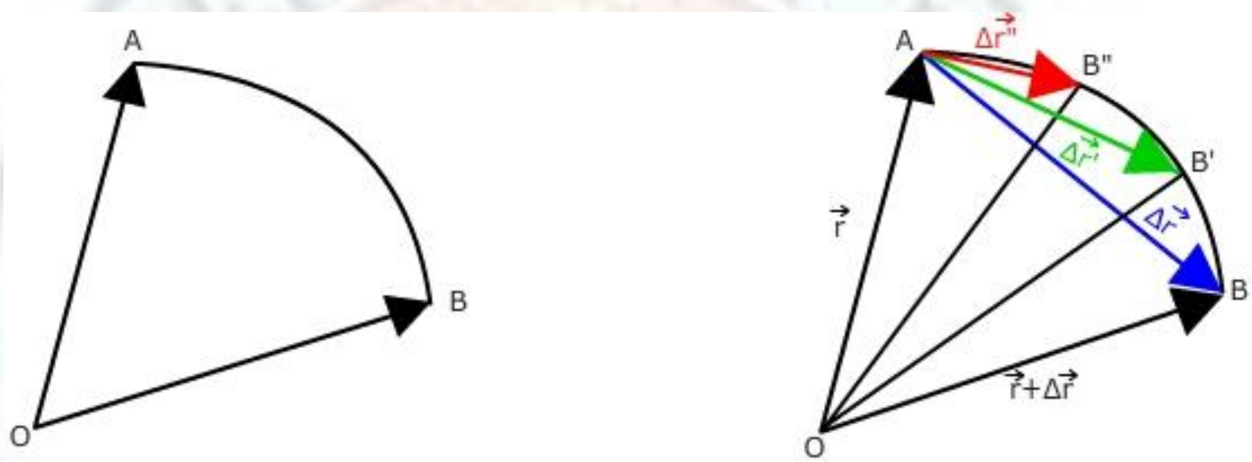


Figure -2.2 Instantaneous Positions of particle at various points in space

Displacement, Velocity and Acceleration: We now consider the motion of a single particle in space (fig2.2)**.** Let it be at the position A at the instant of time t and at B at the instant of time $t + \Delta t$. As described in above paragraph the position of a particle in a particular frame of reference is given by a position vector drawn from the origin of the coordinate axes in that frame to the position of the particle. Let the position vectors of A and B with respect to origin O be **r** and **r + Δr**, respectively. The displacement of the particle in the time Δt, is given by

$$\mathbf{V}_{av} = \frac{\Delta \mathbf{r}}{\Delta t},$$

Since Δt is a scalar quantity the direction of $\mathbf{V}_{av}$ is the same as that of $\Delta\mathbf{r}$, $\mathbf{V}_{av}$ is the velocity at which the particle would have travelled distance AB in uniform and rectilinear motion during the interval of time Δt.

Let us now represent the instantaneous velocities of the particle in passing through the points A and B of its path (fig2.2). We can see that the velocity at B is different from that at A, i.e. velocity is changing in magnitude and direction. Hence, the particle experiences an acceleration .just as we have defined average and instantaneous velocity, we can define average and instantaneous acceleration.

If the velocity of the particle changes from **v** to **v** + **Δv** within the time interval from t to t +Δt, then the **average acceleration** $\mathbf{a}_{av}$ during this interval of time is given by

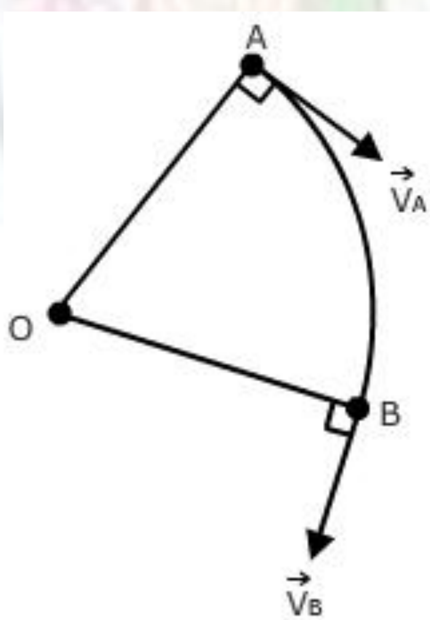
$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta \mathbf{t}}$$

Once again as Δt is a scalar quantity the direction of $\mathbf{a}_{av}$ is along **Δv**. When the interval of time Δt decreases, the ratio $\frac{\Delta \mathbf{v}}{\Delta \mathbf{t}}$ approaches a limit. We define **instantaneous acceleration** of a particle at a given instant of time as,

$$\mathbf{a} = \lim_{\Delta t \to 0} \Delta \mathbf{v}/\Delta t = \frac{d\mathbf{v}}{d\mathrm{t}}$$

So, acceleration is the derivative of **v** w.r.t time, i.e.

$$\mathbf{a} = \frac{d\mathbf{v}}{d\mathrm{t}} = \frac{d^2\mathbf{r}}{dt^2}$$



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Instantaneous velocities at point A and B.

and $a_x = \frac{dVx}{dt}$, $a_y = \frac{dVy}{dt}$, $a_z = \frac{dVz}{dt}$

4. Motion of System of Particles

Till now we have studied the motion of a single particle, however, there are many situations in which we need to deal with the systems of many particles. For example, the Solar System comprising of the sun, the planets, their satellites, comets and asteroids is a many-body system. Gas filled in a cylinder is also a many–body system if its molecules are considered as the point masses. Objects such as exploding stars, an acrobat, a javelin thrown in air, a car, a ball can all be treated as many-body systems.

