Discipline Course-I
Semester -I
Paper: Mathematical PhysicsI IA
Lesson: First Order Exact Differential Equations
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Chapter 5: First Order Exact Differential Equations

Learning Objectives

The student will be able to learn here

- ***what are Exact Differential Equations?***
- ***which first order differential equation will classify as Exact Differential Equations and what are the requirements for the Exact DE?***
- ***how to determine the solution of such Exact DE?***
- ***to know that a separable DE is always exact.***
- ***how to reduce a Non-Exact DE into an Exact DE***
- ***and various ways to find the Integrating Factor (IF) which makes a Non-Exact DE into an Exact DE***
- ***which first order DE will qualify as linear DE and how to find it's solution***
- ***we will also learn to Reduce a special form of DE called the Bernoulli Equations into a linear form.***

First Order Exact Differential Equations

5.1 Exact Equations

Let $u(x,y)$ be a function having continuous partial derivative then its total derivative is,

$$du(x,y) = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{1}$$

Now a *First Order DE* of the form

$$M(x,y)dx + N(x,y)dy = 0 \tag{2}$$

is called an *Exact Differential* if for some function $u(x,y)$ we can write

$$du(x,y) = M(x,y)dx + N(x,y)dy$$

which means

$$du(x,y) = 0$$

By integration we immediately get the solution

$$u(x,y) = c$$

From 1 and 2 we see that 2 is exact if for some $u(x,y) = c$ we can get

$$M(x,y) = \frac{\partial u}{\partial x} \ \& \ N(x,y) = \frac{\partial u}{\partial y}$$

If $M(x,y)$ & $N(x,y)$ of the *Exact DE* have continuous first partial derivative in a region in the $x - y$ plane whose boundary is a closed curve having no self-intersections then

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \ \& \ \frac{\partial N(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

However, since $u(x,y)$ satisfies

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y}$$

the condition for 2 to be *Exact DE* is thus

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

5.2 Determination of Solution of Exact DE

If $M(x,y)dx + N(x,y)dy = 0$ is Exact the function $u(x,y)$ can be found by the following any one of the two ways

A. Step 1

Since $M(x,y) = \frac{\partial u}{\partial x}$ we integrate *wrt* x to get

$$u(x,y) = \int M(x,y)dx + h(y)$$

regarding y as a constant and where $h(y)$ is the constant of integration and not known yet. This $h(y)$ can be determined in the next step.

Step 2

Since $N(x,y) = \frac{\partial u}{\partial y}$ we differentiate RHS of Step 1 *wrt* y and equate it to $N(x,y)$to get

$$N(x,y) = \frac{\partial(\int M(x,y)dx)}{\partial y} + h'(y)$$

$$h'(y) = N(x,y) - \frac{\partial(\int M(x,y)dx)}{\partial y}$$

which can now be integrated *wrt* y to get $h(y)$.

Step 3

Having found $h(y)$ the *General Solution* would be

$$u(x,y) = \int M(x,y)dx + h(y) = c$$

where c is an arbitrary constant.

B. Step 1

Since $N(x,y) = \frac{\partial u}{\partial y}$ we integrate *wrt* y to get

$$u(x,y) = \int N(x,y)dy + l(x)$$

regarding x as a constant and where $l(x)$ is the constant of integration and not known yet. This $l(x)$ can be determined in the next step.

Step 2

Since $M(x,y) = \frac{\partial u}{\partial x}$ we differentiate RHS of Step 1 and equate it to $M(x,y)$ *wrt* x to get

$$M(x,y) = \frac{\partial(\int N(x,y)dy)}{\partial x} + l'(x)$$

$$l'(x) = M(x,y) - \frac{\partial(\int N(x,y)dy)}{\partial x}$$

which can now be integrated *wrt* x to get $l(x)$.

Step 3

Having found $l(x)$ the *General Solution* would be

$$u(x,y) = \int N(x,y)dy + l(x) = c$$

where c is an arbitrary constant.

Examples of Exact First Order DE:

Example 5.2.1 Determine if the following *First Order DE* is *Exact*

$$\mathbf{(3x^2y + y^3)y' + (3y^2x + x^3) = 0}$$

and find its *General Solution.* Is the solution explicit or implicit.

Solution:
(i) Checking for *Exactness*
We know that a *First Order DE* can be written as $M(x, y)dx + N(x, y)dy = 0$ and for this equation to be *Exact*, the necessary condition would be

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

So we first write the *DE* in the Differential Form

$$(3y^2x + x^3)dx + (3x^2y + y^3)dy = 0$$

We note that for the given equation $M(x, y) = 3y^2x + x^3$ and $N(x, y) = 3x^2y + y^3$. So $\frac{\partial M(x,y)}{\partial y} = 6yx$ and $\frac{\partial N(x,y)}{\partial x} = 6xy$ leading to $\frac{\partial N(x,y)}{\partial x} = \frac{\partial M(x,y)}{\partial y}$. Hence the given *DE* is *Exact.*

(i) Finding the *General Solution*
Since the equation is exact we can find $u(x, y)$ such that $du = 0$. For this we retrace the steps already discussed :
Step 1
Since $M(x, y) = \frac{\partial u}{\partial x}$ we integrate *wrt* x to get

$$u(x, y) = \int M(x, y)dx + h(y)$$

$$u(x, y) = \int (3y^2x + x^3)dx + h(y) = \left(3y^2\frac{x^2}{2} + \frac{x^4}{4}\right) + h(y)$$

Step 2
Since $N(x, y) = \frac{\partial u}{\partial y}$ we differentiate RHS of Step 1 *wrt* y and equate it to $N(x, y)$to get

$$N(x, y) = \frac{\partial(\int M(x, y)dx)}{\partial y} + h'(y)$$

$$(3x^2y + y^3) = \frac{\partial\left(3y^2\frac{x^2}{2} + \frac{x^4}{4}\right)}{\partial y} + h'(y) = 3yx^2 + h'(y)$$

$$h'(y) = (3x^2y + y^3) - 3yx^2 = y^3$$

$$h(y) = \int y^3\, dy = \frac{y^4}{4}$$

Step 3
Having found $h(y)$ the *General Solution* would be

$$u(x, y) = \left(3y^2\frac{x^2}{2} + \frac{x^4}{4}\right) + \frac{y^4}{4} = c$$

$$6x^2y^2 + x^4 + y^4 = A$$

where $A = 4c$ is an arbitrary constant. We can see that the solution is *Implicit*.

Example 5.2.2 Is the following *First Order DE Exact*

$$\mathbf{3y^2dx + xdy = 0}$$

and find its *Particular Solution* with the initial value $y(1) = 1/2$.

Solution:
(i) Checking for *Exactness*
We know that a *First Order DE* can be written as $M(x,y)dx + N(x,y)dy = 0$ and for this equation to be *Exact*, the necessary condition would be

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

We note that for the given equation $M(x,y) = 3y^2$ and $N(x,y) = x$. So $\frac{\partial M(x,y)}{\partial y} = 6y$ and $\frac{\partial N(x,y)}{\partial x} = 1$ leading to $\frac{\partial N(x,y)}{\partial x} \neq \frac{\partial M(x,y)}{\partial y}$. Hence the given *DE* is *Not Exact.*

(i) Finding the *General Solution*
The equation in the present form is *Not Exact* but we can rewrite it in a *Separable Form* as

$$\frac{1}{x}dx = -\frac{1}{3y^2}dy$$

Since every *Separable DE* is *Exact* we can find a $u(x,y)$ such that $du = 0$ by simply integrating the *Separable DE*

$$\int \frac{1}{x}dx = -\int \frac{1}{3y^2}dy$$

$$\ln x = \frac{1}{3y} + A$$

$$x = Be^{\frac{1}{3y}}$$

where $c = e^A$ is an arbitrary constant.The *General Solution* would be

$$xe^{-\frac{1}{3y}} = c$$

$$u(x,y) = xe^{-\frac{1}{3y}} = c$$

Its *Particular Solution* with the initial value $y(1) = 1/2$ would require

$$c = 1 \times e^{-\frac{1}{3\times 1/2}} = e^{-\frac{2}{3}}$$

and so

$$x = e^{\frac{1}{3}\left(\frac{1}{y}-2\right)}$$

5.3 Value Addition: A Separable DE is always Exact

Result 1. Prove that a *Separable DE* is always *Exact*!

