Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: Euler Cauchy Equation
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Learning Objectives

The student learns to solve Non-Homogeneous DE where the D-Operator method of the previous lesson is not applicable. In particular the student will learn about the

- **Euler Cauchy Equation**
  - **find that the function x^m is a solution of the *Cauchy-Euler Equation*.**
  - **the exponent m in the solution must be root of the *Auxiliary Equation or Characteristics Equation***
  - **and the general solution would be a linear combination of such independent solutions $y = C_1x^{m_1} + C_2x^{m_2} + \cdots + C_nx^{m_n}$**
- **Euler Cauchy Equation in Second Order**
- **How to write the Euler Cauchy Equation in the *Normal Form* or *Standard Form* and solve it corresponding to the nature of the roots of the Auxiliary equation.**

Euler Cauchy Equation

10.1 Euler Cauchy Equation

We know that the derivative of $e^{\lambda x}$ gives $\lambda e^{\lambda x}$. Thus, an operation of *derivative* $\frac{d}{dx}$ over the function $e^{\lambda x}$ yields a constant λ times the original function $e^{\lambda x}$. The real beauty lies in the fact that an n^{th} operation of $\frac{d}{dx}$ over the function $e^{\lambda x}$ yields a constant λ^n times the original function $e^{\lambda x}$. Thus, a *DE* with constant coefficients

$$a_n \frac{d^n}{dx^n} y + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} y + \cdots + a_1 \frac{d}{dx} y + a_0 y = 0$$
$$\left(a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx} + a_0\right) y = 0$$

could very easily be represented in the *non-differential* form for the solution the type $e^{\lambda x}$ as

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0) y = 0$$

Now if the *DE* admits a non-trivial solution $y \neq 0$ then we have to solve for the *Auxiliary Equation or Characteristics Equation*

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0$$

to find the n values of λ_i each of which will yield a solution $y_i = e^{\lambda_i x}$. We check that $\frac{y_i}{y_j} = e^{(\lambda_i - \lambda_j)x}$ and y_i, y_j are independent if $\lambda_i \neq \lambda_j$. Presuming that the roots λ_i of the *AE* are distinct, the general solution would be a linear combination of such independent solutions

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$$

If we now observe the function $y = x^m$, then the derivative of x^m gives $m x^{m-1}$. Thus, an operation of *derivative* $\frac{d}{dx}$ over the function x^m doesn't the original function x^m. Thus a choice of x^m as a solution to an equation with constant coefficient is not feasible. However, if we chose an operation $x\frac{d}{dx}$ instead of $\frac{d}{dx}$ to be operated on x^m then we get

$$x \frac{d}{dx} x^m = x \times m x^{m-1} = m x^m$$

which has the same property of yielding a constant times the original function. Let's now try $x^2 \frac{d^2}{dx^2}$ on x^m

$$x^2 \frac{d^2}{dx^2} x^m = x^2 \frac{d}{dx} \times m x^{m-1} = m x^2 \frac{d}{dx} x^{m-1} = m x^2 \times (m-1) x^{m-2} = m(m-1) x^m$$

Again, the real beauty lies in the fact that an n^{th} operation $x^n \frac{d^n}{dx^n}$ over the function x^m yields a constant $m(m-1)\ldots(m-n+1)$ times the original function x^m. Thus, with the proper choice of operation we can construct a *DE* of the type

$$a_n x^n \frac{d^n}{dx^n} x^m + a_{n-1} x^{n-1} \frac{d^{n-1}}{dx^{n-1}} x^m + \cdots + a_1 x \frac{d}{dx} x^m + a_0 x^m = 0$$
$$\left(a_n x^n \frac{d^n}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 x \frac{d}{dx} + a_0\right) x^m = 0$$

Such form of *DE* will be written as

$$\left(a_n x^n \frac{d^n}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 x \frac{d}{dx} + a_0\right) y = 0$$

is known as the *Euler-Cauchy Equation*. And as we developed it we can expect x^m to be a solution of the *Cauchy-Euler Equation*. Drawing parallel from the *DE* with constant coefficients, this could very easily be represented in the *non-differential* form for the solution the type x^m as

$$[a_n m(m-1)\ldots(m-n+1) + \cdots + a_2 m(m-1) + a_1 m + a_0] y = 0$$

Now if the *Cauchy-Euler DE* admits a non-trivial solution $y \neq 0$ then we have to solve for the *Auxiliary Equation or Characteristics Equation*