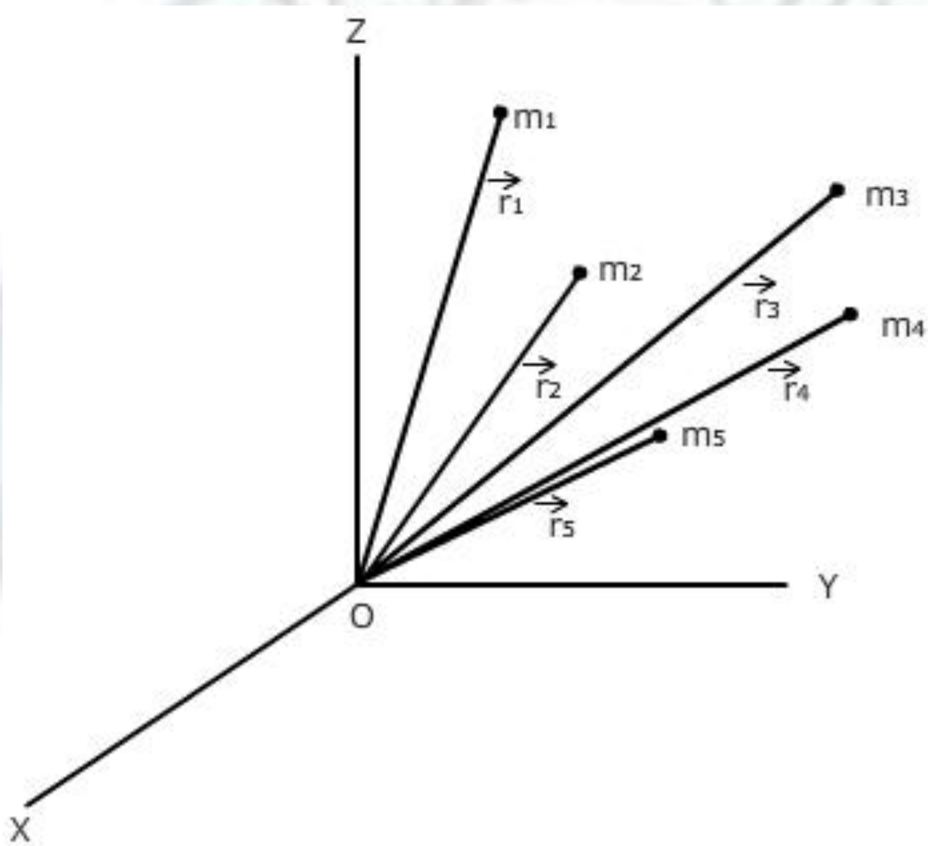

Figure 2.4 System of five particles

Consider a system of five particles as shown in fig 2.4. We can represent the position of each of these particles by position vectors $\mathbf{r_1}$,$\mathbf{r_2}$,$\mathbf{r_3}$, $\mathbf{r_4}$, $\mathbf{r_5}$, respectively as shown in the figure. Now we describe the various physical quantities of motion as we have done for the single particle motion.

Displacement: The individual displacement of each particle from the origin is given by their respective position vectors, i.e., $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$, $\mathbf{r_4}$, $\mathbf{r_5}$, respectively. Where we can have

$\mathbf{r_i} = x_i\,\mathbf{i} + y_i\,\mathbf{j} + z_i\,\mathbf{k}$

and the magnitude of each displacement vector is given by

$r_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}$.

Velocity: The individual velocity of each particle from the origin is represented as velocity vectors, i.e., $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$, respectively; they are related with the position vectors as following:

$$\mathbf{v_1} = \frac{d\,\mathbf{r_1}}{dt}, \ \mathbf{v_2} = \frac{d\,\mathbf{r_2}}{dt}, \ \mathbf{v_3} = \frac{d\,\mathbf{r_3}}{dt}, \ \mathbf{v_4} = \frac{d\,\mathbf{r_4}}{dt}, \ \mathbf{v_5} = \frac{d\,\mathbf{r_5}}{dt}$$

with each velocity vector given by

$\mathbf{v_i} = v_{xi}\,\mathbf{i} + v_{yi}\,\mathbf{j} + v_{zi}\,\mathbf{k}$ with magnitude $v_i = (v_{xi}^2 + v_{yi}^2 + v_{zi}^2)^{1/2}$