Solution: A differential equation in differential form

$$M(x,y)dx + N(x,y)dy = 0$$

will be separable if

$$M(x,y) = M(x) \;\; \& \;\; N(x,y) = N(y)$$

So a *Separable DE* is

$$M(x)dx + N(y)dy = 0$$

The necessary condition for a *DE* to be *Exact* is

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

For the *Separable DE* present situation

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial M(x)}{\partial y} = 0 \;\; \& \;\; \frac{\partial N(x,y)}{\partial y} = \frac{\partial N(y)}{\partial x} = 0$$

will always be true. Hence, it will always be that

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Hence a *Separable DE* is always *Exact*.

5.4 Integrating Factor : Reduction of Non-Exact DE to Exact DE

We know that a *First Order DE* can be written as $M(x,y)dx + N(x,y)dy = 0$ and for this equation to be *Exact*, the necessary condition would be

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

However, if we have a *First Order DE,* of the form $P(x,y)dx + Q(x,y)dy = 0$ that is *Not Exact*, then to make it *Exact* we would like to assume that there exists a function $F(x,y)$ which when multiplied to the *Not Exact DE* yields an equation which is *Exact*

$$F(x,y)\{P(x,y)dx + Q(x,y)dy\} = 0$$
$$[F(x,y)P(x,y)]dx + [F(x,y)Q(x,y)]dy = 0$$

The function $F(x,y)$ is often known as the *Integrating Factor IF.* It's obvious that

$$M(x,y) = F(x,y)P(x,y) \;\&\; N(x,y) = F(x,y)Q(x,y)$$

and the condition for exactness yields

$$\frac{\partial F(x,y)P(x,y)}{\partial y} = \frac{\partial F(x,y)Q(x,y)}{\partial x}$$

Example 5.4.1 Show that y, xy^3 and x^2y^5 are *IF* of the following *First Order DE*

$$\mathbf{ydx + 2xdy = 0}$$

and solve for the its *General Solution*.

Solution:
(i) Checking for *Exactness*
We know that a *First Order DE* can be written as $M(x,y)dx + N(x,y)dy = 0$ and for this equation to be *Exact*, the necessary condition would be

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

We note that for the given equation $M(x,y) = y$ and $N(x,y) = 2x$. So $\frac{\partial M(x,y)}{\partial y} = 1$ and $\frac{\partial N(x,y)}{\partial x} = 2$ leading to $\frac{\partial N(x,y)}{\partial x} \neq \frac{\partial M(x,y)}{\partial y}$. Hence the given *DE* is *Not Exact.*

(ii) Checking for *Exactness* by multiplying with the given *IFs*

Case A: For $IF = y$ we get the *DE* as

$$y^2dx + 2xydy = 0$$

Now $M(x,y) = y^2$ and $N(x,y) = 2xy$. So $\frac{\partial M(x,y)}{\partial y} = 2y$ and $\frac{\partial N(x,y)}{\partial x} = 2y$ leading to $\frac{\partial N(x,y)}{\partial x} = \frac{\partial M(x,y)}{\partial y}$.
Hence the given *DE* is now *Exact.*
The solution can now be found as

Step 1

$$u(x,y) = \int y^2dx + h(y) = y^2x + h(y)$$

Step 2

$$N(x,y) = 2xy = \frac{\partial(y^2x)}{\partial y} + h'(y) = 2xy + h'(y)$$
$$h'(y) = 0$$
$$h(y) = A$$

Step 3

$$u(x,y) = y^2x + h(y) = c$$
$$xy^2 = B$$

where $B = c - A$ is an arbitrary constant.

Case B: For $IF = xy^3$ we get the *DE* as

$$xy^4dx + 2x^2y^3dy = 0$$

Now $M(x,y) = xy^4$ and $N(x,y) = 2x^2y^3$. So $\frac{\partial M(x,y)}{\partial y} = 4xy^3$ and $\frac{\partial N(x,y)}{\partial x} = 4xy^3$ leading to $\frac{\partial N(x,y)}{\partial x} = \frac{\partial M(x,y)}{\partial y}$. Hence the given *DE* is now *Exact.*

The solution can now be found as

Step 1

$$u(x,y) = \int xy^4dx + h(y) = y^4\frac{x^2}{2} + h(y)$$

Step 2

$$N(x,y) = 2x^2y^3 = \frac{\partial\left(\frac{y^4x^2}{2}\right)}{\partial y} + h'(y) = 2y^3x^2 + h'(y)$$

$$h'(y) = 0$$

$$h(y) = A$$

Step 3

$$u(x,y) = \frac{y^4x^2}{2} + h(y) = c$$

$$x^2y^4 = B$$

where $B = 2(c - A)$ is an arbitrary constant.

Case C: For $IF = x^2y^5$ we get the *DE* as

$$x^2y^6dx + 2x^3y^5dy = 0$$

Now $M(x,y) = x^2y^6$ and $N(x,y) = 2x^3y^5$. So $\frac{\partial M(x,y)}{\partial y} = 6x^2y^5$ and $\frac{\partial N(x,y)}{\partial x} = 6x^2y^5$ leading to $\frac{\partial N(x,y)}{\partial x} = \frac{\partial M(x,y)}{\partial y}$. Hence the given *DE* is now *Exact.*

The solution can now be found as

Step 1

$$u(x,y) = \int x^2y^6dx + h(y) = y^6\frac{x^3}{3} + h(y)$$

Step 2

$$N(x,y) = 2x^3y^5 = \frac{\partial\left(\frac{y^6x^3}{3}\right)}{\partial y} + h'(y) = 2y^5x^3 + h'(y)$$

$$h'(y) = 0$$

$$h(y) = A$$

Step 3

$$u(x,y) = \frac{y^6x^3}{3} + h(y) = c$$

$$x^3y^6 = B$$

where $B = 3(c - A)$ is an arbitrary constant.

Having worked some examples with given IF we would like to find some general rules of finding the IF. For this we have a closer look at the condition for exactness

$$\frac{\partial F(x,y)P(x,y)}{\partial y} = \frac{\partial F(x,y)Q(x,y)}{\partial x}$$

$$P(x,y)\frac{\partial F(x,y)}{\partial y} + F(x,y)\frac{\partial P(x,y)}{\partial y} = Q(x,y)\frac{\partial F(x,y)}{\partial x} + F(x,y)\frac{\partial Q(x,y)}{\partial x}$$

$$F(x,y)\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]=Q(x,y)\frac{\partial F(x,y)}{\partial x}-P(x,y)\frac{\partial F(x,y)}{\partial y}$$

Since solving this in general would be complicated, we follow a *Golden Rule*

The rule is to look for an *IF* $F(x,y)$ such that

5.4.1 The IF $F(x,y)$ is only a function of x

The *IF* is only a function of variable x then $F(x,y)=F(x)$ and so

$$F(x)\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]=Q(x,y)\frac{\partial F(x)}{\partial x}-P(x,y)\frac{\partial F(x)}{\partial y}=Q(x,y)\frac{\partial F(x)}{\partial x}$$

$$F(x)\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]=Q(x,y)\frac{\partial F(x)}{\partial x}$$

$$\frac{1}{F(x)}\frac{\partial F(x)}{\partial x}=\frac{\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]}{Q(x,y)}$$