$$a_n m(m-1)\ldots(m-n+1) + \cdots + a_2 m(m-1) + a_1 m + a_0 = 0$$

to find the n values of m_i each of which will yield a solution $y_i = x^{m_i}$. We check that $\frac{y_i}{y_j} = x^{m_i - m_j}$ and y_i, y_j are independent if $m_i \neq m_j$. Presuming that the roots m_i of the *AE* are distinct, the general solution would be a linear combination of such independent solutions

$$y = C_1 x^{m_1} + C_2 x^{m_2} + \cdots + C_n x^{m_n}$$

10.2 Euler Cauchy Equation in Second Order

Consider the general *second order linear homogeneous DE* in $y(x)$

$$a_2 x^2 \frac{d^2}{dx^2} y + a_1 x \frac{d}{dx} y + a_0 y = 0$$

Dividing throughout by a_2, we get the Cauchy-Euler Equation in the *Normal Form* or *Standard Form*

$$x^2 \frac{d^2}{dx^2} y + a x \frac{d}{dx} y + b y = 0$$

where $a = \frac{a_1}{a_2}$ & $a = \frac{a_0}{a_2}$. Now if the *DE* admits a non-trivial solution $y \neq 0$ then we have to solve for the *Auxiliary Equation or Characteristics Equation*

$$m(m-1)+am+b=0$$
$$m^2+(a-1)m+b=0$$

to find the 2 values of $m_1 \,\&\, m_2$ each of which will yield a solution $y_1 = x^{m_1} \,\&\, y_2 = x^{m_2}$

$$m_{1,2} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4\times 1\times b}}{2\times 1} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}$$

Presuming that the roots $m_1 \,\&\, m_2$ of the *AE* are distinct, the general solution would be a linear combination of such independent solutions

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

10.2.1 Case A : Two Real but Distinct Roots $m_1 \,\&\, m_2$

Here in this case $(a-1)^2 > 4b$

$$m_1 = \frac{-(a-1) + \sqrt{(a-1)^2 - 4b}}{2} \,\&\, m_2 = \frac{-(a-1) - \sqrt{(a-1)^2 - 4b}}{2}$$

We find that

$$\frac{y_1(x)}{y_2(x)} = x^{(m_1-m_2)} = x^{\left(\sqrt{(a-1)^2-4b}\right)}$$

is not a constant and so $y_1 \,\&\, y_2$ are not linearly proportional to one another. Hence, they form a *Basis of the Solutions* of the *DE*. The *general solution* would then be

$$y = x^{-\frac{(a-1)}{2}} \left\{ C_1 x^{+\frac{\sqrt{(a-1)^2-4b}}{2}} + C_2 x^{-\frac{\sqrt{(a-1)^2-4b}}{2}} \right\}$$

Example 10.2.1.1 Solve the *Euler-Cauchy DE*

$$\boldsymbol{x^2y'' + 2xy' - 6y = 0}$$

Solution:

Step 1 The *Auxiliary Equation* for the given *DE* will be obtained by replacing $x^2y'' \rightarrow m(m-1)$, $xy' \rightarrow m$ & $y \rightarrow 1$