Acceleration: Similarly we can define individual accelerations of the particles as following:

$$\mathbf{a_1} = \frac{d\,\mathbf{v_1}}{dt}, \ \mathbf{a_2} = \frac{d\,\mathbf{v_2}}{dt}, \ \mathbf{a_3} = \frac{d\,\mathbf{v_3}}{dt}, \ \mathbf{a_4} = \frac{d\,\mathbf{v_4}}{dt}, \ \mathbf{a_5} = \frac{d\,\mathbf{v_5}}{dt}$$

where each acceleration vector is given by

$\mathbf{a_i} = a_{xi}\,\mathbf{i} + a_{yi}\,\mathbf{j} + a_{zi}\,\mathbf{k}$

and the magnitude of acceleration vector is given by

$a_i = (a_{xi} + a_{yi} + a_{zi})^{1/2}$

so we can also have the relations:

$$\mathbf{v} = \int \mathbf{a}\,dt \quad \text{and} \quad \mathbf{r} = \int \mathbf{v}\,dt .$$

5. Summary

- The frames of references are certain suitable coordinate system relative to which we measure the motion of particles and the system of particles.

- The inertial frames of references are those frames where Newton's laws of motion are applicable.

- The non-inertial frames of references are the accelerated frames where Newton's laws are not valid.

- The Galilean transformations are the rules which provide us a way to calculate various physical quantities in different frames of references, where the second law of Newton remains invariant- Galilean invariance.

- The motion of a single particle is defined by displacement, velocity and acceleration vectors.

- The instantaneous velocity is the velocity of the particle at a particular time, while average velocity is the overall velocity average for an interval of time.

- Similarly we can define instantaneous and average acceleration for a particle.

- When we have a collection or system of particle, we have to find individual displacement, velocity and acceleration of each particle to describe the system completely.

6. Exercise

1. The position vector of a particle is given by **r =** t **i** **–**t^2 **j +** t^3 **k.** Find the velocity and acceleration of the particle at t=5 seconds.

2. Two particles are moving with the velocity vector $\mathbf{v_1}$ **=** 2xt**i** – 2yt^2**j** & $\mathbf{v_2}$ **=-**2xt^2**i**-yt**j,** respectively. Find the instantaneous positions of the particles at t=2sec and x=y=3m.

3. The acceleration of a particle is given by **a =** yz**i-** 2xz**j-**3xy**k.** Find the instantaneous velocity and position of the particle at 1second.

4. A particle moves with velocity **v=**t**i-** t^2 **j**, Find the average velocity if it moves for 10 hours.

5. Suppose a system of particles has 4 particles in it, each having mass of 2kg.If they are at the corner points of a square of side 2m and the whole system is moving with a velocity given by **v=**x^2t **i**+ xyt**j** ,Find the expression for

(1) Position vector of each particle,
(2) Velocity vector of each particle,
(3) Acceleration vector of each particle.

Fill in the blanks:

6. The inertial frames are those, which move with a _______________.

7. The non-inertial frames are ___________________.

8. The Newton's law are same in ___________________ frames.

9. The Galilean transformations are variant for____________________.

10. The Galilean invariance means invariance of ______________ in inertial frames.

State whether the following statements are true or false:

11. The Rotating frames are inertial frames of reference.

12. The velocity is invariant in Galilean transformations.

13. The system of particles does not obey all laws of newton.

14. The acceleration is invariant in inertial frames.

15. All laws of Motion are same in all inertial frames.

Choose the most appropriate option in the following questions:

16. The inertial frames of reference are those in which we have

(A) Zero velocity or Constant velocity.

(B) Constant acceleration.

(C) Varying acceleration.

17. One example of non-inertial frame is

(A) The constant velocity moving frame of reference.

(B) The rotating frame with constant angular velocity.

(C) The stationary frame of reference.

18. The acceleration vector of a system of particles is given by

(A) Taking average of all individual accelerations.

(B) Taking vector sum of individual accelerations.

(C) Taking root mean square of individual accelerations.

19. A projectile has its maximum range given by R_{MAX}. Prove that

(a) The height reached in such case is $(1/4)R_{MAX}$.

(b) The time to reach the maximum height is $(R_{MAX}/2g)^{1/2}$.

(c) The time of flight is $(2R_{MAX}/g)^{1/2}$

20. Find the equation of motion of charged particle in a uniform electric field.

21. Find the equation of motion of charged particle in a uniform magnetic field in z-direction.

22. Find the equation of motion of a charged particle in a mutually perpendicular electric and magnetic field.

23. Find the equation of motion of a massive charged particle in crossed EM fields and the gravitational field of earth.

24. An inclined plane makes an angle α with the horizontal. A projectile is launched from the bottom A of the incline with speed v_o in a direction making an angle β with the horizontal. Prove that the range R up the incline is given by

$$R= (2v_o^2 \sin(\beta-\alpha) \cos\beta)/(g\cos^2\alpha)$$

25. For the above case prove that the maximum range up the incline is given by

$$R_{max} = v_o^2 /g (1+\sin\alpha).$$

And is achieved when

$$\beta=\frac{\pi}{4} + \frac{\alpha}{2} \ .$$