If the RHS happens to be exclusively a function $g(x)$ of x, i.e.,

$$g(x)\equiv\frac{\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]}{Q(x,y)}$$

then we can write

$$\frac{1}{F(x)}\frac{\partial F(x)}{\partial x}=g(x)$$

This can be integrated to give the *IF*

$$\frac{dF(x)}{F(x)}=g(x)dx$$

$$\ln F(x)=\int g(x)dx$$

$$F(x)=exp\left[\int g(x)dx\right]$$

When $\frac{\partial M(x,y)}{\partial y}\neq\frac{\partial N(x,y)}{\partial x}$ *so that the equation is not exact and if it is found that* $\left[\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}\right]/N(x,y)=g(x)$ *then* IF *of the equation is* $IF=F(x)=exp[\int g(x)dx]$

5.4.2 The IF $F(x,y)$ is only a function of y

The *IF* is only a function of variable y then $F(x,y)=F(y)$ and so

$$F(y)\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]=Q(x,y)\frac{\partial F(y)}{\partial x}-P(x,y)\frac{\partial F(y)}{\partial y}=-P(x,y)\frac{\partial F(y)}{\partial y}$$

$$F(y)\left[\frac{\partial P(x,y)}{\partial y}-\frac{\partial Q(x,y)}{\partial x}\right]=-P(x,y)\frac{\partial F(y)}{\partial y}$$

$$\frac{1}{F(y)}\frac{\partial F(y)}{\partial y} = -\frac{\left[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right]}{P(x,y)}$$

If the RHS happens to be exclusively a function $h(y)$ of y, i.e.,

$$h(y) \equiv \frac{\left[\frac{\partial P(x,y)}{\partial y} - \frac{\partial Q(x,y)}{\partial x}\right]}{P(x,y)}$$

then we can write

$$\frac{1}{F(y)}\frac{\partial F(y)}{\partial y} = -h(y)$$

This can be integrated to give the *IF*

$$\frac{dF(y)}{F(y)} = -h(y)dy$$

$$\ln F(y) = -\int h(y)dy$$

$$F(y) = exp\left[-\int h(y)dy\right]$$

When $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$ so that the equation is not exact and if it is found that $\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]/M(x,y) = h(y)$ then IF of the equation is $IF = F(y) = exp[-\int h(y)dy]$

5.4.3 The DE is Homogeneous & $xM + yN \neq 0$

In case the equation is Homogeneous (which means that both $M(x,y)$ & $N(x,y)$ are homogeneous functions of same degree) and $xM(x,y) + yN(x,y) \neq 0$, then the IF of the equation is $IF = 1/[xM(x,y) + yN(x,y)]$

To show this let's rewrite the LHS of the DE in form as

$$0 = Mdx + Ndy = \frac{1}{2}\left\{\left(2xM\frac{dx}{x} + xM\frac{dy}{y} - xM\frac{dy}{y}\right) + \left(2yN\frac{dy}{y} + yN\frac{dx}{x} - yN\frac{dx}{x}\right)\right\}$$

$$0 = Mdx + Ndy = \frac{1}{2}\left\{\left(xM\frac{dx}{x} + xM\frac{dy}{y}\right) + \left(xM\frac{dx}{x} - xM\frac{dy}{y}\right) + \left(yN\frac{dy}{y} + yN\frac{dx}{x}\right) + \left(yN\frac{dy}{y} - yN\frac{dx}{x}\right)\right\}$$

$$0 = Mdx + Ndy = \frac{1}{2}\left\{(xM + yN)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (xM - yN)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

$$0 = \frac{Mdx + Ndy}{(xM + yN)} = \frac{1}{2}\left\{\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{(xM - yN)}{(xM + yN)}\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

$$0 = \frac{Mdx + Ndy}{(xM + yN)} = \frac{1}{2}\left\{d(\ln xy) + \frac{(xM - yN)}{(xM + yN)}d\left(\ln\frac{x}{y}\right)\right\}$$

Since M and N are homogeneous of the same degree we can write

$$f\left(\frac{x}{y}\right) \equiv \frac{(xM - yN)}{(xM + yN)}$$

and so the DE reduces to

$$\frac{Mdx + Ndy}{(xM + yN)} = \frac{1}{2}\left\{d(\ln xy) + f\left(\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right)\right\} = 0$$

which means

$$d(\ln xy) + f\left(\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right) = 0$$
$$d(\ln xy) + f\left(\exp\left[\ln\frac{x}{y}\right]\right) d\left(\ln\frac{x}{y}\right) = 0$$

Now for simplicity we replace the function $f\left(\exp\left[\ln\frac{x}{y}\right]\right)$ by $F\left(\ln\frac{x}{y}\right) = f\left(\exp\left[\ln\frac{x}{y}\right]\right)$

$$d(\ln xy) + F\left(\ln\frac{x}{y}\right) d\left(\ln\frac{x}{y}\right) = 0$$

This can be integrated, by writing a new variable $\tau = \ln\left(\frac{x}{y}\right)$ to give

$$\ln xy = -\int F(\tau)d\tau + c$$

where the function $F = f = \frac{(xM-yN)}{(xM+yN)}$ is to be written as a function of $\tau = \ln\left(\frac{x}{y}\right)$.

Example 5.4.3.1 Solve the *First Order* DE by reducing it to *Exact* equation

$$\boldsymbol{(x^2y - 2y^2x)dx - (x^3 - 3x^2y)dy = 0}$$

Solution: Since the *First Order DE* of the form $Mdx + Ndy = 0$ appears be *Homogeneous*, so first we check for it

$$M(tx, ty) = (tx)^2ty - 2(ty)^2tx = t^3 \times (x^2y - 2y^2x) = t^3 \times M(x, y)$$
$$N(tx, ty) = -(tx)^3 + 3(tx)^2ty = t^3 \times (-x^3 + 3x^2y) = t^3 \times N(x, y)$$

so that M and N are homogeneous of the degree 3.
Now we know it is homogeneous, so can check for

$$x(x^2y - 2y^2x) + y(-x^3 + 3x^2y) = y^2x^2 \neq 0$$

So we can look for the function

$$f\left(\frac{x}{y}\right) \equiv \frac{(xM - yN)}{(xM + yN)} = \frac{x(x^2y - 2y^2x) - y(-x^3 + 3x^2y)}{x(x^2y - 2y^2x) + y(-x^3 + 3x^2y)} = \frac{2x^3y - 5x^2y^2}{y^2x^2}$$
$$F(\tau) \equiv 2\frac{x}{y} - 5 = 2e^\tau - 5$$

where $\tau = \ln\frac{x}{y}$ or $\frac{x}{y} = e^\tau$.
Integration will now yield

$$c + \ln xy = -\int F(\tau)d\tau = -\int (2e^\tau - 5)d\tau = 5\tau - 2e^\tau$$

$$c + \ln x + \ln y = 5\ln\frac{x}{y} - 2\frac{x}{y}$$
$$c + 2\frac{x}{y} = 4\ln x - 6\ln y$$
$$3\ln y - 2\ln x = B - \frac{x}{y}$$

where $A = e^B$ and so is always positive. So, we finally arrive at

$$y^3 = Ax^2 e^{-\left(\frac{x}{y}\right)}$$

which is an *explicit* solution for the DE.

5.4.4 The DE is of the Form $f_1(xy)ydx + f_1(xy)ydx = 0$ & $xM - yN \neq 0$

When $xM(x,y) - yN(x,y) \neq 0$ and the equation can be written in the form $f_1(xy)ydx + f_1(xy)ydx = 0$ then $1/[xM(x,y) - yN(x,y)]$ is the IF of the equation.