$$m(m-1)+2m-6=0$$
$$m^2-m+2m-6=0$$
$$m^2+m-6=0$$

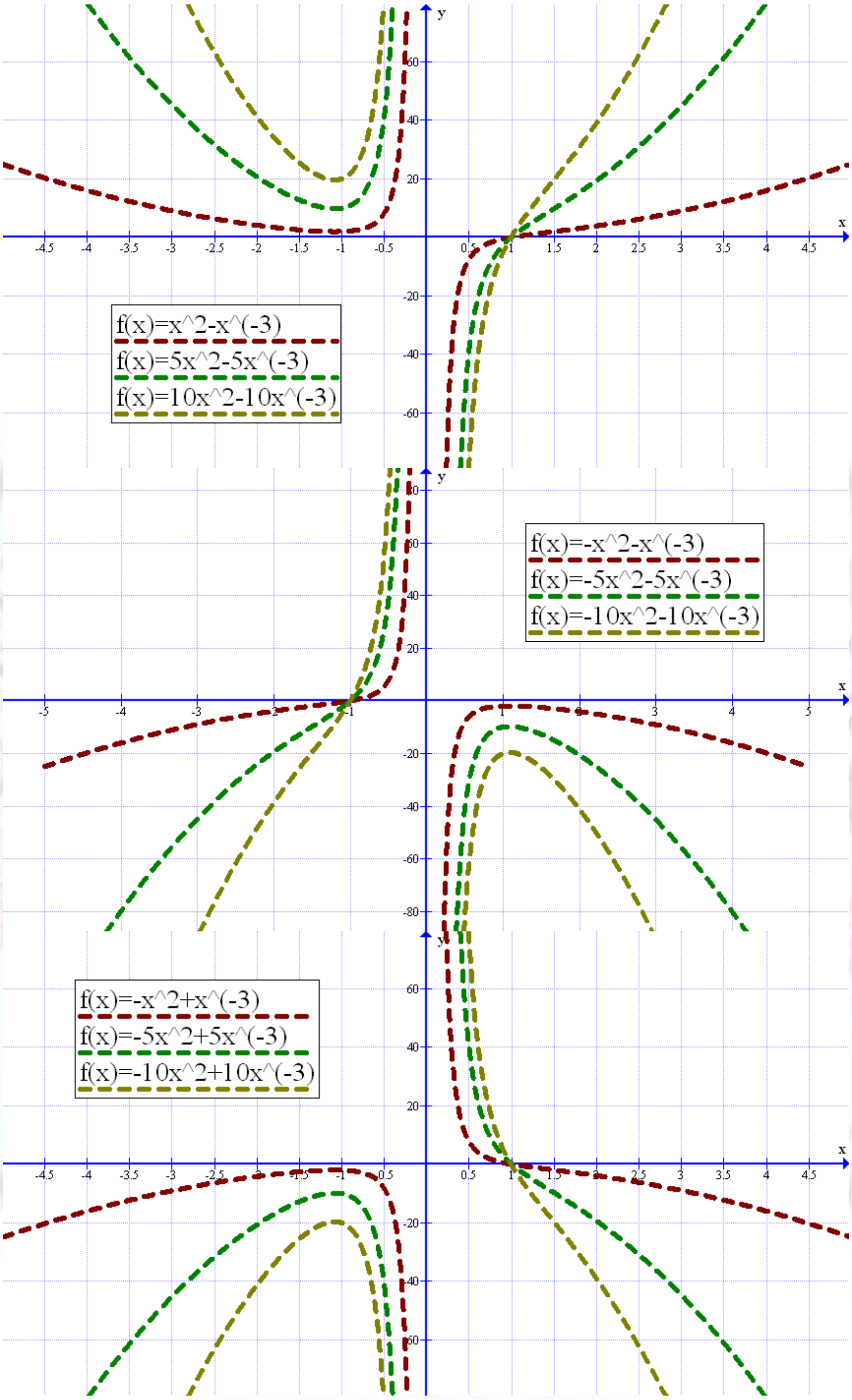
The roots are then found as

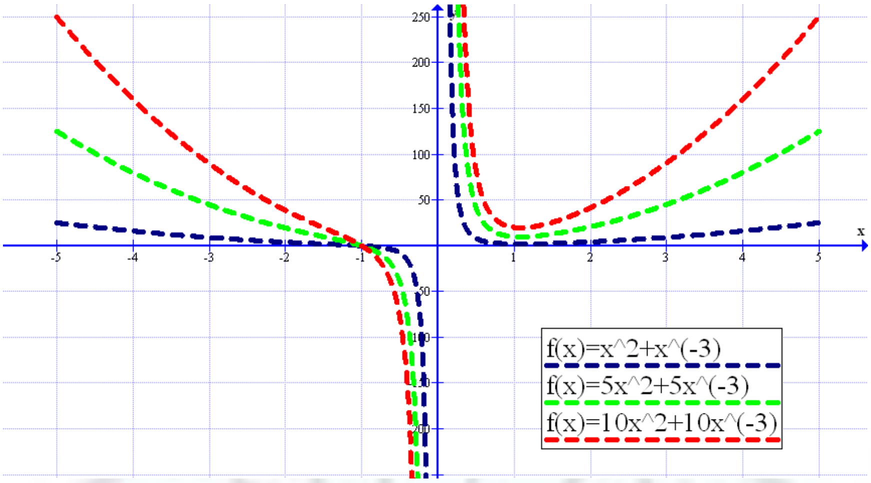
$$m_1 = \frac{-(1) + \sqrt{(1)^2 - 4(-6)}}{2} \,\&\, m_2 = \frac{-(1) - \sqrt{(1)^2 - 4(-6)}}{2}$$
$$m_1 = \frac{-1+\sqrt{25}}{2} = 2 \,\&\, m_2 = \frac{-1-\sqrt{25}}{2} = -3$$

Step 2 The General Solution would be

$$y = C_1x^2 + C_2x^{-3}$$
$$y = C_1x^2 + C_2\frac{1}{x^3}$$

Let's see it on the graph:





Example 10.2.1.2 Solve the *Euler-Cauchy DE*

$$(a+x)^2\frac{d^2y}{dx^2}+(a+x)\frac{dy}{dx}-4y=0$$

Solution:

Step 1 Let $z=(a+x)$ then $\frac{dz}{dx}=1$ and so

$$y'=\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}$$
$$y''=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dy}{dz}\right)=\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}=\frac{d}{dz}\left(\frac{dy}{dz}\right)$$

The *DE* then becomes (with $y'=\frac{dy}{dz}$ & $y''=\frac{d^2y}{dz^2}$)

$$z^2y''+zy'-4y=0$$

The *Auxiliary Equation* for the given *DE* will be obtained by replacing $z^2y''\to m(m-1)$, $zy'\to m$ & $y\to 1$

$$m(m-1)+m-4=0$$
$$m^2-m+m-4=0$$
$$m^2-4=0$$
$$m^2=4$$

The roots are then found as

$$m_1=+2\ \&\ m_2=-2$$

Step 2 The General Solution would be

$$y=C_1z^2+C_2z^{-2}$$
$$y=C_1(a+x)^2+C_2(a+x)^{-2}$$
$$y=C_1(a+x)^2+C_2\frac{1}{(a+x)^2}$$

10.2.2 Case B : Two Real but Equal Roots $m_1 = m_2 = m$

In such case we have $(a-1)^2 = 4b$

$$m_1 = -\frac{(a-1)}{2} = m_2 = m$$

We find that

$$\frac{y_1(x)}{y_2(x)} = x^{(m_1-m_2)} = 1$$

is a constant and so y_1 & y_2 are linearly proportional to one another. Hence, we need to find another solution which will be linearly independent of x^m they form a *Basis of the Solutions* of the *DE*.

To find the second solution y_2 we can apply the *Method of Reduction of Order* and write

$$y_2 = uy_1 = ux^{\frac{(1-a)}{2}}$$

Putting $y_2' = uy_1' + u'y_1$ & $y_2'' = (uy_1'' + u'y_1') + (u''y_1 + u'y_1')$ the *Cauchy-Euler Equation* becomes

$$x^2\{(uy_1'' + u'y_1') + (u''y_1 + u'y_1')\} + ax\{uy_1' + u'y_1\} + buy_1 = 0$$
$$(x^2y_1)u'' + (2x^2y_1' + axy_1)u' + (x^2y_1'' + axy_1' + by_1)u = 0$$
$$(x^2y_1)u'' + (2x^2y_1' + axy_1)u' = 0$$
$$\frac{u''}{u'} = -\frac{(2x^2y_1' + axy_1)}{(x^2y_1)} = -2\frac{y_1'}{y_1} - a\frac{1}{x}$$
$$\frac{du'}{u'} = -2\frac{dy_1}{y_1} - a\frac{dx}{x}$$
$$\ln u' = -2\ln y_1 - a\ln x = -\ln x^a{y_1}^2$$
$$u' = \frac{1}{x^a{y_1}^2}$$

Since $y_1 = x^{\frac{(1-a)}{2}}$ & ${y_1}^2 = x^{(1-a)}$ we get

$$u' = \frac{1}{x^a x^{(1-a)}} = \frac{1}{x}$$
$$du = \frac{dx}{x}$$
$$u = \ln x$$

So the second linearly independent solution would be

$$y_2 = x^{\frac{(1-a)}{2}}\ln x$$

Therefore, we eventually have the *general solution* as

$$y = (C_1\ln x + C_2)x^{\frac{(1-a)}{2}}$$

Example 10.2.2.1 Solve the *Euler Cauchy DE*

$$\boldsymbol{x^2y'' + 3xy' + y = 0}$$

Solution:

Step 1 The *Auxiliary Equation* for the given *DE* will be obtained by replacing $x^2y'' \rightarrow$