To show this let's rewrite the LHS of the DE in differential form as

$$Mdx + Ndy = \frac{1}{2}\left\{\left(2xM\frac{dx}{x} + xM\frac{dy}{y} - xM\frac{dy}{y}\right) + \left(2yN\frac{dy}{y} + yN\frac{dx}{x} - yN\frac{dx}{x}\right)\right\}$$
$$Mdx + Ndy = \frac{1}{2}\left\{\left(xM\frac{dx}{x} + xM\frac{dy}{y}\right) + \left(xM\frac{dx}{x} - xM\frac{dy}{y}\right) + \left(yN\frac{dy}{y} + yN\frac{dx}{x}\right) + \left(yN\frac{dy}{y} - yN\frac{dx}{x}\right)\right\}$$
$$Mdx + Ndy = \frac{1}{2}\left\{(xM + yN)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (xM - yN)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$
$$\frac{Mdx + Ndy}{(xM - yN)} = \frac{1}{2}\left\{\frac{(xM + yN)}{(xM - yN)}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$
$$\frac{Mdx + Ndy}{(xM - yN)} = \frac{1}{2}\left\{\frac{(xM + yN)}{(xM - yN)}d(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\}$$

Since M and N are such that $M = yf_1(xy)$ and $N = xf_2(xy)$ we find that

$$\frac{(xM - yN)}{(xM + yN)} = \frac{(xyf_1(xy) - yxf_2(xy))}{(xyf_1(xy) + yxf_2(xy))} = \frac{(f_1(xy) - f_2(xy))}{(f_1(xy) + f_2(xy))} \equiv f(xy)$$

and so the DE reduces to

$$\frac{Mdx + Ndy}{(xM - yN)} = \frac{1}{2}\left\{f(xy)d(\ln xy) + d\left(\ln\frac{x}{y}\right)\right\} = 0$$

which means

$$f(xy)d(\ln xy) + d\left(\ln\frac{x}{y}\right) = 0$$
$$f(\exp[\ln xy])d(\ln xy) + d\left(\ln\frac{x}{y}\right) = 0$$

Now for simplicity we replace the function $f(\exp[\ln xy])$ by $F(\ln[xy]) = f(\exp[\ln xy])$

$$F(\ln[xy])d(\ln[xy]) + d\left(\ln\left[\frac{x}{y}\right]\right) = 0$$

This can be integrated, by writing a new variable $\tau = \ln[xy]$, to give

$$\ln\left(\frac{x}{y}\right) = -\int F(\tau)d\tau + c$$

where the function $F = f = \frac{(xM+yN)}{(xM-yN)}$ is to be written as a function of $\tau = \ln(xy)$.

Example 5.4.4.1 Solve the *First Order* DE by reducing it to *Exact* equation

$$(1 + 2yx)ydx + (1 + 2xy - x^3y^3)xdy = 0$$

Solution: Since the *First Order DE* is of the form $Mydx + Nxdy = 0$, so first we check for

$$xM - yN = x(1 + 2yx)y - y(1 + 2xy - x^3y^3)x = x^4y^4 \neq 0$$

So we can look for the function

$$f(xy) \equiv \frac{(xM + yN)}{(xM - yN)} = \frac{(1 + 2yx)yx + (1 + 2xy - x^3y^3)xy}{x^4y^4} = \frac{(2xy + 4x^2y^2 - x^4y^4)}{y^4x^4}$$

$$f(xy) = \frac{2}{y^3x^3} + \frac{4}{y^2x^2} - 1$$

$$F(\tau) \equiv \frac{2}{e^{3\tau}} + \frac{4}{e^{2\tau}} - 1 = 2e^{-3\tau} + 4e^{-2\tau} - 1$$

where $\tau = \ln xy$ or $xy = e^\tau$.
Integration will now yield

$$c + \ln\left(\frac{x}{y}\right) = -\int F(\tau)d\tau = -\int (2e^{-3\tau} + 4e^{-2\tau} - 1)d\tau = -\left(-\frac{2}{3}e^{-3\tau} - \frac{4}{2}e^{-2\tau} - \tau\right)$$

$$c + \ln\left(\frac{x}{y}\right) = \frac{2}{3}e^{-3\tau} + \frac{4}{2}e^{-2\tau} + \tau = \frac{2}{3x^3y^3} + \frac{2}{x^2y^2} + \ln xy$$

$$\ln y = -\left(\frac{1}{3x^3y^3} + \frac{1}{x^2y^2}\right) + B = -\left(\frac{1 + 3xy}{3x^3y^3}\right) + B$$

So, we finally arrive at

$$y = Ae^{-\left(\frac{1+3xy}{3x^3y^3}\right)}$$

which is an *explicit* solution for the *DE* and where A is the arbitrary constant.

5.4.5 The DE is of the Form $x^\alpha y^\beta(mydx + nydx) = 0$

If the equation can be written in the form $x^\alpha y^\beta(mydx + nxdy) = 0$ then $x^{-\alpha+km-1}y^{-\beta+kn-1}$ is the IF of the equation.

To show this let's rewrite the DE in differential form as

$$x^{-\alpha+km-1}y^{-\beta+kn-1}x^\alpha y^\beta(mydx + nxdy) = 0$$

$$x^{km-1}y^{kn-1}(mydx+nxdy)=0$$
$$mx^{km-1}y^{kn}dx+nx^{km}y^{kn-1}dy=0$$
$$\frac{1}{k}y^{kn}(kmx^{km-1}dx)+\frac{1}{k}x^{km}(kny^{kn-1}dy)=0$$
$$\frac{1}{k}y^{kn}d(x^{km})+\frac{1}{k}x^{km}d(y^{kn})=0$$
$$d\left(\frac{x^{km}y^{kn}}{k}\right)=0$$

This can be integrated to give the *General Solution* to the *DE*

$$x^{km}y^{kn}=c$$

However, usually equations come in *some combination* such as a *composite* of two differentials

$$\left[x^{\alpha}y^{\beta}(m_1ydx+n_1xdy)\right]+\left[x^{\gamma}y^{\delta}(m_2ydx+n_2xdy)\right]=0$$

From the first general solution $\frac{x^{km}y^{kn}}{k}=A$ for a DE of the form $x^{\alpha}y^{\beta}(mydx+nxdy)=0$ the *IF* was $x^{-\alpha+km-1}y^{-\beta+kn-1}$, therefore it is tempting to consider two factors $x^{-\alpha+k_1m_1-1}y^{-\beta+k_1n_1-1}$ & $x^{-\gamma+k_2m_2-1}y^{-\delta+k_2n_2-1}$ as the *IFs* of the individual components of the *composite DE*. So as to get

$$x^{-\alpha+k_1m_1-1}y^{-\beta+k_1n_1-1}\left[x^{\alpha}y^{\beta}(m_1ydx+n_1xdy)\right]+x^{-\gamma+k_2m_2-1}y^{-\delta+k_2n_2-1}\left[x^{\gamma}y^{\delta}(m_2ydx+n_2xdy)\right]=0$$

and the left side is easily integrated to give the *General Solution* to the *DE*

$$\frac{x^{k_1m_1}y^{k_1n_1}}{k_1}+\frac{x^{k_2m_2}y^{k_2n_2}}{k_2}=c$$

However, the *composite* DE can't have two different *IFs* for its different components therefore the two *IFs* must be equal

$$x^{-\alpha+k_1m_1-1}y^{-\beta+k_1n_1-1}=x^{-\gamma+k_2m_2-1}y^{-\delta+k_2n_2-1}$$

But this would mean the following quantities be equated

$$-\alpha+k_1m_1-1=-\gamma+k_2m_2-1$$
$$-\beta+k_1n_1-1=-\delta+k_2n_2-1$$

$$k_1m_1-k_2m_2=(\alpha-\gamma)$$
$$k_1n_1-k_2n_2=(\beta-\delta)$$

$$\begin{pmatrix}m_1 & -m_2\\ n_1 & -n_2\end{pmatrix}\begin{pmatrix}k_1\\ k_2\end{pmatrix}=\begin{pmatrix}\gamma-\alpha\\ \delta-\beta\end{pmatrix}$$