$m(m-1)$, $xy' \rightarrow m$ & $y \rightarrow 1$

$$m(m-1) + 3m + 1 = 0$$
$$m^2 - m + 3m + 1 = 0$$
$$m^2 + 2m + 1 = 0$$

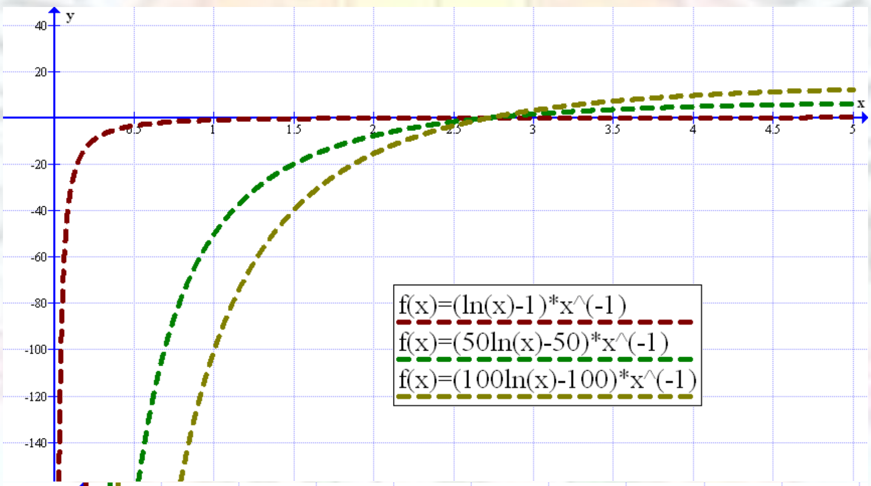
The roots are then found as

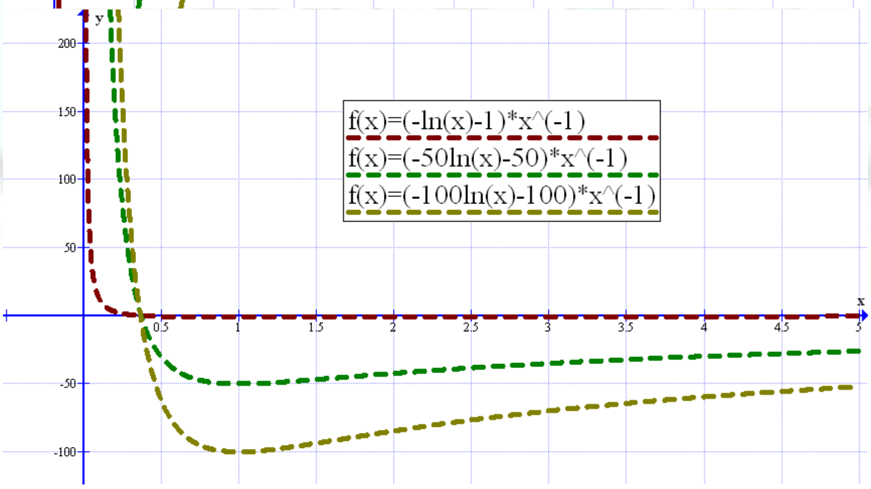
$$m_1 = \frac{-(2) + \sqrt{(2)^2 - 4(1)}}{2} \;\&\; m_2 = \frac{-(2) - \sqrt{(2)^2 - 4(1)}}{2}$$
$$m_1 = \frac{-2 + \sqrt{0}}{2} = -1 \;\&\; m_2 = \frac{-2 - \sqrt{0}}{2} = -1$$
$$m_1 = m_2 = m = -1$$