This can be solved to find values of k_1 *&* k_2 *and put in the solution*

$$\frac{x^{k_1m_1}y^{k_1n_1}}{k_1}+\frac{x^{k_2m_2}y^{k_2n_2}}{k_2}=c$$

Example 5.4.5.1 Solve the *First Order* DE by reducing it to *Exact* equation

$$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$$

Solution: Although this *First Order DE* is of the form $Mydx + Nxdy = 0$, we can rewrite it as

$$y^2dx + 2x^2ydx + 2x^3dy - xydy = 0$$
$$y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$$

and identified with the equation

$$x^\alpha y^\beta(m_1ydx + n_1xdy) + x^\gamma y^\delta(m_2ydx + n_2xdy) = 0$$

We find that

$m_1 = 1$	$m_2 = 2$	$\alpha = 0$	$\gamma = 2$
$n_1 = -1$	$n_2 = 2$	$\beta = 1$	$\delta = 0$
$k_1m_1 = k_1$	$-k_2m_2 = -2k_2$	$\alpha - \gamma = -2$	
$k_1n_1 = -k_1$	$-k_2n_2 = -2k_2$	$\beta - \delta = 1$	

By using $\frac{x^{k_1m_1}y^{k_1n_1}}{k_1} + \frac{x^{k_2m_2}y^{k_2n_2}}{k_2} = c$, the *General Solution* would be

$$\frac{1}{k_1}x^{k_1}y^{-k_1} + \frac{1}{k_2}x^{2k_2}y^{2k_2} = c$$

where again it follows from $\begin{pmatrix} m_1 & -m_2 \\ n_1 & -n_2 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \gamma - \alpha \\ \delta - \beta \end{pmatrix}$ that $m_1k_1 - m_2k_2 = \gamma - \alpha$ & $n_1k_1 - n_2k_2 = \delta - \beta$ and so

$$k_1 - 2k_2 = -2$$
$$-k_1 - 2k_2 = 1$$

So, we finally arrive at $k_1 = -\frac{3}{2}$ & $k_2 = \frac{1}{4}$ and

$$-\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c$$
$$\left(\frac{y}{x}\right)^{\frac{3}{2}} - 6\sqrt{xy} = A$$

as the *General solution* where A is the arbitrary constant.

5.5 Linear Equations

A first order differential equation in standard form $y' = f(x, y)$ is said to be linear if $f(x, y)$ is a linear function of y

$$f(x, y) = -p(x)y + q(x)$$

i.e., a function p of x times y plus another function q of x.
It is also often written as

$$y' + p(x)y = q(x) \quad (1)$$

If the RHS of *Equation 1* is zero i.e., $q(x) = 0$ the equation is said to be *Homogeneous* otherwise it is said to be *Non-Homogeneous*.

5.6 Determination of Solution of First Order Linear DE

a) Solution to the *Homogeneous* Equation

Now a *First Order DE* of the form

$$y' + p(x)y = 0$$

is *Separable* and separation of the variable yields

$$\frac{dy}{y} = -p(x)dx$$

On integration, we immediately get the solution

$$\ln|y| = -\int p(x)dx + A$$

and upon taking exponential on both sides (with $C = \pm e^A$ when $y > 0$ & $y < 0$ respectively)

$$y(x) = Ce^{-\int p(x)dx}$$

This solution of the *Homogeneous DE* is also part of the solution of the *Non-Homogeneous DE*.
It is to be reminded that for $C = 0$ we have the trivial solution $y(x) = 0$.

b) Solution to the *Non-Homogeneous* Equation

Now a *First Order Linear DE* of the form

$$y' + p(x)y = q(x)$$

can be shown to possess an *IF* $F(x)$ which is only a function of x. For this multiply the *Equation 2* with yet to be determined *IF* $F(x)$, which then yields

$$F(x)y' + F(x)p(x)y = F(x)q(x)$$

For the *DE* to be equivalent to the original *DE* we require $F(x)$ to be non-zero on the x interval of interest.
If we require $F(x)$ be such that $F'(x) = F(x)p(x)$ then the *LHS* of the *DE* becomes the full derivative so that

$$F(x)y' + F'(x)y = \frac{d}{dx}[F(x)y] = F(x)q(x)$$

On integration we immediately get the solution

$$F(x)y = \int F(x)q(x)dx + A$$
$$y(x) = \frac{1}{F(x)}\left\{\int F(x)q(x)dx + A\right\}$$

To find $F(x)$ we solve $F'(x) = F(x)p(x)$ to get

$$\frac{dF}{F} = p(x)dx$$
$$F(x) = Ae^{\int p(x)dx}$$

the constant A is assumed to be unity since it gets cancelled in the solution to the equation. Thus,

$$IF = F(x) = e^{\int p(x)dx}$$

and the General Solution will be

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + A\right]$$

If we wish to solve for A to satisfy an initial condition $y(x = a) = y_a$ then we replace the definite integral $\int_a^\tau p(\tau)d\tau$ in place of indefinite integral taking the lower limit as $x = a$,

$$y(x) = e^{-\int_a^\tau p(\varphi)d\varphi}\int_a^x e^{\int_a^\tau p(\varphi)d\varphi}q(\tau)d\tau$$

Examples of First Order Linear DE:

Example 5.6.1 Determine if the following *First Order DE* is *Linear*

$$\boldsymbol{dy + (5y - x)dx = 0}$$

and find its *General Solution.* Is the solution explicit or implicit.

Solution:
(i) Checking for *Linearity*
We know that a *First Order DE* is *Linear* if y' is a linear function of y, for the given equation

$$y' = -5y + x$$

(ii) Finding the *General Solution*
Since the equation is linear, so we first write the *DE* in the Form

$$y' + 5y = x$$

We note that for the given equation $p(x) = 5$ and $q(x) = x$.
Now we retrace the steps already discussed :
Step 1
The *IF* would be

$$F(x) = e^{\int p(x)dx} = e^{\int 5dx} = e^{5x}$$

Step 2
The *General Solution* would therefore be

$$y(x) = \frac{1}{F(x)}\left[\int F(x)q(x)dx + A\right]$$

$$y(x) = \frac{1}{e^{5x}}\left[\int e^{5x}\, x\, dx + A\right]$$

$$y(x) = e^{-5x}\left[\left\{x\left(\int e^{5x}\, dx\right) - \int 1\left(\int e^{5x}\, dx\right) dx\right\} + A\right]$$

$$y(x) = e^{-5x}\left[\left\{x\left(\frac{e^{5x}}{5}\right) - \int 1\left(\frac{e^{5x}}{5}\right) dx\right\} + A\right]$$

$$y(x) = e^{-5x}\left[\frac{xe^{5x}}{5} - \frac{e^{5x}}{25} + A\right]$$

$$y(x) = \frac{x}{5} - \frac{1}{25} + Ae^{-5x}$$

where A is an arbitrary constant.

Example 5.6.2 Determine if the following *First Order DE* is *Linear*

$$\boldsymbol{xdy - 2ydx = (x-2)e^x dx}$$

and find its *General Solution.* Is the solution explicit or implicit.