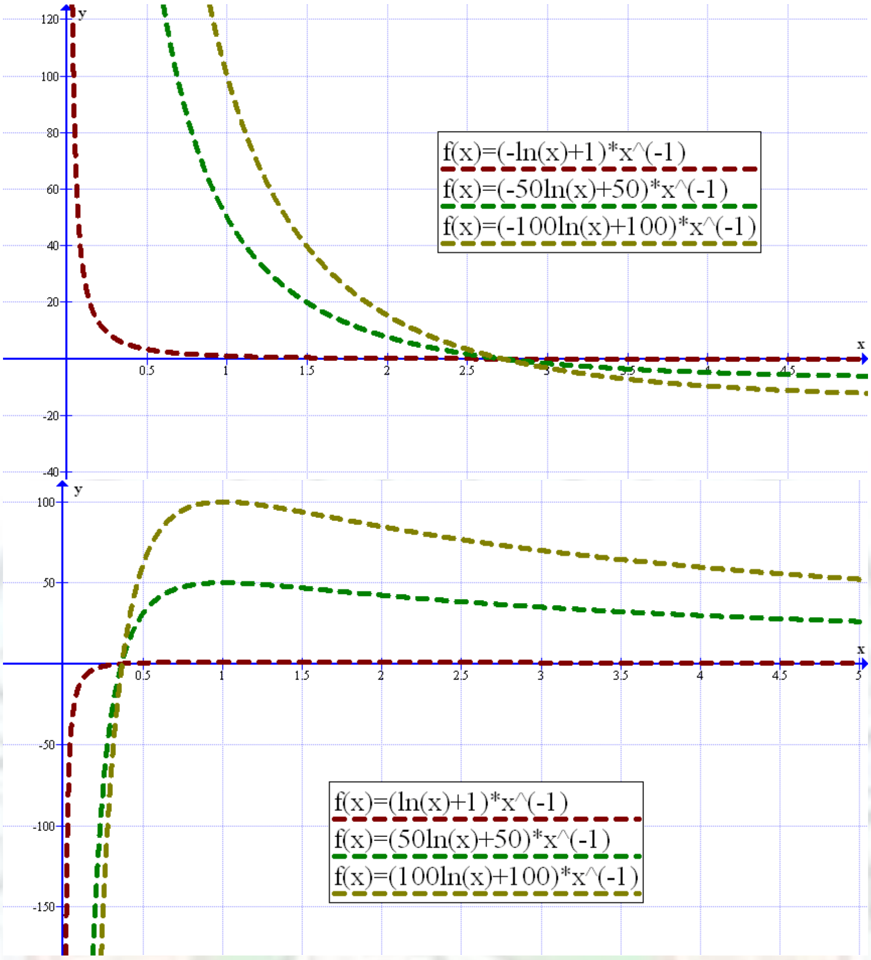
Step 2 The *General Solution* would be

$$y = (C_1 \ln x + C_2)x^{-1}$$
$$y = (C_1 \ln x + C_2)\frac{1}{x}$$

Let's see it on the graph:







Example 10.2.2.2 **Solve the *Euler Cauchy DE***

$$\boldsymbol{x^2y'' - xy' + y = e^{-x}}$$

Solution:

Step 1 The *Auxiliary Equation* for the given *DE* will be obtained by replacing $x^2y'' \to m(m-1)$, $xy' \to m$ & $y \to 1$

$$m(m-1) - m + 1 = 0$$
$$m^2 - m - m + 1 = 0$$
$$m^2 - 2m + 1 = 0$$

The roots are then found as

$$m_1 = \frac{-(-2) + \sqrt{(-2)^2 - 4(1)}}{2} \ \& \ m_2 = \frac{-(-2) - \sqrt{(-2)^2 - 4(1)}}{2}$$
$$m_1 = \frac{2 + \sqrt{0}}{2} = 1 \ \& \ m_2 = \frac{2 - \sqrt{0}}{2} = 1$$
$$m_1 = m_2 = m = 1$$

Step 2 The *CF* would be

$$y = (C_1 \ln x + C_2)x^1$$

$$y = (C_1 \ln x + C_2)x$$

Step 3 To find PI we first write the DE as

$$[D(D-1) - D + 1]y = e^{-x}$$
$$(D-1)^2 y = e^{-x}$$
$$L(D)y = e^{-x}$$

The PI would now be obtained as

$$PI = \frac{1}{L(D)} e^{-x}$$

Since $f(x) = e^{-x}$ is an exponential function we will use the rule $\frac{1}{L(D)} e^{ax} = \frac{e^{ax}}{L(a)}$ to find the PI

$$PI = \frac{1}{L(D)} e^{-x} = \frac{e^{-x}}{L(-1)}$$
$$PI = \frac{e^{-x}}{\big((-1)-1\big)^2}$$
$$PI = \frac{e^{-x}}{4}$$

Step 4 The *General Solution* would therefore be

$$y = CF + PI = (C_1 \ln x + C_2)x + \frac{1}{4} e^{-x}$$

10.2.3 Case C : Two Complex but Conjugate Roots $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$

Here in this case $(a-1)^2 < 4b$

$$m_1 = \frac{-(a-1) + \sqrt{(a-1)^2 - 4b}}{2} = \frac{-(a-1)}{2} + i\frac{\sqrt{4b-(a-1)^2}}{2}$$
$$m_2 = \frac{-(a-1) - \sqrt{(a-1)^2 - 4b}}{2} = \frac{-(a-1)}{2} - i\frac{\sqrt{4b-(a-1)^2}}{2}$$

So that we can write

$$m_1 = \alpha + i\beta \,\&\, m_2 = \alpha - i\beta$$

where $\alpha = -(a-1)/2$ & $\beta = \sqrt{4b-(a-1)^2}/2$.
We find that

$$\frac{y_1(x)}{y_2(x)} = x^{(m_1 - m_2)} = x^{2i\beta} = (e^{\ln x})^{2i\beta} = e^{i2\beta \ln x}$$

is not a constant and so y_1 & y_2 are not linearly proportional to one another. Hence, they form a *Basis of the Solutions* of the *DE*. The *general solution* would then be

$$y = x^{\alpha}\{C_1 x^{+i\beta} + C_2 x^{-i\beta}\}$$
$$y = x^{\alpha}\{C_1 (e^{\ln x})^{+i\beta} + C_2 (e^{\ln x})^{-i\beta}\}$$
$$y = x^{\alpha}\{C_1 e^{i\beta \ln x} + C_2 e^{-i\beta \ln x}\}$$