Solution:
(i) Checking for *Linearity*
We know that a *First Order DE* is *Linear* if y' is a linear function of y, for the given equation

$$y' = \frac{2}{x}y + \frac{(x-2)e^x}{x}$$

(ii) Finding the *General Solution*
Since the equation is linear, so we first write the *DE* in the Form

$$y' - \frac{2}{x}y = \frac{(x-2)e^x}{x}$$

We note that for the given equation $p(x) = -2/x$ and $q(x) = (x-2)e^x/x$.
Now we retrace the steps already discussed :

Step 1

The *IF* would be

$$F(x) = e^{\int(-2/x)dx} = e^{-2\ln x} = \frac{1}{x^2}$$

Step 2

The *General Solution* would therefore be

$$y(x) = \frac{1}{F(x)}\left[\int F(x)q(x)dx + A\right]$$

$$y(x) = x^2\left[\int \frac{1}{x^2}\frac{(x-2)e^x}{x}\,dx + A\right]$$

$$y(x) = x^2\left[\int \frac{1}{x^2}\,e^x\,dx - \int \frac{2}{x^3}\,e^x\,dx + A\right]$$

$$y(x) = x^2\left[\frac{1}{x^2}\int e^x\,dx - \int\left(-\frac{2}{x^3}\right)e^x\,dx - \int \frac{2}{x^3}\,e^x\,dx + A\right]$$

$$y(x) = x^2\left[\frac{1}{x^2}\int e^x\,dx + A\right] = \left[\int e^x\,dx + Ax^2\right]$$

$$y(x) = e^x + Ax^2$$

where A is an arbitrary constant.

5.7 Equation Reducible to Linear Form (Bernoulli Equations)

A first order differential equation in standard form $y' = f(x,y)$ is said to be Bernoulli if $f(x,y)$ is a function of y such as

$$y' = f(x,y) = -p(x)y + q(x)y^n$$

i.e., a function $p(x)$ of x times y plus another function $q(x)$ of x times y^n where n is a real number other then $\{0,1\}$.
It is also often written as

$$y' + p(x)y = q(x)y^n$$

or

$$y^{-n}y' + p(x)y^{-n+1} = q(x)$$

If the RHS of *Equation 1* is zero i.e., $q(x) = 0$ the equation is said to be *Homogeneous* otherwise it is said to be *Non-Homogeneous*.

5.8 Determination of Solution of Bernoulli's DE

The Bernoulli's DE is non-linear in y but can be reduced a form which is Linear in another variable v by the transformation

$$v(y) = y^{-n+1}$$

We can see that this yields $dv/dx = dv/dy \times dy/dx = (-n+1)y^{-n}\,dy/dx$ and so the DE reduces to

$$\frac{1}{(-n+1)}\frac{dv}{dx} + p(x)v = q(x)$$
$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x)$$

The equation is a *First Order Linear DE* of the form

$$v' + P(x)v = Q(x)$$

and it's General Solution will be

$$v(x) = e^{-\int P(x)dx}\left[\int e^{\int P(x)dx} Q(x)dx + A\right]$$
$$v(x) = e^{-(1-n)\int p(x)dx}\left[(1-n)\int e^{(1-n)\int p(x)dx} q(x)dx + A\right]$$

Replacing the variable $v(x)$ by y we finally get,

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\left[(1-n)\int e^{(1-n)\int p(x)dx} q(x)dx + A\right]$$

Examples of Bernoulli's DE:

Example 5.8.1 Determine the *General Solution of the Bernoulli's DE*

$$\boldsymbol{y' - y = xy^5}$$

Solution:
Since the equation is of Bernoulli's type, we retrace the steps already discussed :
Step 1

Write the DE as

$$y^{-5}y' - y^{-4} = x$$

Introduce the variable v by the transformation

$$v(y) = y^{-4}$$

We can see that this yields $dv/dx = (-4)y^{-5}\, dy/dx$ and so the DE reduces to

$$\frac{1}{(-4)}\frac{dv}{dx} - v = x$$
$$\frac{dv}{dx} + 4v = -4x$$

which is Linear in v.

Step 2

Since $p(x) = 4$ & $q(x) = -4x$, the IF would be

$$F = e^{\int 4dx} = e^{4x}$$

The *General Solution* would therefore be

$$v = e^{-4x}\left[\int e^{4x}\,(-4x)\,dx + A\right]$$

$$v = e^{-4x}\left[\left\{(-4x)\int e^{4x}\,dx - \int\,(-4)\left(\int e^{4x}\,dx\right)\,dx\right\} + A\right]$$

$$v = e^{-4x}\left[\left\{(-4x)\times\frac{1}{4}e^{4x} - \int\,(-4)\left(\frac{1}{4}e^{4x}\right)\,dx\right\} + A\right]$$

$$v = e^{-4x}\left[\left\{-xe^{4x} + \int\,e^{4x}\,dx\right\} + A\right]$$

$$v = e^{-4x}\left[\left\{-xe^{4x} + \frac{1}{4}e^{4x}\right\} + A\right]$$

$$v = -x + \frac{1}{4} + Ae^{-4x}$$

Step 3

Substituting back y, the *General Solution* would be

$$\frac{1}{y^4} = -x + \frac{1}{4} + Ae^{-4x}$$

where A is an arbitrary constant.

Example 5.8.2 Determine the *General Solution of the Bernoulli's DE*

$$\mathbf{\sin y\, y' = \cos x\,(2\cos y - \sin^2 x)}$$

Solution:

Lets retrace the steps discussed :

Step 1

Write the DE as

$$\sin y\, y' - (2\cos x)\cos y = -\cos x \sin^2 x$$

and note that it is Linear in $\cos y$ and we can substitute $t = \cos y$ & $\frac{dt}{dx} = -\sin y\, y'$ to get

$$t' + (2\cos x)t = \cos x \sin^2 x$$

Step 2

Since $p(x) = 2\cos x$ & $q(x) = \cos x \sin^2 x$, the IF would be

$$F = e^{\int 2\cos x dx} = e^{2\sin x}$$

The *General Solution* would therefore be

$$t = e^{-2\sin x}\left[\int e^{2\sin x}\,(\cos x \sin^2 x)\,dx + A\right]$$

To solve the integral put $\tau = \sin x$

$$t = e^{-2\tau}\left[\int e^{2\tau}\tau^2\, d\tau + A\right]$$

$$t = e^{-2\tau}\left[\left\{\tau^2 \int e^{2\tau}\, d\tau - \int 2\tau\left(\int e^{2\tau}\, d\tau\right) d\tau\right\} + A\right]$$

$$t = e^{-2\tau}\left[\left\{\tau^2\left(\frac{e^{2\tau}}{2}\right) - \int 2\tau\left(\frac{e^{2\tau}}{2}\right) d\tau\right\} + A\right]$$

$$t = e^{-2\tau}\left[\frac{\tau^2 e^{2\tau}}{2} - \int \tau e^{2\tau}\, d\tau + A\right]$$

$$t = e^{-2\tau}\left[\frac{\tau^2 e^{2\tau}}{2} - \left\{\tau \int e^{2\tau}\, d\tau - \int 1\left(\int e^{2\tau}\, d\tau\right) d\tau\right\} + A\right]$$

$$t = e^{-2\tau}\left[\frac{\tau^2 e^{2\tau}}{2} - \left\{\tau\frac{e^{2\tau}}{2} - \int \frac{e^{2\tau}}{2} d\tau\right\} + A\right]$$

$$t = e^{-2\tau}\left[\frac{\tau^2 e^{2\tau}}{2} - \frac{\tau e^{2\tau}}{2} + \frac{1}{2}\frac{e^{2\tau}}{2} + A\right]$$

$$t = \frac{\tau^2}{2} - \frac{\tau}{2} + \frac{1}{4} + Ae^{-2\tau}$$

Substituting back $\sin x$ for τ, the *General Solution* would be

$$t = \frac{\sin^2 x}{2} - \frac{\sin x}{2} + \frac{1}{4} + Ae^{-2\sin x}$$

Step 3

Substituting back $\cos y$ for t, the *General Solution* would be

$$\cos y = \frac{\sin^2 x}{2} - \frac{\sin x}{2} + \frac{1}{4} + Ae^{-2\sin x}$$

where A is an arbitrary constant.

Example 5.8.3 Determine the *General Solution of the Bernoulli's DE*

$$\boldsymbol{y' + x\sin 2y = x^3\cos^2 y}$$

Solution:

Lets retrace the steps discussed :

Step 1

Try to write the DE in a more familiar form

$$y' + x\, 2\cos y \sin y = x^3 \cos^2 y$$
$$\sec^2 y\, y' + 2x\tan y = x^3$$
$$\frac{d}{dx}\tan y + 2x\tan y = x^3$$

and note that it is Linear in $\tan y$.

Step 2

Since $p(x) = 2x$ & $q(x) = x^3$, the IF would be

$$F = e^{\int 2x dx} = e^{x^2}$$

Step 3

The *General Solution* would therefore be

$$\tan y = e^{-x^2}\left[\int e^{x^2} x^3\, dx + A\right]$$

To solve the integral put $\tau = x^2$ & $d\tau = 2xdx$

$$\tan y = e^{-\tau}\left[\int e^{\tau}\tau \frac{d\tau}{2} + A\right]$$
$$\tan y = e^{-\tau}\left[\frac{1}{2}\left\{\tau\int e^{\tau}\, d\tau - \int 1\left(\int e^{\tau}\, d\tau\right) d\tau\right\} + A\right]$$
$$\tan y = e^{-\tau}\left[\frac{1}{2}\left\{\tau e^{\tau} - \int e^{\tau}\, d\tau\right\} + A\right]$$
$$\tan y = e^{-\tau}\left[\frac{1}{2}\{\tau e^{\tau} - e^{\tau}\} + A\right]$$
$$\tan y = \frac{1}{2}\{\tau - 1\} + Ae^{-\tau}$$

Substituting back x^2 for τ, the *General Solution* would be

$$\tan y = \frac{x^2 - 1}{2} + Ae^{-x^2}$$

where A is an arbitrary constant.

Example 5.8.4 Determine the *General Solution of the Bernoulli's DE*

$$\boldsymbol{y' = \frac{1}{6e^y - 2x}}$$

Solution:

Lets retrace the steps discussed :

Step 1

Try to write the DE in a more familiar form

$$\frac{dy}{dx} = \frac{1}{6e^y - 2x}$$
$$\frac{dx}{dy} = 6e^y - 2x$$
$$\frac{dx}{dy} + 2x = 6e^y$$

and note that it is Linear in x.

Step 2

Since $p(y) = 2$ & $q(y) = 6e^y$, the IF would be

$$F = e^{\int 2dy} = e^{2y}$$

Step 3

The *General Solution* would therefore be

$$x = e^{-2y}\left[\int e^{2y}\,(6e^{y})\;dx + A\right]$$

To solve the integral put $\tau = \sin x$

$$x = e^{-2y}\left[6\int e^{3y}\;\;dx + A\right]$$
$$x = e^{-2y}\left[6 \times \frac{e^{3y}}{3} + A\right]$$
$$x = 2 + Ae^{-2y}$$
$$e^{2y} = \frac{A}{(x-2)}$$

Rewriting, the *General Solution* would be

$$y = \frac{1}{2}\ln\frac{A}{(x-2)}$$

where A is an arbitrary constant.

Example 5.8.5 Determine the *General Solution of the Bernoulli's DE*

$$\boldsymbol{xdy = y\ln y\,(\ln y - 1)dx}$$

Solution:
Lets retrace the steps discussed :
Step 1
Try to write the DE in a more familiar form

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ln^2 y - \frac{1}{x}\ln y$$
$$\frac{d}{dx}\ln y + \frac{1}{x}\ln y = \frac{1}{x}\ln^2 y$$

Let $\tau = \ln y$ then

$$\frac{d\tau}{dx} + \frac{1}{x}\tau = \frac{1}{x}\tau^2$$
$$\tau^{-2}\frac{d\tau}{dx} + \frac{1}{x}\tau^{-1} = \frac{1}{x}$$
$$\frac{d}{dx}\tau^{-1} - \frac{1}{x}\tau^{-1} = -\frac{1}{x}$$

and note that it is Linear in τ^{-1}.

Step 2
Since $p(x) = -\frac{1}{x}$ & $q(x) = -\frac{1}{x}$, the IF would be

$$F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = -\frac{1}{x}$$

Step 3
The *General Solution* would therefore be

$$\tau^{-1} = (-x)\left[\int \left(-\frac{1}{x}\right)\left(-\frac{1}{x}\right) dx + A\right]$$
$$\tau^{-1} = (-x)\left[\int \frac{1}{x^2}\, dx + A\right]$$
$$\tau^{-1} = (-x)\left[\left(-\frac{1}{x}\right) + A\right]$$
$$\tau^{-1} = 1 - Ax$$

Putting $\tau = \ln y$, the *General Solution* would be

$$(\ln y)^{-1} = 1 - Ax$$
$$\ln y = \frac{1}{(1 - Ax)}$$

Rewriting, the *General Solution* would be

$$y = e^{\left[\frac{1}{(1-Ax)}\right]}$$

where A is an arbitrary constant.

Example 5.8.6 Determine the *General Solution of the Bernoulli's DE*

$$\mathbf{y' = 1 - e^{x-y}}$$

Solution:
Lets retrace the steps discussed :
Step 1

Try to write the DE in a more familiar form

$$e^y \frac{dy}{dx} = e^y - e^x$$
$$\frac{d}{dx} e^y - e^y = -e^x$$

Let $\tau = e^y$ then

$$\frac{d\tau}{dx} - \tau = -e^x$$

and note that it is Linear in τ.

Step 2

Since $p(x) = -1$ & $q(x) = -e^x$, the IF would be

$$F = e^{\int -dx} = e^{-x}$$

Step 3

The *General Solution* would therefore be

$$\tau = e^x\left[\int e^{-x}\,(-e^x)\, dx + A\right]$$

$$\tau = e^x \left[-\int dx + A \right]$$
$$\tau = e^x(-x + A)$$

Putting $\tau = e^y$, the *General Solution* would be

$$e^y = e^x(-x + A)$$
$$e^{y-x} = (A - x)$$
$$y - x = \ln(A - x)$$

Rewriting, the *General Solution* would be

$$y = x + \ln(A - x)$$

where A is an arbitrary constant.

5.9 Mixed Bag of Examples

Example 5.9.1 A loaded paratrooper weighing m Kg jumps off a helicopter with initial velocity from an altitude of d m with negligible side wind. Assume the air resistance $R(t)$ the paratrooper encountered with is: $R(t) = cv$ in which the viscous coefficient c. Determine: (a) The appropriate equation for the instantaneous descending velocity of the paratrooper and its solution and (b) the distance travelled in time τ.

Solution: By Newton's Law of Motion we get,

$$m\frac{dv}{dt} = W + R(t) = mg - cv$$

Lets retrace the steps discussed :
Step 1
Write the DE as

$$\frac{dv}{dt} + \frac{c}{m}v = g$$

and note that it is Linear in v.

Step 2
Since $p(t) = \frac{c}{m}$ & $q(t) = g$, the IF would be

$$F = e^{\int \frac{c}{m}dt} = e^{ct/m}$$

Step 3
The *General Solution* would therefore be

$$v = e^{-ct/m}\left[\int e^{ct/m}\, g\, dt + A\right]$$

$$v = e^{-ct/m}\left[\frac{mg}{c}\int e^{ct/m}\, d\left(\frac{c}{m}t\right) + A\right]$$
$$v = e^{-ct/m}\left[\frac{mg}{c}e^{ct/m} + A\right] = \frac{mg}{c} + Ae^{-\frac{ct}{m}}$$

Rewriting, the *General Solution* would be

$$v(t) = v_T + Ae^{-\gamma t}$$

where A is an arbitrary constant and $v_T = \frac{mg}{c}$ & $\gamma = \frac{c}{m}$.