Sometimes it is convenient to use the *Euler's formula* $e^{i\beta x} = \cos\beta x + i\sin\beta x$ to write

$$y = x^{\alpha}\{C_1[\cos(\beta \ln x) + i\sin(\beta \ln x)] + C_2[\cos(\beta \ln x) - i\sin(\beta \ln x)]\}$$
$$y = x^{\alpha}\{(C_1 + C_2)\cos(\beta \ln x) + i(C_1 - C_2)\sin(\beta \ln x)\}$$
$$y = x^{\alpha}\{A\cos(\beta \ln x) + B\sin(\beta \ln x)\}$$

if $C_1 + C_2$ be any constant A and $i(C_1 - C_2)$ be any constant B. We find that $C_1 = (A - iB)/2$ and $C_1 = (A + iB)/2$ are themselves complex conjugate of each other.

Therefore, we eventually have

$$y = Ax^{\alpha}\cos(\beta \ln x) + Bx^{\alpha}\sin(\beta \ln x)$$

Putting

$$y_1(x) = x^{\alpha}\cos(\beta \ln x) \;\&\; y_2(x) = x^{\alpha}\sin(\beta \ln x)$$

Note again that $y_2/y_1 = \tan(\beta \ln x)$ and they are not linearly proportional to one other. Hence, they form a *Basis of the Solutions* of the *DE*. The *general solution* would then be

$$y = Ay_1(x) + By_2(x) = x^{\alpha}\{A\cos(\beta \ln x) + B\sin(\beta \ln x)\}$$

where $\alpha = -(a-1)/2$ & $\beta = \sqrt{4b - (a-1)^2}/2$.

Example 10.2.3.1 Solve the *Euler Cauchy DE*

$$\boldsymbol{x^2 y'' - xy' + 2y = 0}$$

Solution:

Step 1 The *Auxiliary Equation* for the given *DE* will be obtained by replacing $x^2y'' \rightarrow m(m-1)$, $xy' \rightarrow m$ & $y \rightarrow 1$