Now if the paratrooper just drops himself down then $v(0) = 0$ and so $A = -v_T$ and then

$$v(t) = v_T(1 - e^{-\gamma t})$$

The distance d the paratrooper descends in time τ will be

$$d = \int_0^\tau v dt = \int_0^\tau v_T(1 - e^{-\gamma t})dt$$
$$d = v_T\tau + \frac{v_T}{\gamma}(e^{-\gamma\tau} - 1)$$
$$d = v_T\tau - \frac{v_T}{\gamma}(1 - e^{-\gamma\tau})$$

Example 5.9.2 A solid body at temperature T is submerged in a fluid of temperature T_{fluid}. Derive a mathematical model for heat transfer from a submerged solid to the surrounding fluid given that ρ is the density, V is the volume, A is the surface area & c is the specific heat of the solid. The heat transfer co-efficient between the solid and the surrounding fluid is k.

Solution: From the First Law of Thermodynamics, the energy equivalent of heat released by the solid leading to its cooling by a temperature change in a solid ΔT during time period Δt is given to the surrounding fluid,

$$\rho V \times c \times (-\Delta T) = k \times A \times \left(T - T_{fluid}\right) \times \Delta t$$

Lets retrace the steps discussed :

Step 1

Write the equation as a DE

$$-\frac{dT}{dt} = \frac{kA}{c\rho V} \times \left(T - T_{fluid}\right)$$
$$\frac{dT}{dt} + \left(\frac{kA}{c\rho V}\right)T = \left(\frac{kA}{c\rho V}\right)T_{fluid}$$

and note that it is Linear in T.

Step 2

Since $p(t) = \left(\frac{kA}{c\rho V}\right)$ & $q(t) = \left(\frac{kA}{c\rho V}\right)T_{fluid}$, the IF would be

$$F = e^{\int\left(\frac{kA}{c\rho V}\right)dt} = e^{\left(\frac{kA}{c\rho V}\right)t}$$

Step 3

The *General Solution* would therefore be

$$T = e^{-\left(\frac{kA}{c\rho V}\right)t}\left[\int e^{\left(\frac{kA}{c\rho V}\right)t}\left(\frac{kA}{c\rho V}\right)T_{fluid}\,dt + A\right]$$
$$T = e^{-\left(\frac{kA}{c\rho V}\right)t}\left[T_{fluid}\,e^{\left(\frac{kA}{c\rho V}\right)t} + A\right]$$

Rewriting, the *General Solution* would be

$$T(t) = T_{fluid} + Ae^{-\left(\frac{kA}{c\rho V}\right)t}$$

where A is an arbitrary constant.

Now if the initial temperature if T_o then $T(0) = T_o$ and so $A = T_o - T_{fluid}$ and then

$$T(t) = T_{fluid} + (T_o - T_{fluid})e^{-\left(\frac{kA}{c\rho V}\right)t}$$
$$T(t) = T_{fluid}\left(1 - e^{-\left(\frac{kA}{c\rho V}\right)t}\right) + T_o e^{-\left(\frac{kA}{c\rho V}\right)t}$$

Example 5.9.3 Determine the *General Solution of the Bernoulli's DE*

$$\tan x\, y' + y = 5\tan x\, e^{\cos x}$$

Solution:
Lets retrace the steps discussed :
Step 1

Write the DE as

$$y' + (\cot x)y = 5e^{x}$$

and note that it is Linear in y.

Step 2

Since $p(x) = \cot x$ & $q(x) = 5e^{\cos x}$, the IF would be

$$F = e^{\int \cot x dx} = e^{\int\left(\frac{\cos x}{\sin x}\right)dx} = e^{\int d(\ln \sin x)} = e^{\ln \sin x} = \sin x$$

Step 3

The *General Solution* would therefore be

$$y = \frac{1}{\sin x}\left[\int \sin x\ (5e^{\cos x})\,dx + A\right]$$
$$y = \frac{1}{\sin x}\left[-\int 5e^{\cos x} d\cos x + A\right]$$

Putting $\tau = \cos x$ in the integral we get,

$$y = \frac{1}{\sin x}\left[-\int 5e^{\tau}\,d\tau + A\right] = -\frac{5e^{\tau}}{\sin x} + \frac{A}{\sin x}$$

the *General Solution* would be

$$y = -\frac{5e^{\cos x}}{\sin x} + \frac{A}{\sin x}$$

Rewriting, the *General Solution* would be

$$y = \frac{(A - 5e^{\cos x})}{\sin x}$$

where A is an arbitrary constant.

Summary

Exact Equations

- A *First Order DE* of the form $M(x,y)dx + N(x,y)dy = 0$ is called an *Exact Differential Equation* if for some function $u(x,y)$ we can write $du(x,y) = M(x,y)dx + N(x,y)dy$
- For which the solution is $u(x,y) = c$
- To be *Exact DE* the requirement is thus $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$

Determination of Solution of Exact DE

- **Either** get $u(x,y) = \int M(x,y)dx + h(y)$ then integrate $h'(y) = N(x,y) - \frac{\partial(\int M(x,y)dx)}{\partial y}$
- the *General Solution* would be $u(x,y) = \int M(x,y)dx + h(y) = c$ where c is an arbitrary constant.
- **Or** get $u(x,y) = \int N(x,y)dy + l(x)$ then integrate $l'(x) = M(x,y) - \frac{\partial(\int N(x,y)dy)}{\partial x}$
- the *General Solution* would be $u(x,y) = \int N(x,y)dy + l(x) = c$ where c is an arbitrary constant.
- Note that : A Separable DE is always Exact

Integrating Factor (IF)

Reduction of Non-Exact DE to Exact DE

- *When* $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$ *we know that the equation is not exact and (but) if it is found that* $\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]/N(x,y) = g(x)$ *then* IF *of the equation is* $IF = F(x) = exp[\int g(x)dx]$
- *When* $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$ *so that the equation is not exact and if it is found that* $\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]/M(x,y) = h(y)$ *then* IF *of the equation is* $IF = F(y) = exp[-\int h(y)dy]$

The DE is Homogeneous & $xM + yN \neq 0$

- *In case the equation is Homogeneous (which means that both* $M(x,y)$ & $N(x,y)$ *are* *homogeneous functions of same degree) and $xM(x,y) + yN(x,y) \neq 0$, then the IF of the equation is $IF = 1/[xM(x,y) + yN(x,y)]$*

The DE is of the Form $f_1(xy)ydx + f_1(xy)ydx = 0$ & $xM - yN \neq 0$

- *When $xM(x,y) - yN(x,y) \neq 0$ and the equation can be written in the form $f_1(xy)ydx + f_1(xy)ydx = 0$ then $1/[xM(x,y) - yN(x,y)]$ is the IF of the equation.*

The DE is of the Form $x^{\alpha}y^{\beta}(mydx + nydx) = 0$

- *If the equation can be written in the form $x^{\alpha}y^{\beta}(mydx + nxdy) = 0$ then $x^{-\alpha+km-1}y^{-\beta+kn-1}$ is the IF of the equation.*
- This can be integrated to give the *General Solution* to the *DE* $x^{km}y^{kn} = c$

Linear Equations

- A first order differential equation in standard form $y' = f(x,y)$ is said to be linear if $f(x,y)$ is a linear function of y i.e., $f(x,y) = -p(x)y + q(x)$
- Thus, $IF = F(x) = e^{\int p(x)dx}$
- and the General Solution will be

$$y(x) = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx + A\right]$$

Equation Reducible to Linear Form (Bernoulli Equations)

- A first order differential equation in standard form $y' = f(x,y)$ is said to be Bernoulli if $f(x,y)$ is a function of y such as $y' = f(x,y) = -p(x)y + q(x)y^n$
- The Bernoulli's DE is non-linear in y but can be reduced to a form which is Linear in another variable v by the transformation $v(y) = y^{-n+1}$
- The equation is a *First Order Linear DE* of the form $v' + P(x)v = Q(x)$

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