$$m(m-1) - m + 2 = 0$$
$$m^2 - m - m + 2 = 0$$
$$m^2 - 2m + 2 = 0$$

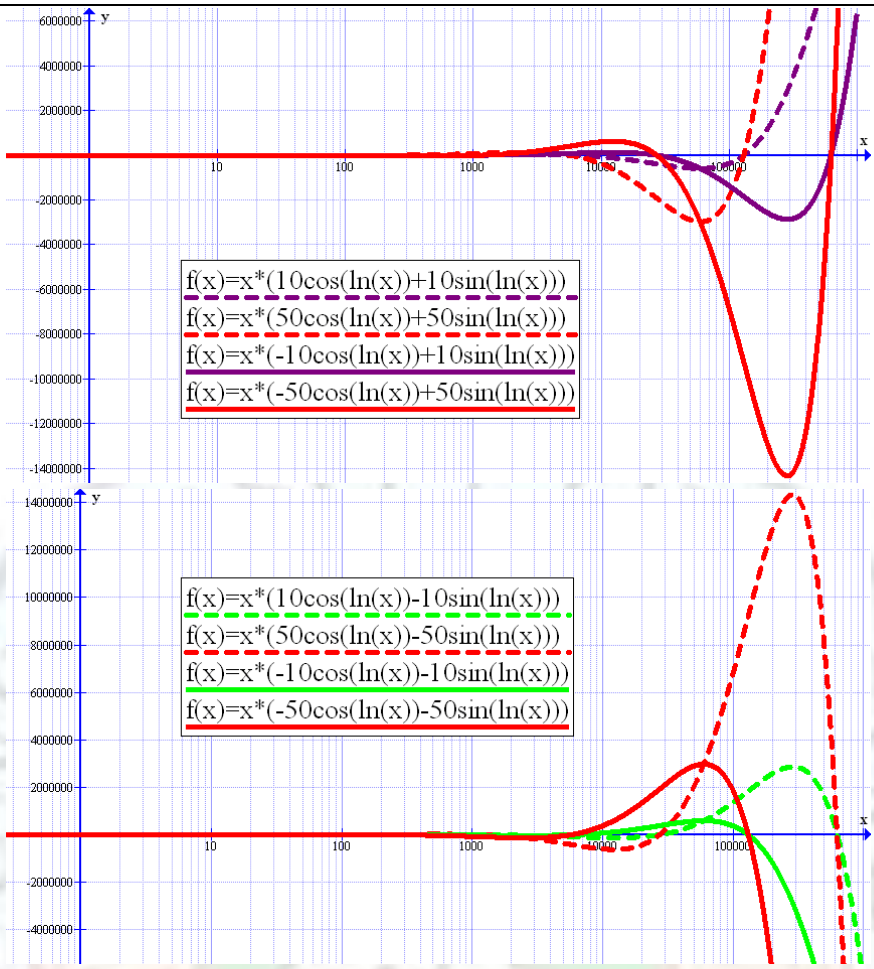
The roots are then found as

$$m_1 = \frac{-(-2) + \sqrt{(-2)^2 - 4(2)}}{2} \;\&\; m_2 = \frac{-(-2) - \sqrt{(-2)^2 - 4(2)}}{2}$$
$$m_1 = \frac{2 + \sqrt{-4}}{2} = 1 + i \;\&\; m_2 = \frac{2 - \sqrt{-4}}{2} = 1 - i$$
$$\alpha = 1 \;\&\; \beta = 1$$

Step 2 The General Solution would be

$$y = Ay_1(x) + By_2(x) = x\{A\cos(\ln x) + B\sin(\ln x)\}$$

Let's see it on the graph:



Example 10.3.4 Solve the equation

$$x^2y'' - (2a - 1)xy' + (a^2 + b^2)y = 0$$

Solution:

Step 1 The *Auxiliary Equation* for the given *DE* will be obtained by replacing $x^2y'' \to m(m-1)$, $xy' \to m$ & $y \to 1$

$$m(m-1) - (2a-1)m + (a^2+b^2) = 0$$
$$m^2 - m - 2am + m + (a^2+b^2) = 0$$
$$m^2 - 2am + a^2 + b^2 = 0$$

The roots are then found as

$$m_1 = \frac{-(-2a) + \sqrt{(-2a)^2 - 4(a^2+b^2)}}{2} \;\&\; m_2 = \frac{-(-2a) - \sqrt{(-2a)^2 - 4(a^2+b^2)}}{2}$$
$$m_1 = \frac{2a + \sqrt{-4b^2}}{2} = a + ib \;\&\; m_2 = \frac{2a - \sqrt{-4b^2}}{2} = a - ib$$
$$\alpha = a \;\&\; \beta = b$$

Step 2 The General Solution would be

$$y = Ay_1(x) + By_2(x) = x^a\{A\cos(b\ln x) + B\sin(b\ln x)\}$$

Summary

Euler Cauchy Equation

- The *DE* will written as $\left(a_n x^n \frac{d^n}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 x\frac{d}{dx} + a_0\right)y = 0$ are known as the *Euler-Cauchy Equation*.
- The function x^m is a solution of the *Cauchy-Euler Equation*.
- The exponent m in the solution must be root of the *Auxiliary Equation or Characteristics Equation* $a_n m(m-1)\dots(m-n+1) + \cdots + a_2 m(m-1) + a_1 m + a_0 = 0$
- Presuming that the roots m_i of the *AE* are distinct, the general solution would be a linear combination of such independent solutions $y = C_1 x^{m_1} + C_2 x^{m_2} + \cdots + C_n x^{m_n}$

Euler Cauchy Equation in Second Order

- The *Normal Form* or *Standard Form* is $x^2\frac{d^2}{dx^2}y + ax\frac{d}{dx}y + by = 0.$
- Presuming that the roots m_1 & m_2 of the *AE* are distinct, the general solution would be a linear combination of such independent solutions $y = C_1 x^{m_1} + C_2 x^{m_2}$

  Case A: Two Real but Distinct Roots m_1 & m_2 then

$$y = x^{-\frac{(a-1)}{2}}\left\{C_1 x^{+\frac{\sqrt{(a-1)^2-4b}}{2}} + C_2 x^{-\frac{\sqrt{(a-1)^2-4b}}{2}}\right\}$$

  Case B: Two Real but Equal Roots $m_1 = m_2 = m$ then

$$y = (C_1 \ln x + C_2)x^{\frac{(1-a)}{2}}$$

  Case C: Two Complex but Conjugate Roots $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$ then

$$y = x^{\alpha}\{A\cos(\beta\ln x) + B\sin(\beta\ln x)\}$$

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